Questão 4. (2,0 pontos)

a) Seja $n \geq 1$ um inteiro. Determine $P_n(x)$ o Polinômio de Taylor de ordem n de $f(x) = e^{2x}$ em torno do ponto $x = 0$. Obtenha uma expressão para o erro $E_n(x) = f(x) - P_n(x)$, em que $x \in \mathbb{R}$.

b) Use o polinômio do item anterior para estimar $\int_0^{\frac{1}{2}} x^{10} e^{2x} dx$ com erro menor que 10^{-5}.

a)
$$f(x) = e^{2x} \Rightarrow f(0) = 1$$
$$f'(x) = 2e^{2x} \Rightarrow f'(0) = 2$$
$$f''(x) = 2^2 e^{2x} \Rightarrow f''(0) = 2^2$$
$$\vdots$$
$$f^{(n)}(x) = 2^n e^{2x} \Rightarrow f^{(n)}(0) = 2^n$$

$$\therefore \underset{x_0=0}{P_n(x)} = \sum_{i=0}^{n} \frac{f^{(i)}(0)\, x^i}{i!} = \sum_{i=0}^{n} \frac{2^i x^i}{i!} = 1 + 2x + \frac{2^2 x^2}{2!} + \frac{2^3 x^3}{3!} + \dots + \frac{2^n x^n}{n!}$$

Resto de Lagrange: $E_n(x) = f(x) - P_n(x) = f^{(n+1)}(\bar{x}) \dfrac{x^{n+1}}{(n+1)!} = \dfrac{2^{n+1} e^{2\bar{x}} x^{n+1}}{(n+1)!}$

onde $\begin{cases} \bar{x} \in]0, x[& \text{se } x > 0 \\ \bar{x} \in]x, 0[& \text{se } x < 0 \end{cases}$ OBS: $\bar{x}$ depende de x

b)
$$\left| \int_0^{1/2} x^{10} e^{2x} dx - \int_0^{1/2} x^{10} P_n(x) dx \right| = \left| \int_0^{1/2} x^{10} E_n(x) dx \right|$$

$$= \left| \int_0^{1/2} x^{10} \frac{2^{n+1} \overbrace{e^{2\bar{x}}}^{\text{depende de } x} x^{n+1}}{(n+1)!} dx \right| \underset{(2\bar{x}<1)}{\leq} \int_0^{1/2} \frac{x^{n+11} \cdot 3 \cdot 2^{n+1}}{(n+1)!} dx$$

$$= \frac{1}{(n+12)(n+1)!} \frac{1}{2^{n+12}} 2^{n+1} \cdot 3 = \frac{3}{(n+12)(n+1)!\, 2^{11}} < 10^{-5}$$

se $n \geq 3$

$$\therefore \int_0^{1/2} x^{10} e^{2x} dx \simeq \int_0^{1/2} x^{10} \left(1 + 2x + 2x^2 + \frac{4}{3} x^3\right) dx$$

$$= \frac{1}{11 \cdot 2^{11}} + \frac{2}{12 \cdot 2^{12}} + \frac{2}{13 \cdot 2^{13}} + \frac{4}{3 \cdot 14 \cdot 2^{14}}$$ com erro $< 10^{-5}$