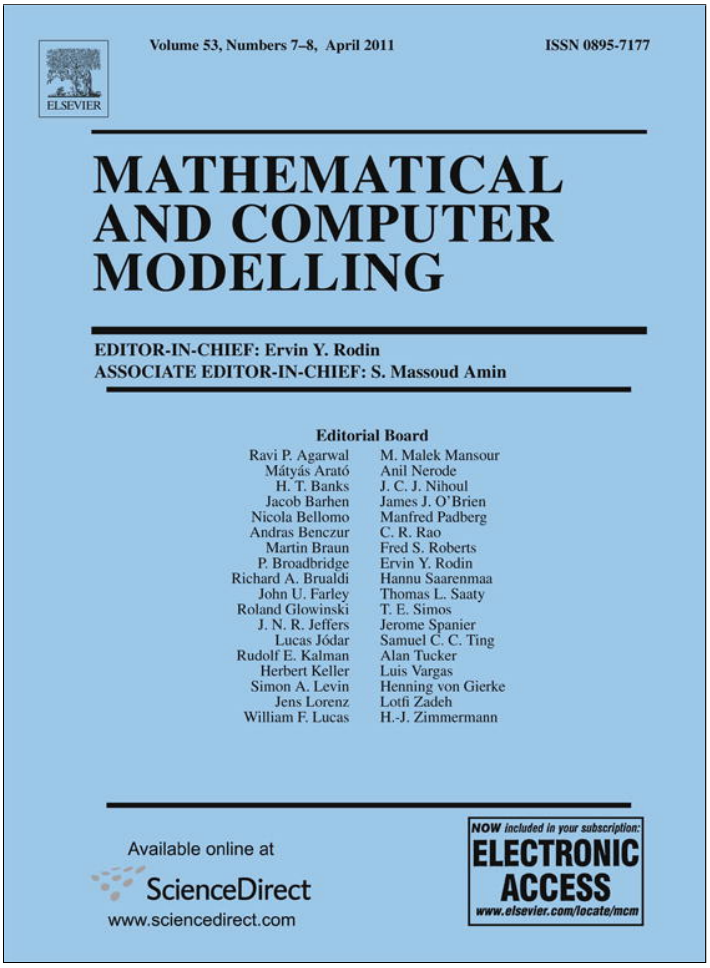

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Symmetry breaking in the genetic code: Finite groups[☆]

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ABSTRACT

We investigate the possibility of interpreting the degeneracy of the genetic code, i.e., the feature that different codons (base triplets) of DNA are transcribed into the same amino acid, as the result of a symmetry breaking process, in the context of finite groups. In the first part of this paper, we give the complete list of all codon representations (64-dimensional irreducible representations) of simple finite groups and their satellites (central extensions and extensions by outer automorphisms). In the second part, we analyze the branching rules for the codon representations found in the first part by computational methods, using a software package for computational group theory. The final result is a complete classification of the possible schemes, based on finite simple groups, that reproduce the multiplet structure of the genetic code.

1. Introduction

The discovery of the molecular structure of DNA in 1953 by Watson and Crick has been a landmark of science, laying the foundation for understanding the physico-chemical basis for the storage and transfer of genetic information. DNA is a macro-molecule in the form of a double helix, encoding all genetic information in a language with 64 three-letter words built from an alphabet with a set of four different letters (A, C, G and T – the four nucleic bases attached to the backbone of a DNA molecule). These words are called codons and form sentences called genes. Each codon can be translated into one of twenty amino acids or a termination signal. This leads to a degeneracy of the code in the sense that different codons represent the same amino acid, that is, different words have the same meaning. In fact, the codons which code for the same amino acids form multiplets as follows:

- 3 sextets — Arg, Leu, Ser
- 5 quadruplets — Ala, Gly, Pro, Thr, Val
- 2 triplets — Ile, Term
- 9 doublets — Asn, Asp, Cys, Gln, Glu, His, Lys, Phe, Tyr
- 2 singlets — Met, Trp

When a protein is synthesized, an appropriate segment of one of the two strings in the DNA molecule (or more precisely, the mRNA molecule built from it) is read, the corresponding amino acids are assembled sequentially and the resulting chain, when complete, is released from the ribosome. The linear chain thus obtained will then fold to the final configuration of the protein.

Historically, the pathway towards the discovery of these now well-known facts has not been nearly so simple. Initially, despite the enormous interest triggered by the work of Crick and Watson, no experimental information on the specific

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mechanisms of cellular protein synthesis was available. In the absence of concrete data, people resorted to mathematically oriented models, among them Gamow's "diamond code" and Crick's proposal of non-overlapping codes, which were intensively debated in the period between 1953 and 1960. For a more detailed account of this early phase of mathematical modelling in molecular biology, see the delightful review [1]. Unfortunately, when experimental work realized between 1960 and 1966 finally unravelled the real structure and functioning of the genetic code, all these models – brilliantly devised as they were – turned out to be utterly wrong. This dramatic failure of mathematically oriented reasoning has probably contributed substantially to the currently widespread belief among biologists that mathematics is not an effective tool for handling problems in genetics and molecular biology – a prejudice that is only gradually beginning to be overcome.

On the other hand, the experimental facts by themselves – well established as they now are – provide no explanation as to why just this special language has been chosen by nature. Even though there have already been, during the 1970's and 1980's, several attempts to find mathematical structures underlying the assignment between amino acids and codons, these have either remained incomplete [2] or fallen into the same trap of contradicting experimentally established biological facts [3]. Thus for a long time, the genetic code has essentially remained a table connecting codons (base triplets) with the amino acids that they represent, but a complete understanding of its structure is still lacking.

A new approach to the question was suggested in 1993 by Hornos and Hornos [4] who proposed explaining the degeneracy of the genetic code as the result of a symmetry breaking process. The demand of this approach, which differs radically from the previous ones in that it is consistent with all known biological facts, can be compared to that of explaining the arrangement of the chemical elements in the periodic table as the result of an underlying dynamical symmetry which is reflected in the electronic shell structure of atoms. Another comparable example is the explanation of the multiplet structure of hadrons as a result of a "flavor" $SU(3)$ symmetry, which led to the quark model and to the prediction of new particles. An interesting and important feature of this "flavor" symmetry is its internal or dynamical nature, that is, it is an internal property of the dynamical equations of the system, rather than being related to the structure of space–time.

In the same spirit, the idea of the above mentioned authors was to explain the multiplet structure of the genetic code through the multiplets found in a codon representation (i.e., an irreducible 64-dimensional representation) of an appropriate simple Lie algebra and its branching rules into irreducible representations of its semi-simple subalgebras. They checked the tables of branching rules of McKay and Patera [5] for semi-simple subalgebras of simple Lie algebras of rank ≤ 8. The most suitable multiplet structure found is derived from the codon representation of the symplectic algebra $\mathfrak{sp}(6)$ by the following sequence of symmetry breakings:

$$\begin{array}{lcll} \mathfrak{sp}(6) & \supset & \mathfrak{sp}(4) \oplus \mathfrak{su}(2) & I \\ & \supset & \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) & II \\ & \supset & \mathfrak{su}(2) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) & III/IV/V \end{array}$$

The sequence of steps I–V is associated with the evolution of the genetic code at the beginning of life. For a more detailed exposition, see Ref. [6].

This work, which had a strong resonance in the scientific community [7,8], has raised a lot of new interesting problems. One of them is that the last step in the symmetry breaking remains incomplete: the lifting of degeneracy by breaking the last two $\mathfrak{su}(2)$ subalgebras to $\mathfrak{u}(1)$ is not followed by all codon multiplets. Only if some of them continue to represent a single amino acid can the actual multiplet structure of the genetic code be obtained. This "freezing" had already been proposed by biologists [9] who claimed that a completely accomplished evolution of the genetic code should have resulted in 28 amino acids [10] – in perfect agreement with the mathematical model. However, the phenomenon that some of the multiplets preserve a symmetry while it is broken in others, even though it does not contradict any biological principle or observation (see Ref. [11] or Ref. [12] for extensive reviews), is somewhat awkward from a mathematical point of view. On the other hand, this property of the model has been explored in order to explain the small deviations observed in non-standard genetic codes; see Ref. [13].

