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CONTINUITY OF GLOBAL ATTRACTORS FOR A CLASS OF NON LOCAL EVOLUTION EQUATIONS

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Abstract. In this work we prove that the global attractors for the flow of the equation

$$\frac{\partial m(r,t)}{\partial t} = -m(r,t) + g(\beta J * m(r,t) + \beta h), \;\; h, \, \beta \geq 0,$$

are continuous with respect to the parameters h and β if one assumes a property implying normal hyperbolicity for its (families of) equilibria.

1. Introduction.

We consider here the non local evolution equation

$$\frac{\partial m(r,t)}{\partial t} = -m(r,t) + g\left(\beta J * m(r,t) + \beta h\right), \tag{1}$$

where $m(r,t)$ is a real function on $\mathbb{R} \times \mathbb{R}_{+}$, h, β are non negative constants and $J \in C^1(\mathbb{R})$ is a non negative even function supported in the interval $[-1,1]$ and integral equal to 1. The $*$ above denotes convolution product, namely:

$$(J * m)(x) = \int_{\mathbb{R}} J(x-y)m(y)dy.$$

There are several works in the literature dedicated to the analysis of the particular case of (1) where $g \equiv \tanh$. (See, for example, [2], [15], [16], [17], [18] and [19]).

In particular (when $g \equiv \tanh$) the existence of a global compact attractor was proved in [2], for the case of bounded domain and $h = 0$ and in [21], for an unbounded domain.

If g is globally Lipschitz, the Cauchy problem for (1) is well posed, for instance, in the space of continuous bounded functions, $C_b(\mathbb{R})$, with the sup norm, since the function given by the right hand side of (1) is uniformly Lipschitz in this space, (see [5] and [7]).

It is an easy consequence of the uniquennes theorem that the subspace $\mathbb{P}_{2\tau}$ of 2τ periodic functions is invariant. We considerer here the equation (1) restricted

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to $\mathbb{P}_{2\tau}$, $\tau > 1$. As shown in a previous work ([22]), this leads naturally to the consideration of the flow generated by (1) in $L^2(S^1)$ where S^1 is the unit sphere and $*$ the convolution product in it. We now describe the assumptions and results of [22]. For the sake of clarity and future reference, it is convenient to start with a list of the hypotheses on g that were used there.

(H1) The function $g : \mathbb{R} \to \mathbb{R}$, is globally Lipschitz, that is, there exists a positive constant k_1 such that

$$|g(x) - g(y)| \leq k_1 |x - y|, \ \ \forall\, x, y \in \mathbb{R}.$$

In particular, there exist non negative constants k_2 and k_3 such that

$$|g(x)| \leq k_2 |x| + k_3, \ \ \forall\, x \in \mathbb{R}. \tag{2}$$

(H2) The function $g \in C^1(\mathbb{R})$ and g' is locally Lipschitz.

(H3) There exist non negative constants k_4 and k_5, such that

$$|g'(x)| \leq k_4 |x| + k_5, \ \ \forall\, x \in \mathbb{R}.$$

(Observe that if (H1) and (H2) hold then (H3) also holds with $k_4 = 0$ and $k_5 = k_1$.

(H4) The function g has positive derivative. In particular it is strictly increasing.

(H5) There exists $a > 0$ such that, for all $x \in \mathbb{R}$, $|g(x)| < a$. In particular, when $a < \infty$ (2) holds with $k_2 = 0$ and $k_3 = a$.

(H6) The function g^{-1} is continuous in $(-a, a)$ and the function

$$f(m) = -\frac{1}{2}m^2 - hm - \beta^{-1} i(m), \ \ m \in [-a, a],$$

where i is defined by

$$i(m) = -\int_0^m g^{-1}(s)ds, \ \ m \in [-a, a],$$

has a global minimum $\overline{m}$ in $(-a, a)$.

Under hypothesis (H1), we proved in [22] that the problem (1) is well posed in $L^2(S^1)$, and the flow thus generated is of class C^1 if one also assumes (H2). Assuming (H1) and (H3), we proved the existence of a global compact attractor in the sense of [10]. We also proved a comparison result under the hypotheses (H1) and (H4). Assuming (H1), (H3), (H4) and (H5), we showed an L_∞ estimate for the attractors. Finally, assuming (H6), we exhibited a continuous Lyapunov functional for the flow of (1) and used it to prove that, under hypotheses (H1), (H3), (H4), (H5) and (H6), the flow is gradient in the sense of [10].

This paper is organised as follows. In Section 2, we prove the upper semicontinuity property of the attractors with respect to the parameter $\lambda = (h, \beta)$ under the hypotheses (H1) and (H3) above.

The much more delicate property of lower semicontinuity is proved in Section 3. To the extent of our knowledge, the proofs of this property available in the literature assume that *the equilibrium points are all hyperbolic* and therefore isolated (see for example [1], [6], [12], [20] and [23]). However, this property cannot hold true in our case, due to the symmetries present in the equation. In fact, it is a consequence of these symmetries that the non constant equilibria arise in families and, therefore, cannot be hyperbolic. To overcome this difficulty we have had to replace the hypothesis of hyperbolicity by *normal hyperbolicity* of curves of equilibria. We then used results of [4] on the permanence of normally hyperbolic invariant manifolds and proved (in the appendix) continuity properties of the local unstable manifolds of the (non necessarily isolated) equilibria with respect to parameter $\lambda = (h, \beta)$, together with results of [11] on the limiting behaviour of trajectories.

In Section 4 we illustrated the results in the important particular case $g \equiv \tanh$.

2. **Upper semicontinuity of the attractor with respect to the parameter $\lambda = (h, \beta)$.** As proved in [22], under hypotheses, (H1) and (H2) the map

$$F(u, \lambda) = -u + g(\beta(J * u) + \beta h) \tag{3}$$

is continuously Frechet differentiable in $L^2(S^1)$ (with $*$ being now the convolution product in $L^2(S^1)$) and, therefore, the problem

$$\frac{\partial u}{\partial t} = F(u, \lambda) = -u + g(\beta(J * u) + \beta h) \qquad (\mathbf{P})_\lambda$$

generates a C^1 flow in $L^2(S^1)$ which depends on the parameter λ. From now on we denote this flow by $T_\lambda(t)$ or $T(\lambda, t)$. It is also proved there that (in a certain range of these parameters) $T_\lambda(t)$ admits a global compact attractor. A natural question to examine is the dependence of this attractor on the parameter λ. We denote by $\mathcal{A}_\lambda$ the global attractor whose existence was proved in [22].

Let us recall that a family of subsets $\{\mathcal{A}_\lambda\}$, is upper semicontinuous at λ_0 if

$$dist(A_\lambda, A_{\lambda_0}) \longrightarrow 0, \text{ as } \lambda \to \lambda_0,$$

where

$$dist(A_\lambda, A_{\lambda_0}) = \sup_{x \in A_\lambda} dist(x, A_{\lambda_0}) = \sup_{x \in A_\lambda} \inf_{y \in A_{\lambda_0}} \|x - y\|_{L^2}. \tag{4}$$

Analogously, $\{A_\lambda\}$ is lower semicontinuous at λ_0 if

$$dist(A_{\lambda_0}, A_\lambda) \longrightarrow 0, \text{ as } \lambda \to \lambda_0.$$

In this section, we prove that the family of attractors is upper semicontinuous with respect to parameter λ at $\lambda_0 \in R$, where R is the semi-bounded strip $0 \leq h \leq \infty$, $0 \leq \beta \leq \beta^*$, with $\beta^* < \frac{1}{k_2}$ and k_2 is the constant given in (2). We denote by $\|\lambda\|$ the norm of the sum in $\mathbb{R}^2_+$.

Lemma 2.1. *Under the assumptions (H1) and (H3), the flow $T_\lambda(t)$ is continuous with respect to λ, uniformly for u in bounded sets and $t \in [0, b]$ with $b < \infty$.*

Proof As shown in [22] the solutions of $(\mathbf{P})_\lambda$ satisfy the 'variations of constants formula',

$$T_\lambda(t)u = e^{-t}u + \int_0^t e^{-(t-s)} g(\beta(J * T_\lambda(s)u + \beta h) ds.$$

Let $\lambda_0 \in R$, $b > 0$ and C a bounded set in $L^2(S^1)$. Given $\varepsilon > 0$, we want to find $\delta > 0$ such that $\|\lambda - \lambda_0\| < \delta$ implies

$$\|T_\lambda(t)u - T_{\lambda_0}(t)u\|_{L^2} < \varepsilon,$$

for $t \in [0, b]$ and u in C. Since g is globally Lipschitz, for any $t > 0$ and $u \in C$, it follows that

$$\begin{aligned}\|T_\lambda(t)u - T_{\lambda_0}(t)u\|_{L^2} &\leq \int_0^t e^{-(t-s)}\|g(\beta(J * T_\lambda(s)u + \beta h)\\ &- g(\beta_0(J * T_{\lambda_0}(s)u + \beta_0 h_0)\|_{L^2} ds\\ &\leq \int_0^t e^{-(t-s)} k_1[\|\beta J * (T_\lambda(s)u) - \beta_0(J * T_{\lambda_0}(s)u)\|_{L^2}\\ &+ \|\beta h - \beta_0 h_0\|_{L^2}]ds.\end{aligned}$$

Subtracting and summing the term $\beta_0 J * T_\lambda(s)u$ and using Young's inequality, we obtain

$$\begin{aligned}\|T_\lambda(t)u - T_{\lambda_0}(t)u\|_{L^2} &\leq \int_0^t e^{-(t-s)} k_1\|\beta - \beta_0\|\|J\|_{L^1}\|T_\lambda(s)u\|_{L^2} ds\\ &+ \int_0^t e^{-(t-s)} k_1\beta_0\|J\|_{L^1}\|T_\lambda(s)u - T_{\lambda_0}(s)u\|_{L^2} ds\\ &+ \int_0^t e^{-(t-s)} k_1\|\beta h - \beta_0 h_0\|_{L^2} ds.\end{aligned}$$

From Theorem 3.3 of [22], it follows that, for all $\lambda \in R$ and $t \in [a, b]$, $\|T_\lambda(t)u\|_{L^2}$ is bounded by a positive constant L depending only of C. Thus, since $\|J\|_{L^1} = 1$, we obtain

$$\begin{aligned}\|T_\lambda(t)u - T_{\lambda_0}(t)u\|_{L^2} &\leq \{Lk_1\|\beta - \beta_0\| + k_1\|\beta h - \beta_0 h_0\|_{L^2}\}\\ &+ \int_0^t e^{-(t-s)} k_1\beta_0\|T_\lambda(s)u - T_{\lambda_0}(s)u\|_{L^2} ds\\ &\leq C(\lambda) + \int_0^t k_1\beta_0\|T_\lambda(s)u - T_{\lambda_0}(s)u\|_{L^2} ds,\end{aligned}$$

where $C(\lambda) = \{Lk_1|\beta - \beta_0| + k_1\|\beta h - \beta_0 h_0\|\}$. Therefore, by Gronwall's Lemma, it follows that

$$\|T_\lambda(t)u - T_{\lambda_0}(t)u\|_{L^2} \leq C(\lambda)e^{k_1\beta_0 t}.$$

From this, the results follows immediately. □

Theorem 2.2. *Assume the hypotheses (H1) and (H3) hold. Then the family of attractors $\mathcal{A}_\lambda$ is upper semicontinuous with respect to λ at $\lambda_0 \in R$.*

Proof From hypotheses (H1) and (H3), it follows that, for every $\lambda \in R$, the attractor $\mathcal{A}_\lambda$, given by Theorem 3.3 of [22], is in the ball $B\left(0, \frac{2\sqrt{2\tau}(k_2\beta h + k_3)}{1 - k_2\beta}\right)$ in $L^2(S^1)$. Therefore

$$\bigcup_{\lambda \in R} \mathcal{A}_\lambda \subset B\left(0, \frac{2\sqrt{2\tau}(k_2\beta^* h^* + k_3)}{1 - k_2\beta^*}\right).$$

Since $\mathcal{A}_{\lambda_0}$ is a global attractor and $B = B\left(0, \frac{2\sqrt{2\tau}(k_2\beta^* h^* + k_3)}{1 - k_2\beta^*}\right)$ is a bounded set then, for every $\varepsilon > 0$, there exists $t^* > 0$ such that $T_{\lambda_0}(t)B \subset \mathcal{A}_{\lambda_0}^{\frac{\varepsilon}{2}}$, for all $t \geq t^*$, where $\mathcal{A}_{\lambda_0}^{\frac{\varepsilon}{2}}$ is the $\frac{\varepsilon}{2}$-neighbourhood of $\mathcal{A}_{\lambda_0}$.

