3-) Calcule a integral $\int_{-a}^{a} x^2\sqrt{a^2-x^2}dx$.

Resolução:

Considere $g : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ dada por $t \mapsto a\sin t$. Tal função é derivável e tem derivada contínua, dada por $g' : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$, $t \mapsto a\cos t$. Além disso, $g(-\frac{\pi}{2}) = -a$ e $g(\frac{\pi}{2}) = a$. Portanto, segue-se do teorema de mudança de variáveis na integral de Riemann que $\int_{-a}^{a} x^2\sqrt{a^2-x^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(t)^2\sqrt{a^2-g(t)^2}g'(t)\,dt = a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t\cos^2 t\,dt = \frac{a^4}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2t\,dt = \frac{a^4}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos 4t)\,dt \overset{TFC}{=} \frac{a^4}{8}\left[t - \frac{\sin 4t}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi a^4}{8}$.