The proposal of the present project, already stated in Ref. [4], is to study the same problem within its most traditional and natural mathematical context, namely that of finite groups, rather than that of compact Lie groups. In fact, there is no particular reason for using continuous symmetries, rather than discrete ones, in this kind of investigation. Another direction, also contemplated in Ref. [4], is to replace ordinary Lie algebras by other algebras in their "vicinity", such as quantum groups and Lie superalgebras; a systematic analysis for the case of Lie superalgebras can be found in Refs [14,15]. Of course, the reader may wonder why the whole program has been carried out first for Lie algebras and even for Lie superalgebras and not for finite groups, but the reason is quite simple: the case of finite groups is technically the most difficult one. Indeed, until recently, we did not really believe it would be possible to overcome the formidable problems associated with the question of obtaining a complete classification and arrive at a definite conclusion. Fortunately, this was not the case. Moreover, as a by-product, we were able to extend and unify the analysis of Ref. [16] about codon and amino acid assignments in algebraic models of the genetic code; see Ref. [17].

2. Symmetry breaking in the evolution of the genetic code

The central idea of the algebraic approach proposed in Ref. [4] and presented in detail in Ref. [6] is to view the distribution of multiplets found in the standard code as the result of a symmetry breaking process. Starting out from a 64-dimensional irreducible representation – or *codon representation*, for short – of some primordial symmetry group or algebra, the standard code has according to this picture evolved into its present form through a sequence of transitions, each of them accompanied by a reduction of the symmetry group existing at the previous stage to an appropriate maximal subgroup or subalgebra. In the last step, this reduction is allowed to be partial, in the sense that some of the multiplets that would normally break up are allowed to remain intact, or "frozen".

This general strategy can be implemented in various algebraic categories and involves two distinct steps:

(1) determination of the codon representations of all simple groups/algebras,
(2) analysis of their branching rules under reduction to subgroups/subalgebras or, more generally, to chains of subgroups/subalgebras.

Such a program has first been carried out for Lie algebras [4,16,18] (see Ref. [6] for a detailed exposition) and later extended to Lie superalgebras [14,15].

Performing the same program for finite groups, which is perhaps the most natural context for this kind of investigation, is however much more difficult and has for a long time remained a challenging open problem, which has only recently been solved [19]. The results have been announced in Ref. [20] and here we shall present their derivation in more detail.

The first task is the determination of all codon representations. The main difficulty to be overcome here is to establish sufficiently stringent cutoffs on the parameters for the infinite series of groups/algebras. In the case of the finite simple groups, this requires the use of a combination of sophisticated theorems from finite group theory, some of which have only recently become available. The remaining cases can then be handled by using the *Atlas of Finite Groups* [21], which is the basic source of information on representations of simple finite groups and their satellites, as well as the computer program *GAP – Groups, Algorithms and Programming* [22], which calculates character tables of arbitrary finite groups, up to a certain order.

The second task is the determination of the branching rules of all these codon representations in order to see whether any of them, when reduced to an appropriate subgroup, will reproduce the multiplet structure of the genetic code. As in the other algebraic categories, there are a few exceptions: in the case of finite groups these are the huge groups A_{65} and S_{65}, which have been excluded since, obviously, their codon representations can be broken so as to reproduce any distribution of multiplets whatsoever, as well as the covering groups of the symmetric and alternating groups S_{13}, A_{14}, S_{14} and S_{15}, for which the computer calculations were unfeasible at the time. The first and most tedious step consists in calculating, for each of the pertinent groups, the lattice of subgroups, up to conjugacy. Due to the structure of the algorithms for computing maximal subgroups, it turns out that – in contrast to the situation prevailing for Lie algebras and Lie superalgebras – nothing is gained by restricting to maximal subgroups, so it is at this stage more efficient to disregard chains of maximal subgroups and instead proceed directly to the final subgroup or, when "freezing" is involved, to the pair of subgroups formed by the penultimate and the final subgroup in the chain, calculating the corresponding branching schemes.

The methodology employed in the analysis of finite groups suggests a technical definition of an extended form of symmetry breaking which is only partial, allowing for a certain amount of "accidental degeneracies" in the final distribution of multiplets. Such a partial symmetry breaking is described by a group G with a given representation and a *pair* (H, K) of subgroups of G such that K is a maximal subgroup of H. Considering the decomposition of the original representation of G into irreducible representations of H and then the further splitting into irreducible representations of K, some of the irreducible H-multiplets that would normally split into several irreducible K-multiplets are allowed to remain intact, or "frozen". The restriction that we propose to impose on this phenomenon of (partial) freezing is that whenever the same H-multiplet occurs with multiplicity greater than 1, all of its copies should behave in the same way: either they all split or else they all remain unbroken. In other words, the alternative of freezing applies not to single multiplets but rather to *isotypic components*: in the decomposition of a representation into irreducible components, each isotypic subspace is the direct sum of all the irreducible subspaces that are equivalent among themselves, and unlike the individual irreducible subspaces, the isotypic subspaces are unique. This is the rule that has been used in our analysis, for Lie algebras [18], for Lie superalgebras [14,15] and for finite groups [19]: we propose to call it the *Higgs–Crick mechanism*.

3. The finite simple groups and their representations

Our first task in what follows is to specify the class of finite groups within which the search for codon representations has been conducted. This class is formed by the *finite simple groups*, i.e., those that have no non-trivial normal subgroups, together with their so-called *satellites*, obtained as central extensions or extensions by outer automorphisms or a combination thereof.

Our basic sources of information will be the *Atlas of Finite Groups* [21], simply referred to as the ATLAS, and the software package *GAP – Groups, Algorithms and Programming* [22], referred to as GAP.

Extensions by automorphisms and covering groups

Recall that any group G has (at least) two natural normal subgroups, namely its *center* $Z(G)$, consisting of those elements that commute with all elements of G, and its commutator subgroup or *derived subgroup* G', formed by products of elements $g_1 g_2 g_1^{-1} g_2^{-1}$ with g_1 and g_2 running through the elements of G; moreover, a group G is called *perfect* if it is equal to its derived subgroup G'. Of course, when G is simple, there are only two alternatives: either G is abelian ($Z(G) = G$, $G' = \{1\}$) or G is perfect ($Z(G) = \{1\}$, $G' = G$).

Among the finite groups, the simple ones constitute the basic building blocks from which all others can be constructed. Indeed, the simple constituents of any finite group can be determined from its so-called *composition series* – a finite sequence of subgroups such that each of them is a maximal normal subgroup of the previous one, implying that the quotient formed by any two consecutive subgroups of the series is simple. This series is unique in the sense that any two such sequences have the same length and provide (up to isomorphism) the same simple quotient groups, though possibly in a different order. As a result, any finite group can be constructed from simple finite groups by repeated application of an appropriate extension procedure. Various such procedures that allow one to build new groups from given ones are known, the ones of interest here being central extensions and extensions by outer automorphisms, which we proceed to explain briefly.

The general idea of a group extension allows two different interpretations which, in a certain sense, are dual to each other; both of them are most easily formulated in terms of short exact sequences. Namely, given three groups A, B, C forming a short exact sequence

$$\{1\} \longrightarrow C \longrightarrow B \longrightarrow A \longrightarrow \{1\}, \tag{1}$$

one says that B is an *extension of C by A* or an *extension of A by C*, depending on the circumstances. These two options correspond to the idea that an extension of a group G by a group H is either a larger group $\hat{G}$ containing G as a normal subgroup with H appearing as the quotient group $\hat{G}/G$ or a larger group $\tilde{G}$ containing H as a normal subgroup with G appearing as the quotient group $\tilde{G}/H$. Both options play an extensive role in group theory, with a wide range of applications; in particular, extensions by outer automorphisms are of the first type and central extensions are of the second type.