From Lemma 2.1, it follows that $T_\lambda(t)$ is continuous at λ_0, uniformly for u in a bounded set and t in compacts. Thus, there exists $\delta > 0$ such that for every $u \in B\left(0, \frac{2\sqrt{2\tau}(k_2\beta^* h^* + k_3)}{1-k_2\beta^*}\right)$,

$$\|\lambda - \lambda_0\| < \delta \Rightarrow \|T_\lambda(t^*)u - T_{\lambda_0}(t^*)u\|_{L^2} < \frac{\varepsilon}{2}.$$

We will show that if $\|\lambda - \lambda_0\| < \delta$ then $\mathcal{A}_\lambda \subset \mathcal{A}^{\varepsilon}_{\lambda_0}$. In fact, let $u \in \mathcal{A}_\lambda$. Since $\mathcal{A}_\lambda$ is invariant, $v = T_\lambda(-t^*)u \in \mathcal{A}_\lambda \subset B\left(0, \frac{2\sqrt{2\tau}(k_2\beta^* h^* + k_3)}{1-k_2\beta^*}\right)$. Therefore, we have

$$T_{\lambda_0}(t^*)v \in \mathcal{A}^{\frac{\varepsilon}{2}}_{\lambda_0}, \tag{5}$$

and

$$\|T_\lambda(t^*)v - T_{\lambda_0}(t^*)v\|_{L^2} < \frac{\varepsilon}{2}. \tag{6}$$

From (5) and (6), it follows that

$$u = T_\lambda(t^*)T_\lambda(-t^*)u = T_\lambda(t^*)v \in \mathcal{A}^{\varepsilon}_{\lambda_0}$$

and the upper semicontinuity of $\mathcal{A}_\lambda$ follows . □

3. **Lower semicontinuity of the attractors.** As mentioned in the introduction, a difficulty we encounter in the proof of lower semicontinuity is that, due to the symmetries present in our model, the non constant equilibria are not isolated. In fact, as we will see shortly, the equivariance property of the map F defined in (3) implies that the nonconstant equilibria appear in curves. (see Lemma 3.1) and, therefore, cannot be hyperbolic preventing the use of tools like the Implicit Function Theorem to obtain their continuity with respect to parameters.

In order to obtain the lower semicontinuity we will need the following additional hypotheses:

(H7) For each $\lambda_0 \in R$, the set E_{λ_0}, of the equilibria of $T_{\lambda_0}(t)$, is such that $E_{\lambda_0} = E_1 \cup E_2$, where

(a) The equilibria in E_1 are (constant) hyperbolic equilibria;

(b) The equilibria in E_2 are non constant and, for each $u_0 \in E_2$, zero is simple eigenvalue of the derivative with respect to u $DF_u(u_0, \lambda_0) : L^2(S^1) \to L^2(S^1)$, given by

$$DF_u(u_0, \lambda_0)v = -v + g'(\beta_0 J * u_0 + \beta_0 h_0)\beta_0(J * v);$$

(H8) The function $g \in C^2(\mathbb{R})$. (Observe that (H8) implies (H2)).

We start with some observations on the spectrum of the linearization around equilibria.

Remark 1. A simple computation shows that, if u_0 is a non constant equilibria of $T_{\lambda_0}(t)$ then zero is always an eigenvalue of the operator

$$DF_u(u_0, \lambda_0)v = -v + g'(\beta_0 J * u_0 + \beta_0 h_0)\beta_0(J * v)$$

with eigenfunction u_0'. Therefore, the hypothesis (H7)-b says that we are in the 'simplest' possible situation for the linearization around non constant equilibria.

Remark 2. Let $u_0 \in E_2$. It is easy to show that $DF_u(u_0, \lambda_0)$ is a self-adjoint operator with respect to the inner product

$$(u, v) = \int_{S^1} u(w)v(w)d\nu(w),$$

where $d\nu(w) = \frac{dw}{g'(\beta(J*u_0)(w)+\beta h)}$ is equivalent to the Lebesgue measure.

Since

$$v \to g'(\beta_0 J * u_0 + \beta_0 h_0)\beta_0(J * v)$$

is a compact operator in $L^2(S^1)$, it follows from (H7) that

$$\sigma(DF_u(u_0, \lambda_0))\backslash\{0\}$$

contains only real eigenvalues of finite multiplicity with -1 as the unique possible accumulation point.

We now prove a result on the structure of the sets of non constant equilibria.

Lemma 3.1. *Suppose that, for some $\lambda_0 \in R$, (H1), (H7) and (H8) hold. Given $u \in E_2$ and $\alpha \in S^1$, define $\gamma(\alpha; u) \in L^2(S^1)$ by*

$$\gamma(\alpha; u)(w) = u(\alpha w), \ w \in S^1.$$

Then $\Gamma = \gamma(S^1; u)$ is a closed, simple C^2 curve of equilibria of $T_{\lambda_0}(t)$ which is isolated in the set of equilibria, that is, no point of Γ is an accumulation point of $E_{\lambda_0} \setminus \Gamma$.

Proof Let $u \in E_2$, $\alpha, w \in S^1$. Then, since $(J * u)(\alpha w) = (J * \gamma(\alpha; u))(w)$, we obtain

$$\gamma(\alpha; u)(w) - g(\beta(J * \gamma(\alpha; u))(w) + \beta h) = u(\alpha w) - g(\beta(J * u)(\alpha w) + \beta h) = 0,$$

and, therefore, $\gamma(\alpha; u)$ is an equilibrium. It is clear that Γ is a closed curve.

Now, let $u_0 \in \Gamma$. From hypothesis (H7), it follows that zero is a simple eigenvalue of the operator $DF_u(u_0, \lambda_0)$. Since $DF_u(u_0, \lambda_0)$ is a self-adjoint Fredholm operator of index zero, we have the decomposition

$$L^2(S^1) = span\{v\} \oplus Y.$$

where $v \in Ker(DF_u(u_0, \lambda_0))$ and Y is the range of $\mathcal{R}(DF_u(u_0, \lambda_0))$.

Define $\widetilde{F} : \mathbb{R} \times Y \to L^2(S^1)$ by

$$\widetilde{F}(t, y) = F(u_0 + tv + y, \lambda_0).$$

Note that $\widetilde{F}(0, 0) = F(u_0, \lambda_0) = 0$. From hypotheses (H1) and (H8) it follows that $\widetilde{F}(t, \cdot)$ is of class C^2. Now $\frac{\partial}{\partial y}\widetilde{F}(0, 0) = DF_u(u_0, \lambda_0)\pi$, where $\pi : L^2(S^1) \to Y$ is the orthogonal projection (with respect to the new inner product) in Y. As $DF_u(u_0, \lambda_0)|_Y$ is injective and $Y = \mathcal{R}(DF_u(u_0, \lambda_0))$ it follows, from the Open Mapping Theorem that $DF_u(u_0, \lambda_0)|_Y$ is isomorphism onto Y. Therefore $\frac{\partial}{\partial y}\widetilde{F}(0, 0) : Y \to Y$ is an isomorphism. Hence, by the Implicit Function Theorem, there exist open sets $(-\varepsilon_0, \varepsilon_0) \subset \mathbb{R}$, $U \subset Y$ with $0 \in U$ and a unique C^2 function $\xi : (-\varepsilon_0, \varepsilon_0) \to U$ such that $\widetilde{F}(t, y) = 0$ if only if $y = \xi(t)$. As $\widetilde{F}(t, y) = 0$ whenever $u_0 + tv + y \in \Gamma$, it follows that in a neighbourhood of u_0, the curve Γ is given by $u_0 + tv + \xi(t)$ with $t \in (-\varepsilon_0, \varepsilon_0)$. In particular, Γ is C^2 and in a neighbourhood of u_0, there are no zeroes of $\widetilde{F}$ except the zeroes on Γ. Thus Γ is isolated.

Finally, suppose that Γ is not simple curve and let $u_1 \in \Gamma$ a point of self intersection. Then there exist $\alpha_1, \alpha_2 \in S^1$ such that $u_1 = \gamma(\alpha_1; u)$ and $u_1 = \gamma(\alpha_2; u)$ and, therefore $\frac{d}{d\alpha}\gamma(\alpha_1; u)$ and $\frac{d}{d\alpha}\gamma(\alpha_2; u)$ are linearly independent eigenvectors associated to the eigenvalue zero; contradicting (H7). □

Corollary 1. *Let M be a closed connected curve of equilibria in E_2 and $u_0 \in M$. Then $M = \Gamma$, where $\Gamma = \gamma(S^1, u_0)$.*

Proof Suppose that $\Gamma \not\subset M$. Then there exist equilibria in $M \setminus \Gamma$ accumulating at u_0 contradicting Lemma 3.1. Therefore $\Gamma \subseteq M$. Since Γ is a simple closed curve, it follows that $M = \Gamma$. $\square$

In order to prove our main result, we need some preliminary results , which we present in the next three subsections.

3.1. **Lower semicontinuity of the equilibria.** The lower semicontinuity of the *hyperbolic* equilibria is usually obtained via the Implicit Function Theorem. However, this approach fails here since the equilibria may appear in families as we have shown in Lemma 3.1. To overcome this difficulty, we need the concept of normal hyperbolicity, (see [4]). Recall that, if $T(t) : X \to X$ is a semigroup a set $M \subset X$ is *invariant* under $T(t)$ if $T(t)M = M$, for any $t > 0$.

Definition 3.2. Suppose that $T(t)$ is a C^1 semigroup in a Banach space X and $M \subset X$ is an invariant manifold for $T(t)$. We say that M is normally hyperbolic under $T(t)$ if

(i) for each $m \in M$ there is a decomposition

$$X = X^c_m \oplus X^u_m \oplus X^s_m$$

by closed subspaces with X^c_m being the tangent space to M at m.

(ii) for each $m \in M$ and $t \geq 0$, if $m_1 = T(t)(m)$

$$DT(t)(m)|_{X^\alpha_m} : X^\alpha_m \to X^\alpha_{m_1},\ \alpha = c, u, s$$

and $DT(t)(m)|_{X^u_m}$ is an isomorphism from X^u_m onto $X^u_{m_1}$.

(iii) there is $t_0 \geq 0$ and $\mu < 1$ such that for all $t \geq t_0$

$$\mu \inf \left\{\|DT(t)(m)x^u\| \,:\, x^u \in X^u_m,\, \|x^u\| = 1\right\} > \max\{1, \|DT(t)(m)|_{X^c_m}\|\right\}, \qquad (7)$$

$$\mu \min \left\{1, \inf\{\|DT(t)(m)x^c\| \,:\, x^c \in X^c_m,\, \|x^c\| = 1\}\right\} > \|DT(t)(m)|_{X^s_m}\|. \qquad (8)$$

The condition (7) suggests that near $m \in M$, $T(t)$ is expansive in the direction of X^u_m and at rate greater than on M, while (8) suggests that $T(t)$ is contractive in the direction of X^s_m, and at a rate greater than that on M.

The following result has been proved in [4].

Theorem 3.3. (**Normal Hyperbolicity**) *Suppose that $T(t)$ is a C^1 semigroup on a Banach space X and M is a C^2 compact connected invariant manifold which is normally hyperbolic under $T(t)$, (that is (i) and (ii) hold and there exists $0 \leq t_0 < \infty$ such that (iii) holds for all $t \geq t_0$). Let $\widetilde{T}(t)$ be a C^1 semigroup on X and $t_1 > t_0$. Consider $N(\varepsilon)$, the ε-neighbourhood of M, given by*

$$N(\varepsilon) = \{m + x^u + x^s,\ x^u \in X^u_m,\ x^s \in X^s_m,\ \|x^u\|, \|x^s\| < \varepsilon\}.$$

Then, there exists $\varepsilon^* > 0$ such that for each $\varepsilon < \varepsilon^*$, there exists $\sigma > 0$ such that if

$$\sup_{u \in N(\varepsilon)} \left\{\|\widetilde{T}(t_1)u - T(t_1)u\| + \|D\widetilde{T}(t_1)(u) - DT(t_1)(u)\|\right\} < \sigma$$

and

$$\sup_{u \in N(\varepsilon)} \|\widetilde{T}(t)u - T(t)u\| < \sigma, \textit{for } 0 \leq t \leq t_1,$$

there is an unique compact connected invariant manifold of class C^1, $\widetilde{M}$, *in* $N(\varepsilon)$. *Furthermore,* $\widetilde{M}$ *is normally hyperbolic under* $\widetilde{T}(t)$ *and, for each* $t \geq 0$, $\widetilde{T}(t)$ *is a* C^1*-diffeomorphism from* $\widetilde{M}$ *to* $\widetilde{M}$.

