Questão 4. Seja $F(x) = \int_0^{\sqrt[3]{x}} \sqrt[3]{x}\,\mathrm{sen}(t^2)dt$.

(1,0) a) Calcule $F'(x)$ para $x \neq 0$.

(1,5) b) Verifique se F é derivável em $x_0 = 0$. Em caso afirmativo, calcule $F'(0)$.

Note que $F(x) = x^{1/3} \cdot \int_0^{x^{1/3}} \mathrm{sen}(t^2)\,dt$

a) $$F'(x) = \frac{1}{3} x^{-2/3} \cdot \int_0^{x^{1/3}} \mathrm{sen}(t^2)dt + x^{1/3} \cdot \mathrm{sen}(x^{1/3})^2 \cdot \frac{1}{3} x^{-2/3}$$

$$= \frac{1}{3}\left[x^{-2/3} \cdot \int_0^{x^{1/3}} \mathrm{sen}(t^2)dt + \mathrm{sen}(x^{2/3}) \cdot x^{-1/3} \right], \ \forall x \neq 0$$

b) Como $\sqrt[3]{\cdot}$ ñ é derivável em zero:

$$\lim_{x \to 0} \frac{F(x) - F(0)}{x - 0} = \lim_{x \to 0} \frac{F(x)}{x} = \lim_{x \to 0} \frac{x^{1/3} \cdot \int_0^{x^{1/3}} \mathrm{sen}(t^2)dt}{x}$$

$$= \lim_{x \to 0} \frac{\int_0^{x^{1/3}} \mathrm{sen}(t^2)dt}{x^{2/3}} \overset{\text{L'Hospital}}{=} \lim_{x \to 0} \frac{\mathrm{sen}(x^{2/3}) \cdot \frac{1}{3} x^{-2/3}}{\frac{2}{3} \cdot x^{-1/3}}$$

$$= \frac{1}{2} \lim_{x \to 0} \underbrace{\frac{\mathrm{sen}(x^{2/3})}{x^{2/3}}}_{\to 1} \cdot \underbrace{x^{1/3}}_{\to 0} = 0$$

$\therefore F'(0) = 0$