4. **(2,5)** Sejam $f : \mathbb{R}^2 \to \mathbb{R}$, $f = f(x,y)$, uma função de classe $C^2$ e $F : \mathbb{R}^2 \to \mathbb{R}$ dada por

$$F(t,u) = t^2 f(t^2u, 5t+3u).$$

Calcule $\dfrac{\partial^2 F}{\partial u \partial t}(1,2)$ em termos de $f$ e de suas derivadas parciais $\dfrac{\partial f}{\partial x}$ e $\dfrac{\partial f}{\partial y}$.

Como $f \in C^2$, $f$ e suas derivadas parciais de 1ª ordem são deriváveis. Aplicando-se a regra de Leibnitz e a regra da cadeia, conclui-se que, $\forall (t,u) \in \mathbb{R}^2$:

1.) $$\frac{\partial F}{\partial t}(t,u) = 2t \cdot f(t^2u, 5t+3u) + t^2\left[\frac{\partial f}{\partial x}(t^2u, 5t+3u) \cdot 2tu + \frac{\partial f}{\partial y}(t^2u, 5t+3u) \cdot 5\right]$$

2.) $$\frac{\partial^2 F}{\partial u \partial t}(t,u) = 2t \cdot \left[\frac{\partial f}{\partial x}(t^2u, 5t+3u) \cdot t^2 + \frac{\partial f}{\partial y}(t^2u, 5t+3u) \cdot 3\right] + 2t^3 \cdot \left[\frac{\partial f}{\partial x}(t^2u, 5t+3u) + u \cdot \frac{\partial^2 f}{\partial x^2}(t^2u, 5t+3u) \cdot t^2 + u \cdot \frac{\partial^2 f}{\partial y \partial x}(t^2u, 5t+3u) \cdot 3\right] + 5t^2 \cdot \left[\frac{\partial^2 f}{\partial x \partial y}(t^2u, 5t+3u) \cdot t^2 + \frac{\partial^2 f}{\partial y^2}(t^2u, 5t+3u) \cdot 3\right]$$

$$\therefore \frac{\partial^2 F}{\partial u \partial t}(1,2) = 4\frac{\partial f}{\partial x}(2,11) + 6\frac{\partial f}{\partial y}(2,11) + 4\frac{\partial^2 f}{\partial x^2}(2,11) + 17\frac{\partial^2 f}{\partial x \partial y}(2,11) + 15\frac{\partial^2 f}{\partial y^2}(2,11)$$