Proposition 1. *Assume that the hypotheses (H1), (H2) and (H7) hold. Then, for each* $\lambda \in R$, *any curve of equilibria of* $T_\lambda(t)$ *is a normally hyperbolic manifold under* $T_\lambda(t)$.

Proof Let M be a curve of equilibria of $T_\lambda(t)$ and $m \in M$. From (H7) it follows that

$$Ker(DF_u(m,\lambda)) = span\{m'\}.$$

Let $Y = \mathcal{R}(DF_u(m,\lambda))$; the range of $DF_u(m,\lambda)$. Since $DF_u(m,\lambda)$ is self-adjoint and Fredholm of index zero, it follows from (H7) that

$$\sigma(DF_u(u_0,\lambda)|_Y) = \sigma_u \cup \sigma_s,$$

where σ_u, σ_s correspond to the positive and negative eigenvalues respectively.

From (H1) and (H2), it follows that $T_\lambda(t)$ is a C^1 semigroup. Consider the linear autonomous equation

$$\dot{v} = (DF_u(m,\lambda)|_Y)v. \tag{9}$$

Then $DT_\lambda(t)v_0$ is the solution of (9) with initial condition v_0, that is $DT_\lambda(t)(m)v_0 = e^{(DF_u(m,\lambda))t}v_0$. In particular $DT_\lambda(t)(m)|_Y \equiv D(T_\lambda(t)|_Y)(m) = e^{(DF_u(m,\lambda)|_Y)t}$.

Let P_u and P_s be the spectral projections corresponding to σ_u and σ_s. The subspaces $X_m^u = P_uY$, $X_m^s = P_sY$ are then invariant under $DT_\lambda(t)$ and the following estimates hold (see [7], p. 73, 81 or [13], p. 37).

$$\|DT_\lambda(t)|_Y v\| \leq Ne^{-\nu t}\|v\|, \text{ for } v \in X_m^s \text{ and } t \geq 0, \tag{10}$$

$$\|DT_\lambda(t)|_Y v\| \leq Ne^{\nu t}\|v\|, \text{ for } v \in X_m^u \text{ and } t \leq 0, \tag{11}$$

for some positive constant ν and some constant $N > 1$.

Its clear that that $DT_\lambda(t) \equiv 0$ when restricted to $X_m^c = span\{m'\}$. Therefore, we have the decomposition

$$L^2(S^1) = X_m^c \oplus X_m^u \oplus X_m^s.$$

Since $DF_u(m,\lambda)|_Y$ is an isomorphism

$$DF_u(m,\lambda)|_{X_m^\alpha} : X_m^\alpha \to X_m^\alpha, \ \alpha = u, s,$$

is an isomorphism. Consequently, the linear flow

$$DT_\lambda(t)(m)|_{X_m^u} : X_m^u \to X_m^u$$

is also an isomorphism.

Finally, the estimates (7) and (8) follow from estimates (10) and (11) above. □

Proposition 2. *Suppose that the hypotheses (H1)-(H3) hold. Let* $DT_\lambda(t)(u)$ *be the linear flow generated by the equation*

$$\frac{\partial v}{\partial t} = -v + g'(\beta J * u + \beta h)\beta(J * v).$$

Then, for a fixed $\lambda_0 \in R$, *we have*

$$\|T_\lambda(t)u - T_{\lambda_0}(t)u\|_{L^2(S^1)} + \|DT_\lambda(t)(u) - DT_{\lambda_0}(t)(u)\|_{\mathcal{L}(L^2(S^1),\, L^2(S^1))} \to 0, \ \lambda \to \lambda_0,$$

uniformly for u *in bounded sets of* $L^2(S^1)$ *and* $t \in [0,b]$, $b < \infty$.

Proof From Lemma 2.1 it follows that

$$\|T_\lambda(t)u - T_{\lambda_0}(t)u\|_{L^2(S^1)} \to 0,\ \lambda \to \lambda_0,$$

for u in bounded sets of $L^2(S^1)$ and $t \in [0,b]$.

By the variation of constants formula, we have

$$DT_\lambda(t)(u)v = e^{-t}v + \int_0^t e^{-(t-s)} g'(\beta J * u + \beta h)(\beta J * v)ds.$$

Thus

$$\begin{aligned}
\|DT_\lambda(t)(u)v - DT_{\lambda_0}(t)(u)v\|_{L^2} &\leq \int_0^t e^{-(t-s)} \big\|[g'(\beta J * u + \beta h)\beta \\
&- g'(\beta_0 J * u + \beta_0 h_0)\beta_0] J * v\big\|_{L^2} ds \\
&\leq \int_0^t e^{-(t-s)} \big\|[g'(\beta J * u + \beta h)\beta \\
&- g'(\beta_0 J * u + \beta_0 h_0)\beta] J * v\big\|_{L^2} ds \\
&+ \int_0^t e^{-(t-s)} \|g'(\beta_0 J * u + \beta_0 h_0)(J * v)(\beta - \beta_0)\|_{L^2} ds.
\end{aligned}$$

Given $\eta > 0$ there exists $\delta > 0$ such that $\|\lambda - \lambda_0\| < \delta$ implies that $(\beta J * u + \beta h)$ belongs to a ball centred at $(\beta_0 J * u + \beta_0 h_0)$ and radius η in $L^\infty(S^1)$. In fact,

$$\begin{aligned}
|(J * u)(w)| &\leq \int_{S^1} |J(wz^{-1})||u(z)|dz \\
&\leq \int_{S^1} \|J\|_\infty |u(z)|dz \\
&\leq \sqrt{2\tau}\|J\|_\infty \|u\|_{L^2},
\end{aligned} \tag{12}$$

where we have used Hölder's inequality in the last estimate. Therefore

$$\begin{aligned}
|\beta_0(J * u)(w) + \beta_0 h_0 - \beta(J * u)(w) - \beta h| &\leq |\beta - \beta_0||(J * u)(w)| + |\beta_0 h_0 - \beta h| \\
&\leq \sqrt{2\tau}\big(\|J\|_\infty \|u\|_{L^2} |\beta - \beta_0| \\
&+ |\beta_0 h_0 - \beta h|\big) \to 0,\ \lambda \to \lambda_0.
\end{aligned}$$

Thus, from (H2), there exists a positive constant L, which depends only on u, such that

$$|g'(\beta(J*u)(w) + \beta h) - g'(\beta_0(J*u)(w) + \beta_0 h_0)| \leq L(|\beta - \beta_0||(J*u)(w)| + |\beta h - \beta_0 h_0|).$$

Then

$$\begin{aligned}
\| & [g'(\beta J * u + \beta h)\beta - g'(\beta_0 J * u + \beta_0 h_0)\beta](J * v)\big\|_{L^2}^2 \\
= & \int_{S^1} |g'(\beta(J * u)(w) + \beta h) - g'(\beta_0(J * u)(w) + \beta_0 h_0)|^2 \beta^2 |(J * v)(w)|^2 dw \\
\leq & \int_{S^1} L^2(|\beta - \beta_0||(J * u)(w)| + |\beta h - \beta_0 h_0|)^2 \beta^2 |(J * v)(w)|^2 dw.
\end{aligned}$$

Using (12), we obtain

$$\begin{aligned}
\|[g'(\beta J * u + \beta h)\beta - g'(\beta_0 J * u + \beta_0 h_0)\beta](J * v)\| &\leq 2\tau L\big(\|u\|_{L^2}\sqrt{2\tau}\|J\|_\infty |\beta - \beta_0| \\
&+ |\beta h - \beta_0 h_0|\big)\beta\|J\|_\infty \|v\|_{L^2}.
\end{aligned}$$

Therefore

$$\begin{aligned}\sup_{\|v\|_{L^2}=1} \int_0^t & e^{-(t-s)}\|[g'(\beta J*u+\beta h)\beta - g'(\beta_0 J*u+\beta_0 h_0)\beta](J*v)\|_{L^2} ds \\ \leq & \sup_{\|v\|_{L^2}=1} \big\{2\tau L\big(\|u\|_{L^2}\sqrt{2\tau}\|J\|_\infty|\beta-\beta_0| \\ + & |\beta h-\beta_0 h_0|\big)\beta\|J\|_\infty\|v\|_{L^2}\int_0^t e^{-(t-s)}ds\big\} \\ \leq & \sup_{\|v\|_{L^2}=1} \big\{2\tau L\big(\|u\|_{L^2}\sqrt{2\tau}\|J\|_\infty|\beta-\beta_0| + |\beta h-\beta_0 h_0|\big)\beta\|J\|_\infty\|v\|_{L^2}\big\} \\ = & \ 2\tau L\big(\|u\|_{L^2}\sqrt{2\tau}\|J\|_\infty|\beta-\beta_0| + |\beta h-\beta_0 h_0|\big)\beta\|J\|_\infty.\end{aligned}$$

Now

$$\|g'(\beta_0 J*u+\beta_0 h_0)(J*v)(\beta-\beta_0)\|_{L^2}^2 = \int_{S^1}|g'(\beta_0(J*u)(w)+\beta_0 h_0)(J*v)(w)(\beta-\beta_0)|^2 dw.$$

But, using (H3) and (12), we have

$$\begin{aligned}|g'(\beta_0(J*u)(w)+\beta_0 h_0)(J*v)(w)| \ \leq & \ \Bigg(k_4\beta_0\sqrt{2\tau}\|J\|_\infty\|u\|_{L^2} \\ + & \ k_4\beta_0 h_0 + k_5\Bigg)\sqrt{2\tau}\|J\|_\infty\|v\|_{L^2}.\end{aligned}$$

Then

$$\begin{aligned}\| & \ g'(\beta_0 J*u+\beta_0 h_0)(J*v)(\beta-\beta_0)\|_{L^2}^2 \\ \leq & \int_{S^1}\Bigg(k_4\beta_0\sqrt{2\tau}\|J\|_\infty\|u\|_{L^2} + k_4\beta_0 h_0 + k_5\Bigg)^2 2\tau\|J\|_\infty^2\|v\|_{L^2}^2|\beta-\beta_0|^2 dw.\end{aligned}$$

Thus

$$\begin{aligned}\| & \ g'(\beta_0 J*u+\beta_0 h_0)(J*v)(\beta-\beta_0)\|_{L^2} \\ \leq & \ \Bigg(k_4\beta_0\sqrt{2\tau}\|J\|_\infty\|u\|_{L^2} + k_4\beta_0 h_0 + k_5\Bigg)2\tau\|J\|_\infty\|v\|_{L^2}|\beta-\beta_0|.\end{aligned}$$

Hence

$$\begin{aligned}\sup_{\|v\|_{L^2}=1} \int_0^t & e^{-(t-s)}\|g'(\beta_0 J*u+\beta_0 h_0)(J*v)(\beta-\beta_0)\|_{L^2} ds \\ \leq & \sup_{\|v\|_{L^2}=1} \big\{\big(k_4\beta_0\sqrt{2\tau}\|J\|_\infty\|u\|_{L^2} \\ + & \ k_4\beta_0 h_0 + k_5\big)2\tau\|J\|_\infty\|v\|_{L^2}|\beta-\beta_0|\int_0^t e^{-(t-s)}ds\big\} \\ \leq & \sup_{\|v\|_{L^2}=1} \big\{\big(k_4\beta_0\sqrt{2\tau}\|J\|_\infty\|u\|_{L^2} + k_4\beta_0 h_0 + k_5\big)2\tau\|J\|_\infty\|v\|_{L^2}|\beta-\beta_0|\big\} \\ = & \ \big(k_4\beta_0\sqrt{2\tau}\|J\|_\infty\|u\|_{L^2} + k_4\beta_0 h_0 + k_5\big)2\tau\|J\|_\infty|\beta-\beta_0|.\end{aligned}$$

Therefore

$$\begin{aligned}&\|DT_\lambda(t)(u) - DT_{\lambda_0}(t)(u)\|_{\mathcal{L}(L^2(S^1),\ L^2(S^1))} = \sup_{\|v\|=1}\|DT_\lambda(t)(u)v - DT_{\lambda_0}(t)(u)v\|_{L^2(S^1)} \\ &\leq 2\tau L\Bigg(\|u\|_{L^2}\sqrt{2\tau}\|J\|_\infty|\beta-\beta_0| + |\beta h-\beta_0 h_0|\Bigg)\beta\|J\|_\infty \\ &+\Bigg(k_4\beta_0\sqrt{2\tau}\|J\|_\infty\|u\|_{L^2} + k_4\beta_0 h_0 + k_5\Bigg)2\tau\|J\|_\infty|\beta-\beta_0| \\ &= C(\lambda),\end{aligned}$$

with $C(\lambda) \to 0$, as $\lambda \to 0$. This completes the proof. □

Theorem 3.4. *Suppose that the hypotheses (H1)-(H2), (H5)-(H6), with $a < \infty$, and (H7)-(H8) hold. Then the set E_λ of the equilibria of $T_\lambda(t)$ is lower semi-continuous with respect to λ at λ_0.*

Proof The continuity of the constant equilibria follows from the Implicit Function Theorem and the hypothesis of hyperbolicity.