More explicitly, a group $\hat{G}$ containing G as a normal subgroup is said to be an *extension* of G *by outer automorphisms* if the centralizer of G in $\hat{G}$ reduces to the center of G, that is, there is no element of $\hat{G} \setminus G$ commuting with all elements of G. The corresponding short exact sequence of groups has the form

$$\{1\} \longrightarrow G \longrightarrow \hat{G} \longrightarrow A \longrightarrow \{1\}. \tag{2}$$

To motivate this definition, note that as $\hat{G}$ is supposed to contain G as a normal subgroup, $\hat{G}$ acts on G by conjugation and this action provides a homomorphism of $\hat{G}$ into the group $\mathrm{Aut}(G)$ of automorphisms of G which restricts to the standard homomorphism of G into the normal subgroup $\mathrm{Inn}(G)$ of inner automorphisms of G. The extra hypothesis stated above asserts that these two homomorphisms have the same kernel and therefore induce an injective homomorphism of quotient groups, and so $A = \hat{G}/G$ can be considered as a subgroup of the group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ of outer automorphisms of G:

$$A \subset \mathrm{Out}(G). \tag{3}$$

Similarly, a group $\tilde{G}$ is said to be a *central extension* of G if G can be written as the quotient of $\tilde{G}$ by an appropriate subgroup M of its center (note that M is automatically abelian and a normal subgroup of $\tilde{G}$). The corresponding short exact sequence of groups has the form

$$\{1\} \longrightarrow M \longrightarrow \tilde{G} \longrightarrow G \longrightarrow \{1\}. \tag{4}$$

When the hypothesis that M is contained in the center of $\tilde{G}$,

$$M \subset Z(\tilde{G}), \tag{5}$$

is replaced by the stronger hypothesis that M is contained in the intersection between the center and the derived subgroup of $\tilde{G}$,

$$M \subset Z(\tilde{G}) \cap \tilde{G}', \tag{6}$$

one says that $\tilde{G}$ is a *proper central extension* or *covering group* of G. The appropriate concept for describing such extensions is the *Schur multiplier* of G, which we shall denote by $M(G)$: it can be defined abstractly as the second Hochschild cohomology group $H^2(G, \mathbb{C}^*)$ of G with coefficients in the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers:

$$M(G) = H^2(G, \mathbb{C}^*). \tag{7}$$

Following the notation of the ATLAS, we shall use the generic symbol $G.A$ to denote any group obtained as an extension of the group G by some group of outer automorphisms A and the generic symbol $M.G$ to denote any group obtained as a proper central extension of the group G by some abelian group M.

In what follows, we shall consider only finite groups – even though the concepts introduced so far make sense even for infinite groups. In particular, if G is a finite group, so are its automorphism group $\mathrm{Aut}(G)$ and – as it turns out – its Schur multiplier $M(G)$, as well as their various subgroups and quotient groups. In this case, just as all groups A which arise for extensions $G.A$ of a given group G by outer automorphisms are subgroups of a largest one, namely $\mathrm{Out}(G)$, one finds that all abelian groups M which arise for proper central extensions $M.G$ of a given group G are quotient groups of a largest one, namely $M(G)$. For the extensions themselves, however, similar statements can be obtained only under additional assumptions on G:

- If G is a finite group with trivial center, and so $G \cong \mathrm{Inn}(G)$, there is, up to isomorphism, a unique largest extension of G by outer automorphisms, namely the group $\mathrm{Aut}(G)$ itself: it is maximal in the sense that any other extension $G.A$ of

G by outer automorphisms can be obtained as a subgroup of $\mathrm{Aut}(G)$, namely the inverse image of A under the natural projection of $\mathrm{Aut}(G)$ onto $\mathrm{Out}(G)$. When G has a non-trivial center $Z(G)$, the desired extension $G.A$ may not exist at all or there may exist several non-isomorphic versions of it; this will in general depend on the specific nature of the action of A on $Z(G)$ (induced from the obvious action of $\mathrm{Aut}(G)$ on G which leaves $Z(G)$ invariant and factors to an action of $\mathrm{Out}(G)$ on $Z(G)$ since $\mathrm{Inn}(G)$ acts trivially on $Z(G)$).

- If G is a perfect finite group, and so $G' = G$, there is, up to isomorphism, a unique largest covering group $\tilde{G}$ of G, called the *universal covering group* of G: its center is just the Schur multiplier of G,

$$Z(\tilde{G}) = M(G), \tag{8}$$

and it is maximal in the sense that any other covering group $M.G$ of G can be obtained as a quotient group of $\tilde{G}$, namely the one by that subgroup $\tilde{M}$ of $M(G)$ for which $M = M(G)/\tilde{M}$. When G has a non-trivial derived subgroup, the desired extension $M.G$ may not exist at all or there may exist several non-isomorphic versions of it. In the special case where M is the entire Schur multiplier $M(G)$, existence of a covering group of G with center isomorphic to $M(G)$ can still be guaranteed, but uniqueness is lost, so it seems appropriate to replace the term "universal" by the term "maximal", that is, G will in general admit several non-isomorphic *maximal covering groups*. To handle this more general case, it is useful to introduce the notion of *isoclinism* – a more general equivalence relation between groups than that of isomorphism – since any two maximal covering groups of G turn out to be *isoclinic*. For more details, the reader is referred to the introduction to the ATLAS and to Ref. [23].

In particular, when starting out from a perfect simple finite group G or, more generally, from a perfect finite group with trivial center, it is clear how to construct the extensions $G.A$ and $M.G$, for any subgroup A of $\mathrm{Out}(G)$ and any quotient group M of $M(G)$. Not nearly as clear is what is to be meant by the double extension $M.G.A$, since the two interpretations that come to mind are both plagued by ambiguities. In view of the relation $(G.A)' = G.A'$, $G.A$ is no longer perfect (except when A is perfect, which is rarely the case), while $M.G$ has center M, so the two candidate groups $M.(G.A)$ and $(M.G).A$ may not exist or may require additional data in order to be well-defined. At any rate, these groups $G.A$, $M.G$ and $M.G.A$ (insofar as they exist) are collectively known as the *satellites* of G.

Linear and projective representations

The problem of determining all irreducible representations of a finite simple group G and of its satellites with a given dimension fortunately does not require considering all satellite groups of G, but only a certain subset of them. To explain why this is so and what is the subset that must be analyzed, we have to make a digression into the representation theory of finite groups.

Covering groups play a fundamental role in group theory because they allow one to lift projective representations to linear representations – a procedure that, according to well-known theorems going back to Wigner and to Bargmann, is essential for applications to quantum theory. In fact, our desire to determine not only the linear codon representations of the simple finite groups but also the projective ones is motivated by the speculation that the origin of symmetry in the genetic code might in some way be related to quantum theory. Moreover, including the representations of the covering groups of simple groups corresponds to the strategy adopted in earlier investigations of the same subject in different contexts, primarily that of compact Lie groups and Lie algebras.

The maximal covering groups mentioned above are also known as *representation groups*, or more precisely as representation groups over $\mathbb{C}$ since the notion can be extended to contemplate ground fields other than that of complex numbers. This term indicates the fact that passing from G to a representation group $\tilde{G}$ of G allows one to include projective representations: not only every linear representation but also every projective representation of G is induced from a linear representation of $\tilde{G}$. As mentioned before, any two such representation groups of G are isoclinic. Moreover, isoclinic groups have essentially the same representations, since any irreducible representation of one of them acting in a complex vector space can, through multiplication of the representing operators by appropriate scalars in $\mathbb{C}^*$, be converted to an irreducible representation of the other acting in the same vector space; the same conversion rule applies to the irreducible characters. In particular, the answer to the question of whether there exist irreducible projective representations of G of a given dimension does not depend on which representation group $\tilde{G}$ is chosen to lift them to irreducible linear representations. A further simplification arises due to the fact that every irreducible linear representation of $\tilde{G}$ maps the center of $\tilde{G}$ to a finite subgroup of $\mathbb{C}^*$ and that this is necessarily a cyclic group, so after dividing out the kernel of the original representation of $\tilde{G}$, it provides an irreducible linear representation of an appropriate covering group $M.G$ of G for which M is cyclic. It is in this way, namely through the characters for irreducible linear representations of G and of all covering groups of G of the form $\mathbb{Z}_n.G$, that the tables of the ATLAS provide a complete classification of all irreducible representations – linear as well as projective – of G, for a large number of simple finite groups, as well as their extensions by cyclic groups of outer automorphisms.

There is also a close relationship between representations, linear as well as projective, of a finite group G and of any one of its extensions $G.A$ by outer automorphisms. The basic result here is a theorem due to Clifford (for linear representations) and to Mackey (for projective representations), based exclusively on the fact that G is a normal subgroup of $G.A$: it states that an irreducible representation of $G.A$ will under restriction to G decompose into the direct sum of a certain number, say r, of copies of a representation of G which in turn is the direct sum of a certain number, say s, of irreducible representations of G that are mutually inequivalent but conjugate (under an outer automorphism belonging to A) [24, p. 268]; in particular, all of these have the same dimension d, implying that the dimension of the original representation of $G.A$ is rsd. Conversely, this means that the irreducible representations of $G.A$ are obtained by fusing a certain number s of mutually inequivalent but conjugate irreducible representations of G, all of the same dimension d, into a single representation of G of dimension sd which, when repeated with a certain multiplicity r, can finally be extended to an irreducible representation of $G.A$ of dimension rsd. Of particular interest is the case $r = 1$, which can be divided into two subcases:

- $r = 1$ and $s = 1$: This means that when restricted to G, the given irreducible representation of $G.A$ remains irreducible, or conversely, that the given irreducible representation of G can be extended to an irreducible representation of $G.A$. This extension is not unique, but the various inequivalent extensions can be classified, namely by the group $\mathrm{Hom}(A, \mathbb{C}^*)$ of homomorphisms of A into $\mathbb{C}^*$ [24, p. 295]. In the ATLAS, this situation is referred to as the "split case", in the sense that the extension splits the given representation of G into several inequivalent representations of $G.A$.
- $r = 1$ and $s > 1$: This means that when restricted to G, the given irreducible representation of $G.A$ splits into s mutually inequivalent but conjugate irreducible representations, or conversely, that s mutually inequivalent but conjugate irreducible representations of G fuse into a single irreducible representation of $G.A$. In the ATLAS, this situation is referred to as the "fusion case".

Moreover, there are theorems that impose restrictions on the possible values of r, s and d, depending on the structure of A. One of these is the theorem of Conlon [24, p. 276] which states that if A is cyclic and $s = 1$, then $r = 1$ as well, so we are back to the split case. For simplicity, we shall in what follows refer to the case $r > 1$ and $s > 1$ as the "generalized fusion case".

For our investigation, the split case is of less interest than the fusion case since irreducible representations of $G.A$ that stay irreducible under restriction to G may already be detected among the irreducible representations of G of the same dimension and are obtained from these by extension; moreover, the classification of all possible extensions is a simple exercise: given one of them, any other one is obtained by twisting with the corresponding homomorphism of A into $\mathbb{C}^*$. Considering the fusion case and generalized fusion case, we observe first of all that if several irreducible representations of G (equivalent or not) fuse in an extension $G.A$ of G by some group A of outer automorphisms, then they must already fuse, at least partially, in some extension $G.\mathbb{Z}_n$ of G by some cyclic subgroup $\mathbb{Z}_n$ of A, and this kind of information can be read from the tables in the ATLAS.

Classification of the finite simple groups

The task of implementing the program outlined in Section 2 within the class of finite simple groups and their satellites is feasible due to the existence of a classification of simple finite groups – one of the great achievements of mathematics in the 20th century. These groups can be divided into four types: the cyclic groups $\mathbb{Z}_p$ (of prime order p), the alternating groups A_n (for $n \geqslant 5$), the 16 infinite families of simple groups of Lie type and the 26 sporadic groups.

The cyclic groups are the only finite simple groups that are abelian and so they are the fundamental building blocks of the finite solvable groups. For each prime number p the cyclic group $\mathbb{Z}_p$ has exactly p irreducible representations, all of them one-dimensional. The Schur multiplier of $\mathbb{Z}_p$ is trivial and its outer automorphism group is cyclic of order $p - 1$.

The alternating groups A_n (for $n \geqslant 5$) are the best-known examples of non-abelian finite simple groups. The Schur multiplier of A_n is

$$M(A_n) = \begin{cases} \mathbb{Z}_2 & \text{if } n \neq 6, 7 \\ \mathbb{Z}_6 & \text{if } n = 6, 7. \end{cases} \tag{9}$$

The double covering group of A_n is denoted by $2.A_n$. In the cases $n = 6$ and $n = 7$, there are other covering groups, with centers of order 3 and order 6, denoted by $3.A_n$ and $6.A_n$, respectively. The outer automorphism group of A_n is

$$\mathrm{Out}(A_n) = \begin{cases} \mathbb{Z}_2 & \text{if } n \neq 6 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } n = 6. \end{cases} \tag{10}$$

In particular, for $n \neq 6$, $\mathrm{Aut}(A_n) = \mathrm{S}_n = A_n.\mathbb{Z}_2$, whereas for $n = 6$, there are (up to isomorphism) two extensions by outer automorphisms, namely $\mathrm{Aut}(A_n) = A_6.(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $A_n.\mathbb{Z}_2 = \mathrm{S}_n$.

The finite simple groups of Lie type bear this name because they are constructed as groups of automorphisms of simple Lie algebras over finite fields; they are also widely known as *finite Chevalley groups*. Their definition is based on the fact that every simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has a *Chevalley basis* in which all structure constants are integers, so any such Lie algebra admits a so-called $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$ and hence, for any field $\mathbb{F}$, a sibling $\mathfrak{g}_{\mathbb{F}}$ – a simple Lie algebra over $\mathbb{F}$ that, with respect to the Chevalley basis, has the same integer structure constants as the original simple Lie algebra over $\mathbb{C}$. This fact explains why the finite simple groups of Lie type are classified in terms of the Cartan labels $A_n (n \geqslant 1)$, $B_n (n \geqslant 2)$, $C_n (n \geqslant 3)$, $D_n (n \geqslant 4)$ and E_6, E_7, E_8, F_4, G_2 for the classical and exceptional simple Lie algebras over $\mathbb{C}$, respectively, together with a prime power $q = p^f$ to characterize the finite Galois field $\mathbb{F}_q$ used in their definition. The restrictions on the values of n mentioned above are the standard ones, imposed to exclude Lie algebras that are not simple and to avoid repetitions, since D_1 is abelian, D_2 is not simple, $D_3 \cong A_3$, $C_2 \cong B_2$ and $C_1 \cong B_1 \cong A_1$. Among them are six families of *classical finite groups* that can be realized as matrix groups with matrix entries from the field $\mathbb{F}_q$:

$$\begin{aligned} &A_n(q) \cong PSL_{n+1}(q) \quad (n \geqslant 1), \\ &B_n(q) \cong PSO_{2n+1}(q)' \quad (n \geqslant 2), \\ &C_n(q) \cong PSp_{2n}(q) \quad (n \geqslant 3), \\ &D_n(q) \cong PSO^+_{2n}(q)' \quad (n \geqslant 4), \\ &{}^2A_n(q) \cong PSU_{n+1}(q) \quad (n \geqslant 2), \\ &{}^2D_n(q) \cong PSO^-_{2n}(q)' \quad (n \geqslant 4). \end{aligned} \tag{11}$$

Here, as usual, the prime denotes the derived subgroup. The remaining ones can be arranged into ten series of *exceptional finite groups*:

$$\begin{aligned} &E_6(q), E_7(q), E_8(q), F_4(q), G_2(q), {}^3D_4(q), {}^2E_6(q), \\ &{}^2B_2(q)(q = 2^{2l+1}), {}^2F_4(q)(q = 2^{2l+1}), {}^2G_2(q)(q = 3^{2l+1}). \end{aligned} \tag{12}$$