Suppose now that m is a non constant equilibrium and let $\Gamma = \gamma(\alpha; m)$ be the isolated curve of equilibria containing m given by Lemma 3.1. We want to show that, for every $\varepsilon > 0$, there exists $\delta > 0$ so that, if λ is such that $\|\lambda - \lambda_0\| < \delta$, there exists $\Gamma_\lambda \in E_\lambda$ such that $\Gamma \subset \Gamma_\lambda^\varepsilon$, where $\Gamma_\lambda^\varepsilon$ is the ε-neighbourhood of Γ_λ.

From Lemma 3.1 and Propositions 1 and 2, the assumptions of the Normal Hyperbolicity Theorem are met. Thus, by Theorem 3.3, given $\varepsilon > 0$, there is $\delta > 0$ such that, if $\|\lambda - \lambda_0\| < \delta$ there is an unique C^1 compact connected invariant manifold Γ_λ normally hyperbolic under $T_\lambda(t)$, such that Γ_λ is ε-close and C^1-diffeomorphic to Γ.

Since $T_\lambda(t)$ is gradient and Γ_λ is compact, there exists at least one equilibrium $m_\lambda \in \Gamma_\lambda$. In fact, the ω limit of any $u \in \Gamma_\lambda$ is nonempty and belongs to Γ_λ by invariance. From Lemma 3.8.2 of [10], it must contain an equilibrium. Since Γ_λ is ε-close to Γ, there exists $m \in \Gamma$ such that $\|m - m_\lambda\|_{L^2(S^1)} < \varepsilon$.

Let $\tilde{\Gamma}_\lambda$ be the curve of equilibria given by $\tilde{\Gamma}_\lambda \equiv \{\gamma(\alpha; m_\lambda),\ \alpha \in S^1\}$ which is a normally hyperbolic invariant manifold under $T_\lambda(t)$ by Proposition 1. Then, for each $\alpha \in S^1$, we have

$$\begin{aligned}\|\gamma(\alpha; m_\lambda) - \gamma(\alpha; m)\|_{L^2}^2 &= \int_{S^1} |\gamma(\alpha; m_\lambda)(w) - \gamma(\alpha; m)(w)|^2 dw \\ &= \int_{S^1} |m_\lambda(\alpha w) - m(\alpha w)|^2 dw \\ &= \|m_\lambda - m\|_{L^2}.\end{aligned}$$

Thus

$$\begin{aligned}\|\gamma(\alpha; m_\lambda) - \gamma(\alpha; m)\|_{L^2} &= \|m_\lambda - m\|_{L^2} \\ &< \varepsilon.\end{aligned}$$

and Γ is ε-close to $\tilde{\Gamma}_\lambda$. Since there are only a finite number of curves of equilibria the result follows immediately. □

The example below shows that the curves of equilibria of

$$\dot{x} = F(x),$$

generated by the action of a group may disappear even when the symmetry is preserved. In other words, we cannot expect a result of the type of the Implicit Function Theorem without additional hypotheses, (see [8]).

Example 1. (An example with symmetry)

Consider the planar system

$$\begin{aligned} \dot{x} &= x(1 - x^2 - y^2), \\ \dot{y} &= y(1 - x^2 - y^2). \end{aligned} \tag{13}$$

Note that (13) has, besides the origin, the curve of equilibria given by

$$x^2 + y^2 = 1$$

which is generated by the rotation of a fixed equilibrium.

However, for any $\varepsilon \neq 0$, the perturbed system

$$\begin{aligned} \dot{x} &= -\varepsilon y + x(1 - x^2 - y^2), \\ \dot{y} &= \varepsilon x + y(1 - x^2 - y^2) \end{aligned} \tag{14}$$

has no non trivial equilibrium, although the system is still invariant under the action of S^1. We observe that the perturbed system is not gradient as is the case for the class of pertubations we are considering.

3.2. **Existence and continuity of the local unstable manifolds.** Let us return to equation $(\mathbf{P})_\lambda$. Recall that the *unstable set* $W^u_\lambda = W^u_\lambda(u_\lambda)$ of an equilibrium u_λ is the set of initial conditions φ of $(\mathbf{P})_\lambda$, such that $T_\lambda(t)\varphi$ is defined for all $t \leq 0$ and $T_\lambda(t)\varphi \to u_\lambda$ as $t \to -\infty$. For a given neighbourhood V of u_λ, the set $W^u_\lambda \cap V$ is called a *local unstable set* of u_λ.

Using results of appendix we now show that the local unstable sets are actually Lipschitz manifolds in a sufficiently small neighbourhood and vary continuously with λ. More precisely, we have

Lemma 3.5. *If u_0 is a fixed equilibrium of $(\mathbf{P})_\lambda$ for $\lambda = \lambda_0$ then there is a $\delta > 0$ such that, if $|\lambda - \lambda_0| + \|u_0 - u_\lambda\|_{L^2} < \delta$ and*

$$U^\delta_\lambda := \{u \in W^u_\lambda(u_\lambda) \ : \ ||u - u_\lambda||_{L^2} < \delta\}$$

then U^δ_λ is a Lipschitz manifold and

$$dist(U^\delta_\lambda, U^\delta_{\lambda_0}) + dist(U^\delta_{\lambda_0}, U^\delta_\lambda) \to 0 \quad as \quad |\lambda - \lambda_0| + \|u_0 - u_\lambda\|_{L^2} \to 0,$$

with dist defined as in (4).

Proof As already mentioned, assuming (H1) and (H2), the map $F : L^2(S^1) \times R \to L^2(S^1)$,

$$F(u, \lambda) = -u + g(\beta(J * u) + \beta h),$$

defined by the right-hand side of $(\mathbf{P})_\lambda$ is continuously Frechet differentiable. Let u_λ be an equilibrium of $(\mathbf{P})_\lambda$. Writing $u = u_\lambda + v$, it follows that u is a solution of $(\mathbf{P})_\lambda$ if and only if v satisfies

$$\frac{\partial v}{\partial t} = L(\lambda)v + r(u_\lambda, v, \lambda), \tag{15}$$

where $L(\lambda)v = \frac{\partial}{\partial u}F(u_\lambda, \lambda) = -v + g'(\beta(J * u_\lambda) + \beta h)\beta(J * v)$ and $r(u_\lambda, v, \lambda) = F(u_\lambda + v, \lambda) - F(u_\lambda, \lambda) - L(\lambda)v$. We rewrite equation (15) in the form

$$\frac{\partial v}{\partial t} = L(\lambda_0)v + f(v, \lambda), \tag{16}$$

where $f(v,\lambda) = [L(\lambda) - L(\lambda_0)]v + r(u_\lambda, v, \lambda)$ is the "non linear part" of (16). Observe that now the "linear part" of (16) does not depend on the parameter λ, as required by theorems Theorems .2 and .3.

To obtain the needed estimates we first observe that, by Hölder inequality

$$|(J*v)(z)| \leq \sqrt{2\tau}\|J\|_\infty \|v\|_{L^2}, \ \forall z \in S^1 \tag{17}$$

for any $v \in L^2(S^1)$. Therefore, since g is of class C^2, $g'(\beta J * u_\lambda(z) + \beta J * v(z) + \beta h)$ and $g''(\beta J * u_\lambda(z) + \beta J * v(z) + \beta h)$ are bounded by a constant M, for any λ in a neighbourhood of λ_0 and $||v||_{L^2(S^1)} \leq 1$. We then obtain

$$\begin{aligned}
\| \quad & g'(\beta J * u_\lambda + \beta h)\beta(J*v) - g'(\beta_0 J * u_{\lambda_0} + \beta_0 h_0)\beta(J*v)\|_{L^2}^2 \\
= \quad & \int_{S^1} |g'(\beta J * u_\lambda(z) + \beta h)\beta - g'(\beta_0 J * u_{\lambda_0}(z) + \beta_0 h_0)|^2 \beta^2 |(J*v)(z)|^2 dz \\
\leq \quad & \int_{S^1} M^2 |[|\beta J * u_\lambda(z) - \beta_0 J * u_{\lambda_0}(z)| + |\beta h - \beta_0 h_0|]^2 \beta^2 |(J*v)(z)|^2 dz \\
\leq \quad & \int_{S^1} M^2 [|\beta J * u_\lambda(z) - \beta_0 J * u_{\lambda_0}(z)| + |\beta h - \beta_0 h_0|]^2 \beta^2 2\tau \|J\|_\infty^2 \|v\|_{L^2}^2 dz \\
\leq \quad & \int_{S^1} M^2 [|\beta J * u_\lambda(z) - \beta J * u_{\lambda_0}(z)| + |\beta J * u_{\lambda_0}(z) - \beta_0 J * u_{\lambda_0}(z)| \\
+ \quad & |\beta h - \beta_0 h_0|]^2 \beta^2 2\tau \|J\|_\infty^2 \|v\|_{L^2}^2 dz \\
\leq \quad & \int_{S^1} M^2 [\beta \sqrt{2\tau}\|J\|_\infty \|u_\lambda - u_{\lambda_0}\|_{L^2} \\
+ \quad & |\beta - \beta_0|\sqrt{2\tau}\|J\|_\infty \|u_{\lambda_0}\|_{L^2} + |\beta h - \beta_0 h_0|]^2 \beta^2 2\tau \|J\|_\infty^2 \|v\|_{L^2}^2 \\
= \quad & d_1(\lambda)\|v\|_{L^2}^2,
\end{aligned}$$

with $d_1(\lambda) \to 0$, as $\lambda \to \lambda_0$. Analogously

$$\begin{aligned}
\| \quad & g'(\beta_0 J * u_{\lambda_0} + \beta_0 h_0)(\beta - \beta_0)(J*v)\|_{L^2}^2 \\
\leq \quad & \int_{S^1} \|g'(\beta_0 J * u_{\lambda_0} + \beta_0 h_0)\|_\infty^2 |\beta - \beta_0|^2 |J * v(z)|^2 dz \\
\leq \quad & \int_{S^1} \|g'(\beta_0 J * u_{\lambda_0} + \beta_0 h_0)\|_\infty^2 |\beta - \beta_0|^2 2\tau \|J\|_\infty^2 \|v\|_{L^2}^2 dz \\
= \quad & d_2(\lambda)\|v\|_{L^2}^2.
\end{aligned}$$

with $d_2(\lambda) \to 0$, as $\lambda \to \lambda_0$. It follows that

$$\begin{aligned}
\|\left(L(\lambda) - L(\lambda_0)\right)v\|_{L^2} \quad \leq \quad & \|g'(\beta J * u_\lambda + \beta h)\beta(J*v) - g(\beta_0 J * u_{\lambda_0} + \beta_0 h_0)\beta(J*v)\|_{L^2} \\
+ \quad & \|g'(\beta_0 J * u_{\lambda_0} + \beta_0 h_0)(\beta - \beta_0)(J*v)\|_{L^2} \\
\leq \quad & C_1(\lambda)\|v\|_{L^2},
\end{aligned} \tag{18}$$

with $C_1(\lambda) \to 0$, as $\lambda \to 0$.