Further restrictions must be imposed on the range of q in order to exclude groups that are not simple and to avoid repetitions; these are the following:

(1) $A_1(q)$ *with* $q \geqslant 7$ *and* $q \neq 9$: we discard $A_1(2) \cong S_3$ and $A_1(3) \cong A_4$, which are solvable, as well as $A_1(4) \cong A_5, A_1(5) \cong A_5$ and $A_1(9) \cong A_6$, which already occur among the alternating groups.
(2) $A_n(q)$ *with* $n = 2$ *or* $n = 3$ *and* $q \geqslant 3$: we discard $A_2(2) \cong A_1(7)$, which already appears elsewhere in the classification, as well as $A_3(2) \cong A_8$, which already occurs among the alternating groups.
(3) $B_2(q)$ *with* $q \geqslant 3$: we discard $B_2(2) \cong S_6$, which is not simple, and whose derived subgroup $B_2(2)' \cong A_6$ (of index 2) already occurs among the alternating groups.
(4) $C_n(q)$ *with* q *odd*: we discard $C_n(q)$ if q is even, and hence a power of 2, since in this case $C_n(q) \cong B_n(q)$ already appears elsewhere in the classification.
(5) $G_2(q)$ *with* $q \geqslant 3$: we discard $G_2(2)$, which is not simple, and whose derived subgroup $G_2(2)' \cong {}^2A_2(3)$ (of index 2) already appears elsewhere in the classification.
(6) ${}^2A_n(q)$ *with* $n = 2$ *or* $n = 3$ *and* $q \geqslant 3$: we discard ${}^2A_2(2)$, which is not simple, as well as ${}^2A_3(2) \cong B_2(3)$, which already appears elsewhere in the classification.
(7) ${}^2B_2(q)$ *with* $q \geqslant 8$: we discard ${}^2B_2(2)$, which is not simple.
(8) ${}^2F_4(q)$ *with* $q \geqslant 8$: we discard ${}^2F_4(2)$, which is not simple, but retain its derived subgroup ${}^2F_4(2)'$ (of index 2), which is a simple group known as the *Tits group T* that does not appear anywhere else in the classification.
(9) ${}^2G_2(q)$ *with* $q \geqslant 27$: we discard ${}^2G_2(3)$, which is not simple, and whose derived subgroup ${}^2G_2(3)' \cong A_1(8)$ (of index 3) already appears elsewhere in the classification.

4. Determination of codon representations

In order to determine the codon representations not only of finite simple groups G but also of their satellites, we must (a) consider projective representations as well as linear ones and (b) also look for (linear or projective) irreducible representations of G that have dimension < 64 but are capable of fusing into a (linear or projective) codon representation of an appropriate extension. Fortunately, this last possibility is strongly restricted by the fact that 64 is a power of 2 and hence the three numbers r, s and d introduced in the previous section must all be powers of 2 as well. More specifically, in the fusion case or generalized fusion case, $s \geqslant 2$, so the group G must admit at least two inequivalent but conjugate representations of dimension d that fuse in the extension $G.\mathbb{Z}_n$ of G by some outer automorphism of G of even order n, with d assuming one of the values 2, 4, 8, 16 or 32.

The cyclic groups can be discarded immediately because they are abelian and hence all their irreducible representations are one-dimensional. For the remaining cases, we use a series of general results on dimensions of irreducible representations that can be found in the literature, together with the character tables of the ATLAS or the GAP library.

The next easiest case is that of the sporadic groups, whose character tables are completely listed in the ATLAS. The result is that only one sporadic group qualifies, namely the second Janko group J_2: it has two pseudo-real projective codon representations which under extension by the full outer automorphism group $\mathbb{Z}_2$ of J_2 fuse into one irreducible pseudo-real projective representation of $J_2.\mathbb{Z}_2$ of dimension 128.

The other finite simple group types are characterized by the fact that they form infinite families, parametrized either by one natural number n, as in the case of the alternating groups, or by one natural number q or two natural numbers n and q, with the restriction that q has to be a prime power, as in the 16 families of finite simple groups of Lie type, also known as the (untwisted or twisted) finite Chevalley groups. The strategy here is the same as for the classical series of simple Lie algebras or of basic classical Lie superalgebras: one recognizes that the "lowest" possible dimension d_1 for an irreducible representation grows with n and with q; a similar statement holds for the "second-lowest" dimension d_2, the "third-lowest" dimension d_3, etc. Note that there may very well exist several inequivalent representations of dimension $d_1, d_2, d_3, \ldots$; when this is the case, their numbers will be denoted by $N_1, N_2, N_3 \ldots$. More concretely, various authors have given exact formulas or at least lower bounds for d_1, d_2, d_3 …, as functions of n and of q (where applicable), which allow us to impose upper bounds on n and on q (where applicable) in order for the relevant finite simple group or one of its satellites to have any (non-trivial) irreducible representation of dimension ≤ 64 at all. With these cutoffs, the remaining cases can be analyzed

Table 1
Number N_l of linear and N_p of projective codon representations of finite simple groups and their satellites: alternating and symmetric groups (2^* denotes a complex conjugate pair).

G	$\|G\|$	N_l	N_p
A_8	20.160	1	1
A_{10}	1.814.400	0	2
A_{14}	43.589.145.600	0	1
A_{15}	653.837.184.000	0	2^*
A_{65}	65!/2	1	0
S_8	40.320	2	2
S_{13}	6.227.020.800	0	1
S_{14}	87.178.291.200	0	2^*
S_{65}	65!	2	0

Table 2
Number N_l of linear and N_p of projective codon representations of finite simple groups and their satellites: Chevalley groups and sporadic groups ($k \times 2^*$ denotes k complex conjugate pairs).