Observe now that, for any $z \in S^1$

$$\begin{aligned}
& F(u_\lambda(z) + v(z), \lambda) - F(u_{\lambda_0}(z) + v(z), \lambda_0) \\
& \quad = [g(\beta_0 J * u_{\lambda_0}(z) + \beta_0 h_0) - g(\beta_0 J * u_{\lambda_0}(z) + \beta_0 J * v(z) + \beta_0 h_0)] \\
& \quad -[g(\beta J * u_\lambda(z) + \beta h) - g(\beta J * u_\lambda(z) + \beta J * v(z) + \beta h)] \\
& \quad = g'(\beta_0 J * u_{\lambda_0}(z) + \beta_0 J * \bar{v}(z) + \beta_0 h_0)\beta_0 (J*v)(z)] \\
& \quad -g'(\beta J * u_\lambda(z) + \beta J * \bar{\bar{v}}(z) + \beta h)\beta(J*v)(z),
\end{aligned}$$

for some $\bar{v}$ in the segment defined by $J * u_{\lambda_0}$ and $J * (u_{\lambda_0} + v)$ and some $\bar{\bar{v}}$ in the segment defined by $J * u_\lambda$ and $J * (u_\lambda + v)$. Then

$$\begin{aligned}
&|F(u_\lambda(z) + v(z), \lambda) - F(u_{\lambda_0}(z) + v(z), \lambda_0)| \\
&\leq [|g'(\beta_0 J * u_{\lambda_0}(z) + \beta_0 J * \bar{v}(z) + \beta_0 h_0)\beta_0 - g'(\beta_0 J * u_{\lambda_0}(z) + \beta_0 J * \bar{v}(z) + \beta_0 h_0)\beta| \\
&+\beta|g'(\beta_0 J * u_{\lambda_0}(z) + \beta_0 J * \bar{v}(z) + \beta_0 h_0) - g'(\beta J * u_\lambda(z) + \beta J * \bar{\bar{v}}(z) + \beta h)|]|J * v(z)| \\
&\leq [M|\beta - \beta_0| + \beta M|\beta_0 J * u_{\lambda_0}(z) - \beta J * u_\lambda(z)| \\
&+\beta M|\beta_0 J * \bar{v}(z) - \beta J * \bar{\bar{v}}(z)| + \beta M|\beta_0 h_0 - \beta h|]\sqrt{2\tau}\|J\|_\infty \|v\|_{L^2} \\
&\leq [M|\beta - \beta_0| + \beta M|\beta - \beta_0||J * (u_{\lambda_0}(z) - u_\lambda(z))| + \beta M(|\beta - \beta_0||J * \bar{v}(z)| \\
&+\beta|J * (\bar{v} - \bar{\bar{v}})(z)|) + \beta M|\beta h - \beta_0 h_0|]\sqrt{2\tau}\|J\|_\infty \|v\|_{L^2} \\
&\leq [M|\beta - \beta_0| + \beta M|\beta - \beta_0|\sqrt{2\tau}\|J\|_\infty \|u_\lambda - u_{\lambda_0}\|_{L^2} + \beta M|\beta - \beta_0|\sqrt{2\tau}\|J\|_\infty \|\bar{v}\|_{L^2} \\
&+\beta^2 M\sqrt{2\tau}\|J\|_\infty \|\bar{v} - \bar{\bar{v}}\|_{L^2} + \beta M|\beta h - \beta_0 h_0|]\sqrt{2\tau}\|J\|_\infty \|v\|_{L^2}.
\end{aligned}$$

Therefore, since $\|\bar{v} - \bar{\bar{v}}\|_{L^2} \to 0$, as $\lambda \to \lambda_0$,

$$\|F(u_\lambda + v, \lambda) - F(u_{\lambda_0} + v, \lambda_0)\|_{L^2} \leq C_2(\lambda)\|v\|_{L^2}, \tag{19}$$

with $C_2(\lambda) \to 0$, as $\lambda \to 0$.

Since $r(u_\lambda, v, \lambda) = F(u_\lambda + v, \lambda) - L(\lambda)v$, we obtain from (18) and (19) that

$$\|r(u_\lambda, v, \lambda) - r(u_{\lambda_0}, v, \lambda)\| \leq C_3(\lambda)\|v\|_{L^2}. \tag{20}$$

From (18) and (20), it follows that

$$\|f(v, \lambda) - f(v, \lambda_0)\| \leq C_4(\lambda)\|v\|_{L^2},$$

where $C_4(\lambda) \to 0$ as $\lambda \to \lambda_0$.

In a similar way, we obtain for any v, w with $||v||_{L^2(S^1)}$ and $||w||_{L^2(S^1)}$ smaller than 1

$$\begin{aligned}
&|r(u_\lambda(z), v(z), \lambda) - r(u_\lambda(z), w(z), \lambda)| \\
&\leq \|g''(\beta J * u_\lambda(z) + \beta J * \bar{\bar{v}}(z) + \beta h)\beta J * \bar{v}(z)\|_\infty \beta^2 \sqrt{4\tau^2}\|J\|_\infty^2 \|\bar{v}\|_{L^2}\|v - w\|_{L^2}.
\end{aligned}$$

for some $\bar{v}$ in the segment defined by $\beta J * v + \beta h$ and $\beta J * w + \beta h$ and some $\bar{\bar{v}}$ in the segment defined by 0 and $\beta(J * \bar{v}) + \beta h$. As $\|v\|_{L^2}, \|w\|_{L^2} \to 0$, it follows that

$$\|r(u_\lambda(z), v(z), \lambda) - r(u_\lambda(z), w(z), \lambda)\|_{L^2} \leq \nu_1(\rho)\|v - w\|_{L^2},$$

with $\nu(\rho) \to 0$, as $\rho \to 0$ and $\|v\|_{L^2}, \|w\|_{L^2} < \rho$. Furthermore

$$\|[L(\lambda) - L(\lambda_0)]v - [L(\lambda) - L(\lambda_0)]w\|_{L^2} \leq C_1(\lambda)\|\|(v - w)\|_{L^2}$$

Thus

$$\|f(v, \lambda) - f(w, \lambda)\|_{L^2} \leq (\nu(\rho) + C_1(\lambda))\|v - w\|_{L^2}, \tag{21}$$

where $\nu(\rho) \to 0$, as $\rho \to 0$ and $\|v\|_{L^2}, \|w\|_{L^2} \leq \rho$, and $C_1(\lambda) \to 0$, as $\lambda \to \lambda_0$.

Therefore, the conditions of Theorems .2, .3 are satisfied and we obtain the existence of locally invariant sets for (16) near the origin, given as graphs of Lipschitz functions which depend continuously on the parameter λ near λ_0. Using uniquennes of solutions, we can easily prove that these sets coincide with the local unstable manifolds of (16).

Observing now that the translation

$$u \to (u - u_\lambda)$$

sends an equilibrium u_λ of $(\mathbf{P})_\lambda$ into the origin (which is an equilibrium of (16)), the results claimed follow immediately. □

Using the compacity of the set of equilibria, one can obtain an 'uniform version' of Lemma 3.5 that will be needed later.

Lemma 3.6. *Let $\lambda = \lambda_0$ be fixed. Then, there is a $\delta > 0$ such that, for any equilibrium u_0 of $(\mathbf{P})_{\lambda_0}$, if $|\lambda - \lambda_0| + \|u_0 - u_\lambda\|_{L^2} < \delta$ and*

$$U_\lambda^\delta := \{u \in U_\lambda(u_\lambda) \ : \ ||u - u_\lambda||_{L^2(S^1)} < \delta\}$$

then U_λ^δ is a Lipschitz manifold and

$$\sup_{u_0 \in E_{\lambda_0}} dist(U_\lambda^\delta, U_{\lambda_0}^\delta) + dist(U_{\lambda_0}^\delta, U_\lambda^\delta) \to 0 \quad as \quad |\lambda - \lambda_0| + \|u_0 - u_\lambda\|_{L^2} \to 0,$$

with dist defined as in (4)

Proof From Lemma 3.5, we know that, for any $u_0 \in E_{\lambda_0}$, there is a $\delta = \delta(u_0)$ such that U_λ^δ is a Lipschitz manifold, if $|\lambda - \lambda_0| + \|u_0 - u_\lambda\|_{L^2} < 2\delta$. Thus, in particular, U_λ^δ is a Lipschitz manifold, if $|\lambda - \lambda_0| + \|\tilde{u}_0 - u_\lambda\|_{L^2} < \delta$. for any $\tilde{u}_0 \in E_{\lambda_0}$ with $\|\tilde{u}_0 - u_0\|_{L^2} < \delta$. Taking a finite subcovering of the covering of E_{λ_0} by balls $B(u_0, \delta(u_0))$, with u_0 varying in E_{λ_0}, the first part of the result follows with δ chosen as the minimum of those $\delta(u_0)$.

Now, if $\varepsilon > 0$ and $u_0 \in E_{\lambda_0}$, there exists, by Lemma 3.5, $\delta = \delta(u_0)$ such that, if $|\lambda - \lambda_0| + \|u_0 - u_\lambda\|_{L^2} < 2\delta$, then

$$\text{dist}(U_\lambda^\delta, U_{\lambda_0}^\delta) + dist(U_{\lambda 0}^\delta, U_\lambda^\delta) < \varepsilon/2.$$

If $\tilde{u}_0 \in E_{\lambda_0}$ is such that $\|\tilde{u}_0 - u_0\|_{L^2} < \delta$ and $|\lambda - \lambda_0| + \|\tilde{u}_0 - u_\lambda\|_{L^2} < \delta$ then, since $|\lambda - \lambda_0| + \|u_0 - u_\lambda\|_{L^2} < 2\delta$

$$\begin{aligned}
&\text{dist}(U_\lambda^\delta(u_\lambda), U_{\lambda_0}^\delta(\tilde{u}_0)) + \text{dist}(U_{\lambda_0}^\delta(\tilde{u}_0), U_\lambda^\delta(u_\lambda)) \\
&\quad < \text{dist}(U_\lambda^\delta(u_\lambda), U_{\lambda_0}^\delta(u_0)) + \text{dist}(U_{\lambda_0}^\delta(u_0), U_\lambda^\delta(u_\lambda)) + \text{dist}(U_{\lambda_0}^\delta(\tilde{u}_0), U_{\lambda_0}^\delta(u_0)) \\
&\quad + \text{dist}(U_{\lambda_0}^\delta(u_0), U_{\lambda_0}^\delta(\tilde{u}_0)) < \varepsilon
\end{aligned}$$

By the same procedure above of taking a finite subcovering of the covering of E_{λ_0} by balls $B(u_0, \delta(u_0))$, and δ the minimum of those $\delta(u_0)$, we conclude that

$$\text{dist}(U_\lambda^\delta(u_\lambda), U_{\lambda_0}^\delta(\tilde{u}_0)) + \text{dist}(U_{\lambda_0}^\delta(\tilde{u}_0), U_\lambda^\delta(u_\lambda)) < \varepsilon$$

if $|\lambda - \lambda_0| + \|\tilde{u}_0 - u_\lambda\|_{L^2} < \delta$, for any $\tilde{u}_0 \in E_{\lambda_0}$. This proves the result claimed. □

3.3. **Decomposition of the attractor.** As a consequence of its gradient structure, proved in [22], the attractor of the flow generated by $(\mathbf{P})_\lambda$ is given by the union of the unstable set of the set of equilibria (see [10]). Using results of [11], we prove below a more precise result on this direction.

Consider an equation of the form

$$\dot{x} + Bx = g(x), \tag{22}$$

where B is a bounded linear operator on a Banach space X and $g : X \to X$ is a C^2 function. We may write (22) in the form

$$\dot{x} + Ax = f(x), \tag{23}$$

where $A = B - g'(x_0)$ and $f(x) = g(x_0) + r(x)$, with r differentiable and $r(0) = 0$.

The following result has been proven in [11]

Theorem 3.7. *Suppose the spectrum $\sigma(A)$ contains 0 as a simple eigenvalue, while the remainder of the spectrum has real part outside some neighbourhood of zero. Let γ be a curve of equilibria of the flow generated by (23), of class C^2. Then there exists a neighbourhood U of γ such that, for any $x_0 \in U$ whose positive orbit is precompact and whose ω-limit set $\omega(x_0)$ belongs to γ, there exists a unique point $y(x_0) \in \gamma$ with $\omega(x_0) = y(x_0)$. Similarly, for any $x_0 \in U$ with bounded negative orbit and α-limit set $\alpha(x_0)$ in γ, there exists a unique point $y(x_0) \in \gamma$ such that $\alpha(x_0) = y(x_0)$.*

Proposition 3. *Assume the hypotheses (H1), (H2), (H5)-(H6), with $a < \infty$ and (H7) hold. Let E_λ be the set of the equilibria of $T_\lambda(t)$. For $u \in E_\lambda$, let $W^u_\lambda(u)$ be the unstable set of u. Then*

$$\mathcal{A}_\lambda = \bigcup_{u \in E_\lambda} W^u_\lambda(u).$$

Proof From Theorem 5.6 of [22], we have

$$\mathcal{A}_\lambda = \bigcup_{u \in E_\lambda} W^u_\lambda(E_\lambda).$$

There exists only a finite number, $\{u_1, \cdots, u_k\}$ of constant equilibria since they are all hyperbolic. For each non constant equilibrium $u \in E_\lambda$, there is a curve $M_u \subset E_\lambda \subset \mathcal{A}_\lambda$. From Lemma 3.1 these curves M_u are all isolated and, since $\mathcal{A}_\lambda$ is compact, it follows that there exists only a finite number of them; $M_1, \ldots, M_n$. Thus

$$\mathcal{A}_\lambda = \Big(\bigcup_{i=1}^{n} W^u_\lambda(M_i) \Big) \bigcup \Big(\bigcup_{j=1}^{k} W^u_\lambda(u_j) \Big).$$

By Theorem 3.7, it follows that

$$W^u_\lambda(M_i) = \bigcup_{v \in M_i} W^u_\lambda(v), \ \ i = 1, \cdots, n.$$

Therefore

$$\mathcal{A}_\lambda = \bigcup_{v \in E_\lambda} W^u_\lambda(v)$$

as claimed. □

3.4. **Proof of the lower semicontinuity.** We now turn to the proof of our main result, starting with some auxiliary results.