<table>
<tr><th>G</th><th>$|G|$</th><th>$M(G)$</th><th>$\mathrm{Out}(G)$</th><th>N_l</th><th>N_p</th></tr>
<tr><td>$A_2(4) = PSL_3(4)$</td><td>20.160</td><td>$\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$</td><td>$D_{12}$</td><td>1</td><td>$3+6\times 2^*$</td></tr>
<tr><td>$B_2(3) = PO_5(3)$</td><td>25.920</td><td>$\mathbb{Z}_2$</td><td>$\mathbb{Z}_2$</td><td>1</td><td>1</td></tr>
<tr><td>${}^2B_2(8) = Sz(8)$</td><td>29.120</td><td>$\mathbb{Z}_2 \times \mathbb{Z}_2$</td><td>$\mathbb{Z}_3$</td><td>1</td><td>3</td></tr>
<tr><td>${}^2A_2(4) = PSU_3(4)$</td><td>62.400</td><td>$\{1\}$</td><td>$\mathbb{Z}_4$</td><td colspan="2">1</td></tr>
<tr><td>$A_1(64) = PSL_2(64)$</td><td>262.080</td><td>$\{1\}$</td><td>$\mathbb{Z}_6$</td><td colspan="2">1</td></tr>
<tr><td>$A_1(127) = PSL_2(127)$</td><td>1.024.128</td><td>$\mathbb{Z}_2$</td><td>$\mathbb{Z}_2$</td><td>0</td><td>2^*</td></tr>
<tr><td>$C_3(2) = PSp_6(2)$</td><td>1.451.520</td><td>$\mathbb{Z}_2$</td><td>$\{1\}$</td><td>0</td><td>2^*</td></tr>
<tr><td>$G_2(3)$</td><td>4.245.696</td><td>$\mathbb{Z}_3$</td><td>$\mathbb{Z}_2$</td><td>2^*</td><td>0</td></tr>
<tr><td>J_2</td><td>604.800</td><td>$\mathbb{Z}_2$</td><td>$\mathbb{Z}_2$</td><td>0</td><td>2</td></tr>
<tr><th>$G.A$</th><th>$|G.A|$</th><th colspan="2"></th><th>N_l</th><th>N_p</th></tr>
<tr><td>$G_2(2) = {}^2A_2(3).\mathbb{Z}_2$</td><td>12.096</td><td colspan="2"></td><td colspan="2">1</td></tr>
<tr><td>$A_2(4).\mathbb{Z}_{2_1}$</td><td>40.320</td><td colspan="2">$(\mathbb{Z}_2)_1 = Z(\mathrm{D}_{12})$</td><td>2</td><td>6</td></tr>
<tr><td>$A_2(4).\mathbb{Z}_{2_2}$</td><td>40.320</td><td colspan="2">$(\mathbb{Z}_2)_2 \neq Z(\mathrm{D}_{12})$</td><td>6</td><td>$6+12\times 2^*$</td></tr>
<tr><td>$A_2(4).\mathbb{Z}_{2_3}$</td><td>40.320</td><td colspan="2">$(\mathbb{Z}_2)_3 \neq Z(\mathrm{D}_{12})$</td><td>6</td><td>$6+24^*$</td></tr>
<tr><td>$A_2(4).\mathbb{Z}_3$</td><td>60.480</td><td colspan="2"></td><td>$1+2^*$</td><td>0</td></tr>
<tr><td>$A_2(4).\mathbb{Z}_6$</td><td>120.960</td><td colspan="2"></td><td>$2+2\times 2^*$</td><td>0</td></tr>
<tr><td>$B_2(3).\mathbb{Z}_2$</td><td>51.840</td><td colspan="2"></td><td>2</td><td>2^*</td></tr>
<tr><td>${}^2B_2(8).\mathbb{Z}_3$</td><td>87.360</td><td colspan="2"></td><td>$1+2^*$</td><td>0</td></tr>
<tr><td>${}^2A_2(4).\mathbb{Z}_2$</td><td>124.800</td><td colspan="2"></td><td colspan="2">2</td></tr>
<tr><td>${}^2A_2(4).\mathbb{Z}_4$</td><td>249.600</td><td colspan="2"></td><td colspan="2">$2+2^*$</td></tr>
<tr><td>$A_1(64).\mathbb{Z}_2$</td><td>524.160</td><td colspan="2"></td><td colspan="2">2</td></tr>
<tr><td>$A_1(64).\mathbb{Z}_3$</td><td>786.240</td><td colspan="2"></td><td colspan="2">$1+2^*$</td></tr>
<tr><td>$A_1(64).\mathbb{Z}_6$</td><td>1.572.480</td><td colspan="2"></td><td colspan="2">$2+2\times 2^*$</td></tr>
<tr><td>$G_2(3).\mathbb{Z}_2$</td><td>8.491.392</td><td colspan="2"></td><td>2^*+2^*</td><td>0</td></tr>
</table>

explicitly with the help of the character tables of the ATLAS or the GAP library, leading to the list of codon representations shown in Tables 1 and 2 as the final result, where the star indicates pairs of complex conjugate representations.

Alternating groups

The representation theory of the alternating groups A_n and the symmetric groups S_n is presented in many textbooks, so we shall restrict ourselves to briefly commenting on a few aspects that are relevant for our purposes. First of all, we observe that, according to the character tables of the ATLAS, the first three simple alternating groups A_5, A_6 and A_7 do not admit any codon representations, and the same holds for their extensions by outer automorphisms. Therefore, we may without loss of generality assume that $n \geqslant 8$; this guarantees that both the Schur multiplier and the outer automorphism group of A_n are equal to $\mathbb{Z}_2$:

$$M(A_n) = \mathbb{Z}_2 \quad \text{and} \quad \mathrm{Out}(A_n) = \mathbb{Z}_2 \quad (n \geqslant 8). \tag{13}$$

In particular, $S_n = A_n.\mathbb{Z}_2$ is the maximal extension of A_n by outer automorphisms. As we have seen before, the irreducible representations of A_n and of S_n are then related in one of two possible ways:

- The *split case* is that of an irreducible representation of A_n which extends to an irreducible representation of S_n (it then does so in precisely two inequivalent ways), or conversely, of an irreducible representation of S_n which under restriction to A_n remains irreducible. The relation is 1: 2 (one irreducible representation of A_n splitting into two of S_n under extension).

- The *fusion case* is that of two irreducible representations of A_n which fuse to give a single irreducible representation of S_n, or conversely, of an irreducible representation of S_n which under restriction to A_n splits into two irreducible representations of A_n. Obviously, the relation here is 2: 1 (two irreducible representations of A_n fusing into one of S_n under extension).

Exactly the same situation holds not only for linear representations but also for projective ones, which can be lifted to linear representations of the double covering groups $\mathbb{Z}_2.A_n$ and $\mathbb{Z}_2^{\pm}.S_n$ (recall that the latter comes in two isoclinic variants); this happens because $\mathbb{Z}_2.A_n$ turns out to be isomorphic to the derived subgroup of $\mathbb{Z}_2^{\pm}.S_n$, just as A_n is the derived subgroup of S_n [24,25].

In order to exclude the existence of codon representations of A_n or S_n from a certain value of n onwards, it is convenient to distinguish between linear and properly projective representations.

Starting with the linear ones, we use a theorem that can be found in Ref. [26], according to which the three lowest dimensions of irreducible linear representations of S_n are given by

$$d_1(S_n) = n - 1, \qquad d_2(S_n) = \frac{1}{2}\, n(n-3), \qquad d_3(S_n) = \frac{1}{2}\,(n-1)(n-2), \tag{14}$$

provided that $n \geqslant 14$. The only number among these that can take the value 64 or 128 is $d_1(S_n)$, and since the irreducible representation of S_{129} of dimension 128 remains irreducible when restricted to A_{129}, we conclude that there is no linear codon representation of A_n or S_n when $n \geqslant 14$ except for $n = 65$: this is the case where the irreducible linear representation of lowest possible dimension provides a real codon representation of A_{65} which upon extension by its unique outer involution splits into two real codon representations of S_{65}. Of course, this is a gigantic group, as can be seen by comparing its order

$$2^{62} \cdot 3^{30} \cdot 5^{15} \cdot 7^{10} \cdot 11^{5} \cdot 13^{5} \cdot 17^{3} \cdot 19^{3} \cdot 23^{2} \cdot 29^{2} \cdot 31^{2} \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61,$$

which is a number of order $\sim 10^{93}$, with the order of the monster group, the largest of the sporadic groups,

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71,$$

which is "only" a number of order $\sim 10^{55}$.

Turning to the projective representations, we use a theorem that can be found in Ref. [27], according to which the dimension of any properly projective representation of S_n and of A_n is divisible by a certain power of 2, namely by $2^{[(n-s)/2]}$ and by $2^{[(n-s-1)/2]}$, respectively, where the square brackets denote taking the integral part and s is the number of terms in the decomposition of n into powers of 2 ($n = 2^{w_1} + \cdots + 2^{w_s}$), or in other words, the number of digits that, in the binary representation of n, are equal to 1. Now it is easily seen that $n - s$ is a monotonically non-decreasing function of n, and this implies that there is no projective codon representation of A_n or of S_n when $n \geqslant 16$.

With these cutoffs, it is possible to read off the remaining information from the character tables of the ATLAS or, in the cases $n = 14$ and $n = 15$, from general theorems on the basic representations of the alternating groups [25, Theorems 6.8 and 6.9, pp. 70–73], to obtain the results presented in Table 1. It coincides essentially with the list presented by the authors of Ref. [28], except for the projective representations of A_8, S_8 and A_{10} which are neither ordinary nor spin representations and, probably for this reason, do not appear in Ref. [28].

Finite simple groups of Lie type

For the six families of classical finite groups G, exact formulas for the lowest dimension $d_1(G)$ and for the number $N_1(G)$ of irreducible representations (linear as well as projective) of dimension $d_1(G)$ have been given in Ref. [29]. For the ten series of exceptional finite groups G, the literature contains lower bounds $b(G)$ for $d_1(G)$ which were originally derived in Ref. [30] and later improved in Ref. [31].

In what follows, we shall first of all apply the formulas from Refs [29–31] to determine the groups G for which $d_1(G) \leqslant 64$, since it is clear that when $d_1(G) > 64$, neither G itself nor any of its extensions by outer automorphisms has a codon representation. Given the fact that $d_1(G)$ is a polynomial in q whose exponents are affine functions of n, an inequality such as $d_1(G) \leqslant N$, where N is any given number, imposes upper bounds on q and n, so the set of candidate groups is finite. An even sharper requirement is that G should have irreducible representations of dimension 2^k where k takes one of the values 1, 2, 3, 4, 5, 6, since as shown in the previous section, this is a necessary condition for G to admit some extension by outer automorphisms that has a codon representation. As we shall see, this greatly reduces the number of cases that must be analyzed in detail. In fact, it turns out that the question can to a large extent be settled by consulting the character tables of the ATLAS, while in the few cases not covered by the ATLAS, the necessary information can be computed using Ref. [29] which, for the groups of interest, gives the lowest three dimensions for irreducible representations (rather than just the lowest one). These numbers are derived either directly (cf. Tables IV and V) or indirectly from the statement (cf. the beginning of Section 5 and Theorem 5.2) that for $n \geqslant 2$ and q odd, the first five irreducible representations of $C_n(q) = PSp_{2n}(q)$ have dimension $(q^n-1)/2$ (two representations), $(q^n+1)/2$ (two representations) and $q(q^{n-1}-1)(q^n-1)/2(q+1)$ (one representation), while all others must have dimension $\geqslant (q^{2n}-1)/2(q+1)$.

With these general remarks out of the way, we proceed to the analysis of the individual cases. We start by investigating which of the Chevalley groups admit codon representations or, more generally, irreducible representations of dimension 2^k

where k takes one of the values 1, 2, 3, 4, 5, 6. In a second step, we analyze the fate of these representations under extension by outer automorphisms. The results are collected in Table 2.

The remainder of this section will be devoted to presenting the relevant arguments in more detail.

- $A_1(q)$ ($q \geqslant 7$, $q \neq 9$): all the information needed can be extracted from the generic character tables for $A_1(q)$ found, e.g., in Ref. [32], according to which the dimensions of the irreducible representations of $A_1(q)$ are $q-1, q, q+1$ when q is even (i.e., a power of 2) and are $(q-1)/2$, $(q+1)/2$, $q-1, q, q+1$ when q is odd (i.e., a power of some other prime). Thus we have $d_1(A_1(q)) \leqslant 64$ if and only if $q \leqslant 64$ when q is even and $q \leqslant 129$ when q is odd. Moreover, there are only two values of q for which $A_1(q)$ admits codon representations, namely $q = 64$ when q is even and $q = 127$ when q is odd: in the first case, we find one linear codon representation which is real, whereas in the second case, we find two projective codon representations forming a complex conjugate pair. For lower values of q, we still have to consider the possibility of obtaining a codon representation of some extension of $A_1(q)$ by outer automorphisms fusing a certain number of irreducible representations of $A_1(q)$ of dimension $r < 64$ where, as before, r must be one of the numbers 2, 4, 8, 16 or 32. When q is even, this rules out the possibilities of having r equal to $q-1$ or $q+1$, so r must be equal to q. Taking into account that when q is even and $q = 2^f$, the outer automorphism group of $A_1(q)$ is $\mathbb{Z}_f$ which is cyclic, we conclude from Conlon's theorem that the only option would be to fuse several (at most f) inequivalent irreducible representations of $A_1(q)$ of dimension q, which is impossible since there is just one of them. When q is odd, r may be equal to $(q-1)/2$ or $(q+1)/2$ or $q-1$ or $q+1$, and given the fact that q should be a prime power p^f such that $q \geqslant 7$ and $q \neq 9$, there are precisely three solutions, namely $q = 7$, $q = 17$, $q = 31$. Taking into account that when q is odd, the outer automorphism group of $A_1(q)$ is $\mathbb{Z}_2 \times \mathbb{Z}_f$ and that the three solutions just mentioned are prime numbers ($f = 1$) and so this group is just $\mathbb{Z}_2$ which is cyclic, we conclude from Conlon's theorem that the only option would be to fuse two inequivalent irreducible representations of $A_1(q)$ of dimension 32. But $A_1(7)$ and $A_1(17)$ have no irreducible representations of this dimension, whereas $A_1(31)$ has a lot of them but they are all split under the extension to $A_1(31).\mathbb{Z}_2$.
- $A_n(q)$ ($n \geq 2$):
  (1) $n = 2$, $q \geqslant 3$: according to Table II of Ref. [29], we have $d_1(A_2(q)) \leqslant 64$ if and only if $q \leqslant 7$. According to the ATLAS, $A_2(7)$ and $A_2(5)$ have no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, $A_2(4)$ has one linear codon representation which is real and five projective codon representations, of which one is real while the other four form two complex conjugate pairs, apart from also having two irreducible projective representations of dimension 8 which under extension by any cyclic subgroup of its full outer automorphism group D_{12} either split or fuse into one irreducible projective representation of dimension 16, and finally $A_2(3)$ has four irreducible representations of dimension 16 which under extension by its full outer automorphism group $\mathbb{Z}_2$ fuse into two irreducible representations of dimension 32.
  (2) $n = 3$, $q \geqslant 3$: according to Table II of Ref. [29], we have $d_1(A_3(q)) \leqslant 64$ if and only if $q = 3$, and according to the ATLAS, $A_3(3)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
  (3) $n = 4$: according to Table II of Ref. [29], we have $d_1(A_4(q)) \leqslant 64$ if and only if $q = 2$, and according to the ATLAS, $A_4(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
  (4) $n = 5$: according to Table II of Ref. [29], we have $d_1(A_5(q)) \leqslant 64$ if and only if $q = 2$, and $A_5(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
  (5) $n \geqslant 6$: according to Table II of Ref. [29], we have $d_1(A_n(q)) > 64$ when $n \geqslant 6$, for all possible values of q.
- $B_2(q)$ ($q \geqslant 3$): we have $d_1(B_2(q)) \leqslant 64$ if and only if $q = 4$ when q is even or $q \leqslant 11$ when q is odd. Moreover, $B_2(11)$, $B_2(9)$ and $B_2(7)$ have no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, and according to the ATLAS, the same is true for $B_2(5)$ and $B_2(4)$, while $B_2(3)$ has one linear and one projective codon representation, apart from two irreducible projective representations of dimension 4 which under extension by its full outer automorphism group $\mathbb{Z}_2$ fuse into a single irreducible projective representation of dimension 8.
- $B_n(q)$ ($n \geqslant 3$, q even):
  (1) $n = 3$: we have $d_1(B_3(q)) \leqslant 64$ if and only if $q = 2$, and according to the ATLAS, $B_3(2)$ has two projective codon representations, apart from a single irreducible projective representation of dimension 8.
  (2) $n = 4$: we have $d_1(B_4(q)) \leqslant 64$ if and only if $q = 2$, and according to the ATLAS, $B_4(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
  (3) $n \geqslant 5$: we have $d_1(B_n(q)) > 64$ when $n \geqslant 5$, for all possible even values of q.
- $B_n(q)$ ($n \geqslant 3$, q odd):
  (1) $n = 3$: we have $d_1(B_3(q)) \leqslant 64$ if and only if $q = 3$, and according to the ATLAS, $B_3(3)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
  (2) $n \geqslant 4$: we have $d_1(B_n(q)) > 64$ when $n \geqslant 4$, for all possible odd values of q.
- $C_n(q)$ ($n \geqslant 3$, q odd):
  (1) $n = 3$: we have $d_1(C_3(q)) \leqslant 64$ if and only if $q \leqslant 5$. Moreover, $C_3(5)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, and according to the ATLAS, the same is true for $C_3(3)$.
  (2) $n = 4$: we have $d_1(C_4(q)) \leqslant 64$ if and only if $q = 3$, and $C_4(3)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
  (3) $n \geqslant 5$: we have $d_1(C_n(q)) > 64$ when $n \geqslant 5$, for all possible odd values of q.