Lemma 3.8. *Assume the same hypotheses of Lemma 3. Then, given $\varepsilon > 0$, there exists $T > 0$ such that, for all $u \in \mathcal{A}_{\lambda_0} \backslash E^\varepsilon_{\lambda_0}$*

$$T_{\lambda_0}(-t)u \in E^\varepsilon_{\lambda_0},$$

for some $t \in [0, T]$, where $E^\varepsilon_{\lambda_0}$ is the ε-neighbourhood of E_{λ_0}. Furthermore, when ε is sufficiently small,

$$T_{\lambda_0}(-t)u \in U_{\lambda_0}(u_0),$$

for some $u_0 \in E_{\lambda_0}$, where $U_{\lambda_0}(u_0)$ is the local unstable manifold of $u_0 \in E_{\lambda_0}$.

Proof Let $\varepsilon > 0$ be given and $u \in \mathcal{A}_{\lambda_0} \backslash E^\varepsilon_{\lambda_0}$. From Lemma 3, it follows that

$$u \in W^u_{\lambda_0}(\bar{u}) \backslash E^\varepsilon_{\lambda_0}.$$

for some $\bar{u} \in E_{\lambda_0}$. Thus, there exists $t_u = t_u(\varepsilon) < \infty$ such that $T_{\lambda_0}(-t_u)u \in E^\varepsilon_{\lambda_0}$. By continuity of the operator $T_{\lambda_0}(-t_u)$, there exists $\eta_u > 0$ such that $T_{\lambda_0}(-t_u)B(u, \eta_u) \subset$

$E^{\varepsilon}_{\lambda_0}$, where $B(u,\eta_u)$ is the ball of centre u and radius η_u. By compactness, there are $u_1,\cdots,u_n \in \mathcal{A}_{\lambda_0}\backslash E^{\varepsilon}_{\lambda_0}$ such that

$$\mathcal{A}_{\lambda_0}\backslash E^{\varepsilon}_{\lambda_0} \subset \bigcup_{j=1}^{n} B(u_j,\eta_{u_j}),$$

with $T_{\lambda_0}(-t_{u_j})B(u_j,\eta_{u_j}) \subset E^{\varepsilon}_{\lambda_0}$, for $j=1,\ldots,n$. Let $T=\max\{t_{u_1},\cdots,t_{u_n}\}$. Then, for any $u \in \mathcal{A}_{\lambda_0}\backslash E^{\varepsilon}_{\lambda_0}$, $T_{\lambda_0}(-t)u \in E^{\varepsilon}_{\lambda_0}$, for some $t\in[0,T]$. Since $u \in W^{u}_{\lambda_0}(\overline{u})\backslash E^{\varepsilon}_{\lambda_0}$, for some $\overline{u}\in E_{\lambda_0}$ and $T_{\lambda_0}(-t)u \in E^{\varepsilon}_{\lambda_0}$, to conclude that $T_{\lambda_0}(-t)u \in U_{\lambda_0}(\bar{u})$, when ε is sufficiently small, it is enough to show that there exists $\delta>0$ such that $W^{u}_{\lambda_0}(v)\cap B(v,\delta)\subset U_{\lambda_0}(v)$, for all $v\in E_{\lambda_0}$. Therefore, the conclusion follows immediately from Lemma 3.5.

Theorem 3.9. *Assume the hypotheses (H1)-(H2),(H5)-(H6), with $a<\infty$, and (H7) and (H8). Then the family of attractors $\mathcal{A}_\lambda$ is lower semicontinuous with respect to the parameter λ at $\lambda_0 \in R$.*

Proof Let $\varepsilon>0$ be given. From Lemma 3.8, there is $T>0$ such that, for all $u\in\mathcal{A}_{\lambda_0}\backslash E^{\varepsilon}_{\lambda_0}$, there exists $t_u\in[0,T]$ such that

$$\bar{u} := T_{\lambda_0}(-t_u)u \in U_{\lambda_0}(u_0), \tag{24}$$

for some $u_0\in E_{\lambda_0}$. Since $T_{\lambda_0}(t)$ is a continuous family of bounded operators, there exists $\eta>0$ such that, for all $t\in[0,T]$

$$\|z-w\|_{L^2}<\eta \Rightarrow \|T_{\lambda_0}(t)z - T_{\lambda_0}(t)w\|_{L^2} < \frac{\varepsilon}{2}. \tag{25}$$

Now, by the (uniform) continuity of the equilibria and local unstable manifolds with respect to the parameter λ asserted by Theorem 3.4 and Lemma 3.6 , there exists $\delta^*>0$ *independent of* u such that $\|\lambda-\lambda_0\|<\delta^*$ implies the existence of $u_\lambda\in E_\lambda$ and some $\bar{\bar{u}}_\lambda \in U_\lambda(u_\lambda)$ with

$$\|\bar{\bar{u}}_\lambda - \bar{u}\|_{L^2} < \eta, \tag{26}$$

where $U_\lambda(u_\lambda)$ denotes the local unstable manifold of the equilibrium u_λ of $T_\lambda(t)$. Hence, when $\|\lambda-\lambda_0\|<\delta^*$ we obtain, from (25) and (26)

$$\|T_{\lambda_0}(t)\bar{\bar{u}}_\lambda - T_{\lambda_0}(t)\bar{u}\|_{L^2} < \frac{\varepsilon}{2} \quad \text{for any} \quad t\in[0,T]. \tag{27}$$

On the other hand, from continuity of the flow with respect to parameter λ, there exists $\overline{\delta}>0$ such that $\|\lambda-\lambda_0\|<\overline{\delta}$ implies

$$\|T_\lambda(t)(u) - T_{\lambda_0}(t)(u)\|_{L^2} < \frac{\varepsilon}{2}, \tag{28}$$

for any $u\in B(0,2a\sqrt{2\lambda}))$ and $t\in[0,T]$, and thus in particular for $u=\bar{\bar{u}}_\lambda$ and $t=t_u$.

Choose $\delta=\min\{\delta^*,\overline{\delta}\}$ and let $v_\lambda := T_\lambda(t_u)\bar{\bar{u}}_\lambda$. It is clear that $v_\lambda\in\mathcal{A}_\lambda$, since $\bar{\bar{u}}_\lambda\in U_\lambda(u_\lambda)$.

Using (27) and (28) we obtain, when $\|\lambda-\lambda_0\|<\delta$

$$\begin{aligned} \|v_\lambda - u\|_{L^2} &= \|T_\lambda(t_u)\bar{\bar{u}}_\lambda - T_{\lambda_0}(t_u)\bar{u}\|_{L^2} \\ &\le \|T_\lambda(t_u)\bar{\bar{u}}_\lambda - T_{\lambda_0}(t_u)\bar{\bar{u}}_\lambda\|_{L^2} + \|T_{\lambda_0}(t_u)\bar{\bar{u}}_\lambda - T_{\lambda_0}(t_u)\bar{u}\|_{L^2} \\ &< \varepsilon. \end{aligned}$$

When $u\in E^{\varepsilon}_{\lambda_0}\subset\mathcal{A}_{\lambda_0}$ this conclusion follows straightforwardly from the continuity of equilibria. Thus the lower semicontinuity of attractors follows. □

4. **A special case.** We consider now the particular case of (1) where $g \equiv \tanh$ and $\beta > 1$, that is, the equation

$$\frac{\partial m(w,t)}{\partial t} = -m(w,t) + \tanh(\beta J * m(w,t) + \beta h), \tag{29}$$

Equation (29) arises as a continuum limit of one-dimensional spin systems with Glauber dynamics and Kac potentials, (see [2], [15], [16], [17], [18], [19] and [21]); m represents then a magnetisation density and β^{-1} the product of the absolute temperature by the Boltzmann constant.

The Lyapunov functional is now given by

$$\mathbb{F}(u) = \int_{S^1} [f(u(w)) - f(m_\beta^+)]dw + \frac{1}{4}\int_{S^1}\int_{S^1} J(w \cdot z^{-1})[u(w) - u(z)]^2 dwdz \tag{30}$$

where f (the free energy density) is given by

$$f(x) = -\frac{1}{2}x^2 - hx - \beta^{-1} i(x), \;\; x \in [-1,1], \;\; \text{(see Figure 4.2)} \;,$$

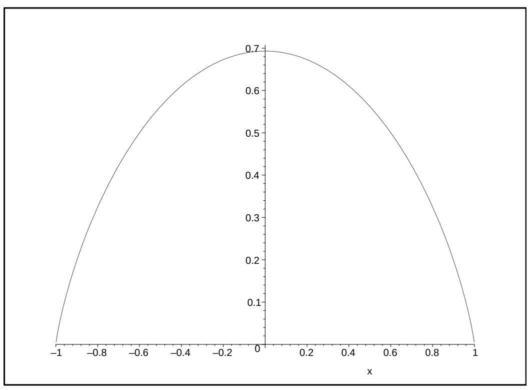

FIGURE 1.

where i is the entropy density, given by

$$i(x) = -\frac{1+x}{2}\ln\left(\frac{1+x}{2}\right) - \frac{1-x}{2}\ln\left(\frac{1-x}{2}\right), \;\; x \in [-1,1], \;\; \text{(see figure 4.1)},$$

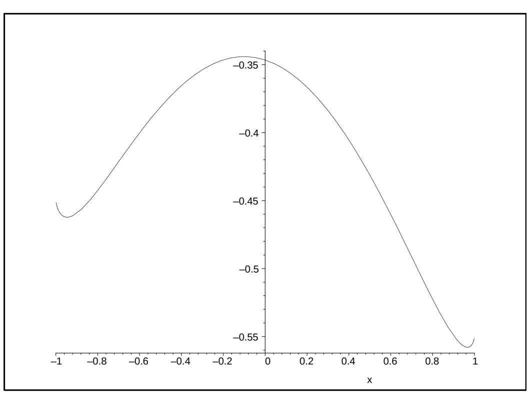

FIGURE 2.

Here $-\frac{x^2}{2}$ represent the inner energy density and $-hx$ energy density of external field h, (see [18]).

Note that the functional given in (30) is defined in the whole phase space. Furthermore m_β^+ is the global minimum of f in $(-1,1)$, (see [18]). Thus the integrand in (30) are non negative. It is easy to show that $\max_{x\in[-1,1]}[i(x)] = \ln(2)$ and $\lim_{x\to\pm 1} i(x) = 0$, (see Figure 4.1).

The function $g(x) = \tanh(x)$ satisfies the hypotheses (H1)-(H6) and (H8) with $k_1 = k_3 = k_5 = a = 1$ and $k_2 = k_4 = 0$. Thus the upper semicontinuity of the family of attractors with respect to $\lambda = (h,\beta)$ follows from Theorem 2.2. If (H7) holds the lower semicontinuity also follows from Theorem 3.9. □

Appendix A. **Continuity of unstable manifolds for an abstract problem.** In this section we prove a result of continuity for unstable manifolds near a (non-hyperbolic) equilibrium adapting the ideas of [3]. The result is well known either in the case of hyperbolic equilibrium (see for example [1]) or discrete finite dimensional systems (see [14]), but we have not been able to find a suitable result under our hypotheses.

Let X be a Banach space, $A : D(A) \subset X \to X$ the generator of a strongly continuous semigroup of linear operators $\{T(t)\}_{t\geq 0}$ on X and consider the problem

$$\begin{aligned} \dot{x} &= Ax + f(x) \\ x(0) &= x_0. \end{aligned} \tag{31}$$

It is well known (see for example [13]) that, if $f : X \to X$ is a continuous and locally Lipschitz continuous function then (31), has a unique local 'mild solution', that is, a solution of the integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s))ds, \tag{32}$$

defined for small positive $0 < t < t_1$, with $x(t) \to x_0$ as $t \to 0+$. If $x_0 \in D(A)$ and f is continuously differentiable then the solution is also a strict solution (i.e $x : (0,t_1) \to X$ is C^1, $x(t) \in D(A)$, for $0 < t < t_1$ and the differential equation (31) is satisfied). If A is bounded, the solution is defined in a *open* interval around 0 (see [7]).

We assume the following hypotheses for the semigroup $T(t)$ generated by A:

(1) (BU - Backwards Uniqueness). For each $t \geq 0$ $T(t)$ is injective;

(2) X has a decomposition such that:

(2a) $X = \pi_- X \oplus \pi_0 X \oplus \pi_+ X$, where π_-, π_0, π_+ are continuous linear projections on X.

(2b) For each $t \geq 0$, $T(t)$ commutes with the operators π_-, π_0, π_+ so that each subspace $\pi_- X$, $\pi_0 X$, $\pi_+ X$ is invariant under $T(t)$. Furthermore, $T(t)$ may be extended to a continuous group of linear operators on $\pi_0 X \oplus \pi_+ X$;

(2c) There exist constants

$$a_-,\ a_0,\ a_+,\ \min\{a_-, a_+\} > a_0 \geq 0 \text{ and } K > 1, \tag{33}$$

such that

(2c.i) $\|T(t)\varphi_-\| \leq Ke^{-a_- t}\|\varphi_-\|$, $\forall\ \varphi \in X$, $t \geq 0$;

(2c.ii) $\|T(t)\varphi_0\| \leq Ke^{a_0|t|}\|\varphi_0\|$, $\forall\ \varphi \in X$, $t \in \mathbb{R}$;

(2c.iii) $\|T(t)\varphi_+\| \leq Ke^{a_+ t}\|\varphi_+\|$, $\forall\ \varphi \in X$, $t \leq 0$.

(Without loss of generality we may assume $a_0 > 0$).

The following theorem on the existence of a unstable manifold near the origin for (31) has been proven in [3].

Theorem A.1. *Suppose the semigroup $T(t)$ satisfies the hypotheses* **(1)** *and* **(2)** *above and $f : X \to X$ is a continuous function satisfying*
(i) $f(0) = 0$*;*
(ii) $\|f(\varphi) - f(\psi))\| \leq \eta(r)\|\varphi - \psi\|,\ \|\varphi\|,\ \|\psi\| < r$*,*
where η is non decreasing continuous real function in $[0, \infty)$ with $\eta(0) = 0$. Suppose also that $\varepsilon > 0$ is such that $a_+ > \varepsilon$. Then, for $\delta > 0$ sufficiently small there exists a locally invariant set

$$U = \{\varphi \in B(0, \delta)\ :\ \|\varphi_+\| < \frac{\delta}{2K},\ \varphi_- + \varphi_0 = q(\varphi_+)\},$$

for (31), where q is a Lipschitz function defined for $\|\varphi_+\| < \frac{\delta}{2K}$. If $\varphi \in U$ then a unique solution $w(t)$ of (32) with $w(0) = \varphi$ exists for $t \leq 0$ and

$$\|w(t)\| \leq 2Ke^{(a_+ - \varepsilon)t}\|w_+(0)\|,\ t \leq 0. \tag{34}$$

Furthermore, U is tangent at zero to $\pi_+ X$ and (q, w_+^q) is the unique solution of the system

$$\begin{aligned} q(\varphi_+) &= \textstyle\int_{-\infty}^{0} T(-s)(\pi_- + \pi_0) f(w_+(s, \varepsilon) + q(w_+(s, \varepsilon)))ds, \\ w_+(t, \varepsilon) &= T(t)\varphi_+ + \textstyle\int_0^t T(t - s)\pi_+ f(w_+(s, \varepsilon) + q(w_+(s, \varepsilon)))ds,\ t \leq 0. \end{aligned} \tag{35}$$

The function q has Lipschitz constant smaller or equal to 1, $q(0) = 0$ and K is the constant given in the hypothesis (2.c).

In the proof of the result above, the hypothesis **(ii)** on f is used to show that, after the usual trick of 'cutting-off' near the origin, one may assume that f satisfies a Lipschitz condition with an arbitrarily small constant. A careful analysis of the proof in [3] reveals, however, that this hypothesis is not necessary in its full force. During the proof, it is only used that the Lipschitz constant of f is *smaller than a constant given in terms of the bound K of the semigroup and the exponential rates $a_-,\ a_0,\ a_+$*. The only part of the result that cannot then be obtained is the tangency to the linear unstable space. More precisely, we have

Theorem A.2. *Suppose the semigroup $T(t)$ satisfies the hypotheses* **(1)** *and* **(2)** *above and $f : X \to X$ is a continuous function satisfying*
(i) $f(0) = 0$*;*
(ii) $\|f(\varphi) - f(\psi))\| \leq L\|\varphi - \psi\|,\ \|\varphi\|,\ \|\psi\| < r$*, with $0 < \frac{4K^2 L}{a_+ - a_0 - 4KL} < 1$ Then, for $\delta > 0$ sufficiently small there exists a locally invariant set*

$$U = \{\varphi \in B(0, \delta)\ :\ \|\varphi_+\| < \frac{\delta}{2K},\ \varphi_- + \varphi_0 = q(\varphi_+)\},$$

for (31), where q is a Lipschitz function defined for $\|\varphi_+\| < \frac{\delta}{2K}$. If $\varphi \in U$ then a unique solution $w(t)$ of (32) with $w(0) = \varphi$ exists for $t \leq 0$ and

$$\|w(t)\| \leq 2Ke^{(a_+ - 2KL)t}\|w_+(0)\|,\ t \leq 0. \tag{36}$$

Furthermore (q, w_+^q) *is the unique solution of the system*

$$\begin{aligned} q(\varphi_+) &= \int_{-\infty}^0 T(-s)(\pi_- + \pi_0) f(w_+(s,\varepsilon) + q(w_+(s,\varepsilon)))ds, \\ w_+(t,\varepsilon) &= T(t)\varphi_+ + \int_0^t T(t-s)\pi_+ f(w_+(s,\varepsilon) + q(w_+(s,\varepsilon)))ds, \ t \le 0. \end{aligned} \tag{37}$$

The function q *has Lipschitz constant smaller or equal to 1,* $q(0) = 0$ *and* K *is the constant given in the hypothesis (2.c).*

The advantage of Theorem .2 for us is that it allows a 'small linearity' in the function f, which will be needed in our applications. We now state the main result of this section.

Theorem A.3. *(Continuity of the unstable manifolds). Suppose that the function* $f = f_\lambda$ *depends on a parameter* $\lambda \in \Lambda$*, where* Λ *is an open set of a Banach space and, satisfies*

$$\|f_\lambda(u) - f_{\lambda_0}(u)\| \le C_1(\lambda)\|u\|, \text{ with } \ C_1(\lambda) \to 0, \text{ as } \lambda \to \lambda_0 \quad \text{for} \quad ||u|| \le r; \tag{38}$$

and the conditions **(i)** *and* **(ii)** *of Theorem .2 above with* $0 < \frac{4K^2L}{a_+ - a_0 - 4KL} < \frac{1}{2}$ *for* λ *in a neighbourhood of* λ_0*.*

Then the unstable manifold U_λ *given by Theorem .2 is continuous with respect to the parameter* λ *at* λ_0*. More precisely, if* δ *is sufficiently small, the Lipschitz functions* $q = q_\lambda$ *given by (37) are defined for* $\|\varphi_+\| < \delta$ *and* $q_\lambda(\varphi_+) \to q_{\lambda_0}(\varphi_+)$*, as* $\lambda \to \lambda_0$ *uniformly for* $||\varphi_+|| < \delta$*.*

Proof After cutting-off near the origin, if necessary, we may suppose that the hypotheses on f_λ hold in the whole space. By Theorem .2, in a neighbourhood of the origin (which can be chosen as the same for all λ sufficiently small), U_λ is the graph of a Lipschitz function $q = q_\lambda$, where (q, w_+^q) is the unique solution of (37). Therefore, we have

$$\begin{aligned} \|w_+(t,\lambda)\| &\le Ke^{a_+t}\|\varphi_+\| + \int_t^0 Ke^{a_+(t-s)}L\|w_+(s,\lambda) + q_\lambda(w_+(s,\lambda))\|ds \\ &\le Ke^{a_+t}\|\varphi_+\| + \int_t^0 Ke^{a_+(t-s)}L[\|w_+(s,\lambda)\| + \|q_\lambda(w_+(s,\lambda))\|]ds \\ &\le Ke^{a_+t}\|\varphi_+\| + \int_t^0 2KLe^{a_+(t-s)}\|w_+(s,\lambda)\|ds. \end{aligned}$$

By Gronwall's Lemma, we obtain

$$\|w_+(t,\lambda)\| \le K\|\varphi_+\|e^{(a_+ - 2KL)t}, \ t \le 0. \tag{39}$$

We will use the metric ρ given by

$$\rho(h_1, h_2) = \sup_{\varphi \in X, \ \varphi_+ \neq 0} \frac{\|h_1(\varphi_+) - h_2(\varphi_+)\|}{\|\varphi_+\|},$$

equipped with which, the set

$$G = \{h : \pi_+X \to \pi_-X \oplus \pi_0X, \ \|h(\varphi_+) - h(\psi_+)\| \le \|\varphi_+ - \psi_+\|, \forall\, \varphi,\, \psi \in X, \ h(0) = 0\}$$

becomes a complete metric space. Let $\theta(t) = \|w_+(t,\lambda) - w_+(t,\lambda_0)\|$, $t \leq 0$. Then

$$\begin{aligned}
\theta(t) &\leq \int_t^0 \|T(t-s)\pi_+\{f_\lambda[w_+(s,\lambda) + q_\lambda(w_+(s,\lambda))] - f_{\lambda_0}[w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))]\}\|ds \\
&\leq \int_t^0 Ke^{(t-s)a_+}\|f_\lambda[w_+(s,\lambda) + q_\lambda(w_+(s,\lambda))] - f_{\lambda_0}[w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))]\|ds \\
&\leq \int_t^0 Ke^{(t-s)a_+}\|f_\lambda[w_+(s,\lambda) + q_\lambda(w_+(s,\lambda))] - f_\lambda[w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))]\|ds \\
&+ \int_t^0 Ke^{(t-s)a_+}\|f_\lambda[w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))] - f_{\lambda_0}[w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))]\|ds.
\end{aligned}$$

From (38) and hypothesis **(ii)**, it follows that

$$\begin{aligned}
\theta(t) &\leq \int_t^0 KLe^{(t-s)a_+}\Big[\|w_+(s,\lambda) - w_+(s,\lambda_0)\| + \|q_\lambda(w_+(s,\lambda)) - q_{\lambda_0}(w_+(s,\lambda_0))\|\Big]ds \\
&+ \int_t^0 Ke^{(t-s)a_+}C_1(\lambda)\|w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))\|ds \\
&= \int_t^0 KLe^{(t-s)a_+}\Big[\theta(s) + \|q_\lambda(w_+(s,\lambda)) - q_{\lambda_0}(w_+(s,\lambda_0))\|\Big]ds \\
&+ \int_t^0 Ke^{(t-s)a_+}C_1(\lambda)\|w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))\|ds.
\end{aligned}$$

Using that q_{λ_0} is Lipschitz with Lipschitz constant ≤ 1 and $q_{\lambda_0}(0) = 0$, we obtain

$$\begin{aligned}
\theta(t) &\leq \int_t^0 KLe^{(t-s)a_+}\Big[\theta(s) + \|q_\lambda(w_+(s,\lambda)) - q_{\lambda_0}(w_+(s,\lambda_0))\|\Big]ds \\
&+ \int_t^0 2Ke^{(t-s)a_+}C_1(\lambda)\|w_+(s,\lambda_0)\|ds.
\end{aligned}$$

Now, using the same argument for q_λ

$$\begin{aligned}
\|q_\lambda(w_+(s,\lambda)) - q_{\lambda_0}(w_+(s,\lambda_0))\| &\leq \|q_\lambda(w_+(s,\lambda)) - q_\lambda(w_+(s,\lambda_0))\| \\
&+ \|q_\lambda(w_+(s,\lambda_0)) - q_{\lambda_0}(w_+(s,\lambda_0))\| \\
&\leq \|w_+(s,\lambda) - w_+(s,\lambda_0)\| \\
&+ \|q_\lambda(w_+(s,\lambda_0)) - q_{\lambda_0}(w_+(s,\lambda_0))\| \\
&= \theta(s) + \|q_\lambda(w_+(s,\lambda_0)) - q_{\lambda_0}(w_+(s,\lambda_0))\| \\
&= \theta(s) + \|w_+(s,\lambda_0)\|\frac{\|q_\lambda(w_+(s,\lambda_0)) - q_{\lambda_0}(w_+(s,\lambda_0))\|}{\|w_+(s,\lambda_0)\|} \\
&\leq \theta(s) + \|w_+(s,\lambda_0)\|\rho\left(q_\lambda, q_{\lambda_0}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\theta(t) &\leq \int_t^0 KLe^{(t-s)a_+}[\theta(s)+\theta(s)+\|w_+(s,\lambda_0)\|\rho\left(q_\lambda,q_{\lambda_0}\right)]ds \\
&+ \int_t^0 2KC_1(\lambda)e^{(t-s)a_+}\|w_+(s,\lambda_0)\|ds \\
&= \int_t^0 2KLe^{(t-s)a_+}\theta(s)ds \\
&+ \int_t^0 KLe^{(t-s)a_+}\|w_+(s,\lambda_0)\|\rho\left(q_\lambda,q_{\lambda_0}\right)ds \\
&+ \int_t^0 2KC_1(\lambda)e^{(t-s)a_+}\|w_+(s,\lambda_0)\|ds.
\end{aligned}$$

Using (39), we obtain

$$\begin{aligned}
\theta(t) &\leq \int_t^0 2KLe^{(t-s)a_+}\theta(s)ds + \int_t^0 KLe^{(t-s)a_+}\rho(q_\lambda,q_{\lambda_0})K\|\varphi_+\|e^{(a_+-2KL)s}ds \\
&+ \int_t^0 2KC_1(\lambda)e^{(t-s)a_+}K\|\varphi_+\|e^{(a_+-2KL)s}ds \\
&= \int_t^0 2KLe^{(t-s)a_+}\theta(s)ds + K^2\|\varphi_+\|L\rho(q_\lambda,q_{\lambda_0})e^{a_+t}\int_t^0 e^{-2KLs}ds \\
&+ 2K^2\|\varphi_+\|C_1(\lambda)e^{a_+t}\int_t^0 e^{-2KLs}ds.
\end{aligned}$$

Thus

$$\begin{aligned}
e^{-a_+t}\theta(t) &\leq \int_t^0 2KLe^{-a_+s}\theta(s)ds + K^2\|\varphi_+\|L\rho(q_\lambda,q_{\lambda_0})\int_t^0 e^{-2KLs}ds \\
&+ 2K^2\|\varphi_+\|C_1(\lambda)\int_t^0 e^{-2KLs}ds \\
&\leq \int_t^0 2KLe^{-a_+s}\theta(s)ds + \frac{K\|\varphi_+\|L}{2L}\rho(q_\lambda,q_{\lambda_0})e^{-2KLt} \\
&+ \frac{K\|\varphi_+\|}{L}C_1(\lambda)e^{-2KLt}.
\end{aligned}$$

From General Gronwall‘s Lemma, (see [9]), it follows that

$$e^{-a_+t}\theta(t) \leq e^{-2KLt}\left[\frac{K\|\varphi_+\|}{2}\rho(q_\lambda,q_{\lambda_0})e^{-2KLt} + \frac{K\|\varphi_+\|C_1(\lambda)}{L}e^{-2KLt}\right].$$

Hence

$$\theta(t) \leq \frac{K\|\varphi_+\|}{2}\rho(q_\lambda,q_{\lambda_0})e^{(a_+-4KL)t} + \frac{K\|\varphi_+\|C_1(\lambda)}{L}e^{(a_+-4KL)t}. \tag{40}$$

Now

$$
\begin{aligned}
\|q_\lambda(\varphi_+) - q_{\lambda_0}(\varphi_+)\| &\leq \int_{-\infty}^0 \|T(-s)\pi_-[f_\lambda(w_+(s,\lambda) + q_\lambda(w_+(s,\lambda)))\\
&- f_{\lambda_0}(w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0)))]\|ds\\
&+ \int_{-\infty}^0 \|T(-s)\pi_0[f_\lambda(w_+(s,\lambda) + q_\lambda(w_+(s,\lambda)))\\
&- f_{\lambda_0}(w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0)))]\|ds\\
&\leq \int_{-\infty}^0 Ke^{a_- s}\|f_\lambda[w_+(s,\lambda) + q_\lambda(w_+(s,\lambda))]\\
&- f_{\lambda_0}[w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))]\|ds\\
&+ \int_{-\infty}^0 Ke^{-a_0 s}\|f_\lambda[w_+(s,\lambda) + q_\lambda(w_+(s,\lambda))]\\
&- f_{\lambda_0}[w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))]\|ds\\
&\leq \int_{-\infty}^0 2Ke^{-a_0 s}\|f_\lambda[w_+(s,\lambda) + q_\lambda(w_+(s,\lambda))]\\
&- f_{\lambda_0}[w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))]\|ds\\
&\leq \int_{-\infty}^0 2Ke^{-a_0 s}\|f_\lambda[w_+(s,\lambda) + q_\lambda(w_+(s,\lambda))]\\
&- f_\lambda[w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))]\|ds\\
&+ \int_{-\infty}^0 2Ke^{-a_0 s}\|f_\lambda[w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))]\\
&- f_{\lambda_0}[w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))]\|ds.
\end{aligned}
$$

Using (38) and **(ii)**, it follows that

$$
\begin{aligned}
\|q_\lambda(\varphi_+) - q_{\lambda_0}(\varphi_+)\| &\leq \int_{-\infty}^0 2KLe^{-a_0 s}\Big\{\|w_+(s,\lambda) - w_+(s,\lambda_0)\|\\
&+ \|q_\lambda(w_+(s,\lambda)) - q_{\lambda_0}(w_+(s,\lambda_0))\|\Big\}ds\\
&+ \int_{-\infty}^0 2Ke^{-a_0 s}C_1(\lambda)\|w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))\|ds\\
&= \int_{-\infty}^0 2KLe^{-a_0 s}\Big\{\theta(s) + \|q_\lambda(w_+(s,\lambda)) - q_{\lambda_0}(w_+(s,\lambda_0))\|\Big\}ds\\
&+ \int_{-\infty}^0 2Ke^{-a_0 s}C_1(\lambda)\|w_+(s,\lambda_0) + q_{\lambda_0}(w_+(s,\lambda_0))\|ds\\
&\leq \int_{-\infty}^0 2KLe^{-a_0 s}\Big\{\theta(s) + \|q_\lambda(w_+(s,\lambda)) - q_{\lambda_0}(w_+(s,\lambda_0))\|\Big\}ds\\
&+ \int_{-\infty}^0 4Ke^{-a_0 s}C_1(\lambda)\|w_+(s,\lambda_0)\|ds.
\end{aligned}
$$

Using once again that

$$\|q_\lambda(w_+(s,\lambda)) - q_{\lambda_0}(w_+(s,\lambda_0))\| \leq \theta(s) + \|w_+(s,\lambda_0)\|\rho(q_\lambda, q_{\lambda_0}),$$

we obtain

$$\begin{aligned}\|q_\lambda(\varphi_+) - q_{\lambda_0}(\varphi_+)\| &\leq \int_{-\infty}^{0} 2KLe^{-a_0 s}\Big[2\theta(s) + \|w_+(s,\lambda_0)\|\rho(q_\lambda, q_{\lambda_0})\Big]ds \\ &+ \int_{-\infty}^{0} 4KC_1(\lambda)e^{-a_0 s}\|w_+(s,\lambda_0)\|ds.\end{aligned}$$

Now, using (39), it follows that

$$\begin{aligned}\|q_\lambda(\varphi_+) - q_{\lambda_0}(\varphi_+)\| &\leq \int_{-\infty}^{0} 4KLe^{-a_0 s}\theta(s)ds \\ &+ \int_{-\infty}^{0} 2K^2L\|\varphi_+\|\rho(q_\lambda, q_{\lambda_0})e^{(a_+-a_0-2KL)s}ds \\ &+ \int_{-\infty}^{0} 4K^2\|\varphi_+\|C_1(\lambda)e^{(a_+-a_0-2KL)s}ds.\end{aligned}$$

Thus,

$$\|q_\lambda(\varphi_+) - q_{\lambda_0}(\varphi_+)\| \leq I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{-\infty}^{0} 4KLe^{-a_0 s}\theta(s)ds,$$

$$I_2 = \int_{-\infty}^{0} 2K^2L\|\varphi_+\|\rho(q_\lambda, q_{\lambda_0})e^{(a_+-a_0-2KL)s}ds$$

and

$$I_3 = \int_{-\infty}^{0} 4K^2\|\varphi_+\|C_1(\lambda)e^{(a_+-a_0-2KL)s}ds.$$

Using the estimate obtained for $\theta(t)$ in (40), we obtain

$$\begin{aligned}I_1 &\leq \int_{-\infty}^{0} 4KLe^{-a_0 s}\left[\frac{K}{2}\|\varphi_+\|\rho(q_\lambda, q_{\lambda_0})e^{(a_+-4KL)s} + \frac{K\|\varphi_+\|}{L}C_1(\lambda)e^{(a_+-4KL)s}\right]ds \\ &= \int_{-\infty}^{0} 2K^2\|\varphi_+\|L\rho(q_\lambda, q_{\lambda_0})e^{(a_+-a_0-4KL)s}ds \\ &+ \int_{-\infty}^{0} 4K^2\|\varphi_+\|C_1(\lambda)e^{(a_+-a_0-4KL)s}ds \\ &= \frac{2K^2L\|\varphi_+\|}{a_+-a_0-4KL}\rho(q_\lambda, q_{\lambda_0}) + \frac{4K^2\|\varphi_+\|}{a_+-a_0-4KL}C_1(\lambda).\end{aligned}$$

Furthermore,

$$\begin{aligned}I_2 &= \int_{-\infty}^{0} 2K^2L\|\varphi_+\|\rho(q_\lambda, q_{\lambda_0})e^{(a_+-a_0-2KL)s}ds \\ &= \frac{2K^2L\|\varphi_+\|}{a_+-a_0-2KL}\rho(q_\lambda, q_{\lambda_0})\end{aligned}$$

and

$$\begin{aligned} I_3 &= \int_{-\infty}^{0} 4K^2\|\varphi_+\|C_1(\lambda)e^{(a_+-a_0-2KL)s}ds \\ &= \frac{4K^2\|\varphi_+\|}{a_+-a_0-2KL}C_1(\lambda). \end{aligned}$$

Therefore

$$\begin{aligned} \|q_\lambda(\varphi_+)-q_{\lambda_0}(\varphi_+)\| &\leq \frac{2K^2L\|\varphi_+\|}{a_+-a_0-4KL}\rho(q_\lambda,q_{\lambda_0}) + \frac{4K^2\|\varphi_+\|}{a_+-a_0-4KL}C_1(\lambda) \\ &+ \frac{2K^2L\|\varphi_+\|}{a_+-a_0-2KL}\rho(q_\lambda,q_{\lambda_0}) + \frac{4K^2\|\varphi_+\|}{a_+-a_0-2KL}C_1(\lambda) \\ &= \left[\frac{2K^2L\|\varphi_+\|}{a_+-a_0-4KL} + \frac{2K^2L\|\varphi_+\|}{a_+-a_0-2KL}\right]\rho(q_\lambda,q_{\lambda_0}) \\ &+ \left[\frac{4K^2\|\varphi_+\|}{a_+-a_0-4KL} + \frac{4K^2\|\varphi_+\|}{a_+-a_0-2KL}\right]C_1(\lambda). \end{aligned}$$

Hence

$$\begin{aligned} \frac{\|q_\lambda(\varphi_+)-q_{\lambda_0}(\varphi_+)\|}{\|\varphi_+\|} &\leq \left[\frac{2K^2L}{a_+-a_0-4KL} + \frac{2K^2L}{a_+-a_0-2KL}\right]\rho(q_\lambda,q_{\lambda_0}) \\ &+ \left[\frac{4K^2}{a_+-a_0-4KL} + \frac{4K^2}{a_+-a_0-2KL}\right]C_1(\lambda), \end{aligned}$$

which implies

$$\begin{aligned} \sup_{\varphi\in X,\,\varphi_+\neq 0}\frac{\|q_\lambda(\varphi_+)-q_{\lambda_0}(\varphi_+)\|}{\|\varphi_+\|} &\leq \left[\frac{2K^2L}{a_+-a_0-4KL} + \frac{2K^2L}{a_+-a_0-2KL}\right]\rho(q_\lambda,q_{\lambda_0}) \\ &+ \left[\frac{4K^2}{a_+-a_0-4KL} + \frac{4K^2}{a_+-a_0-2KL}\right]C_1(\lambda). \end{aligned}$$

Therefore

$$\begin{aligned} \rho(q_\lambda,q_{\lambda_0}) &\leq \left[\frac{2K^2L}{a_+-a_0-4KL} + \frac{2K^2L}{a_+-a_0-2KL}\right]\rho(q_\lambda,q_{\lambda_0}) \\ &+ \left[\frac{4K^2}{a_+-a_0-4KL} + \frac{4K^2}{a_+-a_0-2KL}\right]C_1(\lambda) \\ &< \frac{1}{2}\rho(q_\lambda,q_{\lambda_0}) + C_2(\lambda), \end{aligned}$$

where $C_2(\lambda) = \left[\frac{4K^2}{a_+-a_0-4KL} + \frac{4K^2}{a_+-a_0-2KL}\right]C_1(\lambda)$.

Therefore

$$\rho(q_\lambda,q_{\lambda_0}) < 2C_2(\lambda),$$

where $C_2(\lambda)\to 0$, as $\lambda\to\lambda_0$, concluding the proof. □

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