- $D_n(q)$ $(n \geqslant 4)$:
  (1) $n = 4$: we have $d_1(D_4(q)) \leqslant 64$ if and only if $q = 2$, and according to the ATLAS, $D_4(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, except for a single irreducible projective representation of dimension 8.
  (2) $n \geqslant 5$: we have $d_1(D_n(q)) > 64$ when $n \geqslant 5$, for all possible values of q.
- ${}^2A_n(q)$ $(n \geqslant 2)$:
  (1) $n = 2, q \geqslant 3$: we have $d_1({}^2A_2(q)) \leqslant 64$ if and only if $q \leqslant 8$. According to the ATLAS, ${}^2A_2(8)$, ${}^2A_2(7)$ and ${}^2A_2(5)$ have no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, ${}^2A_2(4)$ has one codon representation, and finally ${}^2A_2(3)$ has two irreducible representations of dimension 32 which under the extension by its full outer automorphism group $\mathbb{Z}_2$ fuse into a single codon representation of ${}^2A_2(3).\mathbb{Z}_2$.
  (2) $n = 3, q \geqslant 3$: we have $d_1({}^2A_3(q)) \leqslant 64$ if and only if $q \leqslant 4$. Moreover, ${}^2A_3(4)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, and according to the ATLAS, the same is true for ${}^2A_3(3)$.
  (3) $n = 4$: we have $d_1({}^2A_4(q)) \leqslant 64$ if and only if $q \leqslant 3$. Moreover, ${}^2A_4(3)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, and according to the ATLAS, the same is true for ${}^2A_4(2)$.
  (4) $n = 5$: we have $d_1({}^2A_5(q)) \leqslant 64$ if and only if $q = 2$, and according to the ATLAS, ${}^2A_5(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
  (5) $n = 6$: we have $d_1({}^2A_6(q)) \leqslant 64$ if and only if $q = 2$, and ${}^2A_6(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
  (6) $n \geqslant 7$: we have $d_1({}^2A_n(q)) > 64$ when $n \geqslant 7$, for all possible values of q.
- ${}^2D_n(q)$ $(n \geqslant 4)$:
  (1) $n = 4$: we have $d_1({}^2D_4(q)) \leqslant 64$ if and only if $q = 2$, and according to the ATLAS, ${}^2D_4(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
  (2) $n \geqslant 5$: we have $d_1({}^2D_n(q)) > 64$ when $n \geqslant 5$, for all possible values of q.
- $E_n(q)$ $(n = 6, 7, 8)$, ${}^2E_6(q)$: these groups have no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
- $F_4(q)$: we have $d_1(F_4(q)) > 64$ when $q \geqslant 3$, and according to the ATLAS, $F_4(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
- $G_2(q)$ $(q \geqslant 3)$: we have $d_1(G_2(q)) > 64$ when $q \geqslant 5$, and according to the ATLAS, $G_2(4)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$, while $G_2(3)$ has two linear codon representations.
- ${}^3D_4(q)$: we have $d_1({}^3D_4(q)) > 64$ when $q \geqslant 3$, and according to the ATLAS, ${}^3D_4(2)$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
- ${}^2B_2(q)$ $(q = 2^{2l+1}$ with $l \geqslant 1)$: we have $d_1({}^2B_2(q)) > 64$ when $q \geqslant 32$, and according to the ATLAS, ${}^2B_2(8)$ has one linear and one projective codon representation.
- ${}^2F_4(q)$ $(q = 2^{2l+1}$ with $l \geqslant 1)$: we have $d_1({}^2F_4(q)) > 64$ when $q \geqslant 8$. Moreover, according to the ATLAS, $F_4(2)$ the Tits group ${}^2F_4(2)'$ has no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.
- ${}^2G_2(q)$ $(q = 3^{2l+1}$ with $l \geqslant 1)$: these groups have no irreducible representations of dimension 2^k with $1 \leqslant k \leqslant 6$.

5. Computation of the branching schemes

A well-known classical result in group theory, called *Cayley's theorem*, states that any finite group is isomorphic to a permutation group and is proved by explicitly constructing the isomorphism into the symmetric group on n symbols, where n is the order of the finite group. Before the advent of the digital computer this theorem was regarded as a purely theoretical result without any practical implications, since it is extremely laborious to do calculations by hand on large symmetric groups. Attempts to prove the existence of some the largest sporadic groups by realizing them as permutations groups provided the motivation to develop efficient computational methods for handling large symmetric groups. Those attempts were very successful and gave birth to the *computational group theory*, by combining computer science and group theory. It is no surprise that today we have highly efficient computer programs to do all sorts of calculations on permutations groups; in fact, this is the most advanced part of computational group theory. In this second part we shall explain some parts of this theory that were necessary to implement our program.

Permutation representations and maximal subgroups

In order to proceed with the second part of our program, namely, to obtain the branching schemes, our first task is to realize all the groups obtained in the last part as permutation groups in a form suitable for performing computer calculations with. On the other hand, the vast majority of the (satellites of the) finite simple groups are naturally defined as *matrix groups*. Only the symmetric and alternating groups are naturally defined as permutation groups – even their covering groups are matrix groups obtained by lifting the natural orthogonal representation over a finite field to the spinor group. Therefore, we must be able to convert a (satellite group of a) finite simple matrix group into a permutation group.

Let us start by explaining how a computer stores a finite group. A naive way to do this would be to simply to store all elements – permutations on n symbols or invertible matrices over a finite fields – of the group. But when the group under consideration is large it would require a lot of storage space and a high cost of time to read this information into the memory. A more efficient approach is to store a permutation group or a matrix group as a *list of generators* $\{g_1, \ldots, g_s\}$, where each g_i is a permutation on n symbols or an n by n invertible matrix over a finite field. All algorithms work with a generating set without needing to compute all the elements of the group. Two properties of a generating set are important when defining a group: the number of generators and the relation with other generating sets.

A set of *standard generators* of a given group G is a list $(g_1, \ldots, g_s)$ of elements $g_i \in G$ satisfying certain conditions (depending on the isomorphism type of G), such that:

(i) $\langle g_1, \ldots, g_s \rangle = G$,
(ii) the list is unique up to automorphisms of G, i.e., for two lists $(g_1, \ldots, g_s)$ and $(h_1, \ldots, h_s)$ of standard generators, the map $g_i \to h_i$ extends to an automorphism of G.

It is a consequence of the classification theorem that all finite simple groups and their satellites have a set of standard generators with two elements, which were explicitly determined in most cases. The standard generating sets for finite simple groups and their satellites are readily available in most of the computational group theory packages or can be downloaded from the Internet database *Atlas of Finite Group Representations* [33]. For example, GAP has a database containing standard generators of all perfect groups of order up to 10^6, as permutation groups in all cases and matrices groups in some cases. This includes all finite simple groups and all proper covering groups of finite simple groups of order up to 10^6. Permutation representations of the groups $\mathbb{Z}_2.J_2$, $\mathbb{Z}_2.A_1(127)$, $\mathbb{Z}_2.C_3(2)$, $G_2(3)$ can be obtained from the Internet database *Atlas of Finite Group Representations* [33].

We are still left with a few groups that should be converted into permutation groups directly from their matrix standard generators, namely, $\mathbb{Z}_2.A_{10}$, $\mathbb{Z}_2.A_{13}\mathbb{Z}_2.A_{14}$, $\mathbb{Z}_2.A_{15}$ and all the extensions by outer automorphisms. Before explaining how to do this conversion let us recall some terminology.

A *permutational representation of degree n* of a finite group G is a homomorphism $\pi : G \to S_n$. A permutational representation $\pi : G \to S_n$ is called *transitive* if the action of $\pi(G)$ on $\{1, \ldots, n\}$ has only one orbit. There is a straightforward way to construct all the transitive permutational representations of a finite group G: let H be a subgroup of G and $X = G/H$ the set of left cosets of H in G, and consider the homomorphism $\pi : G \to X$ given by $\pi(g)(g'H) = (g\,g')H$. This permutation representation has degree $[G : H] = G/H$ and the stabilizer G_x of a left coset $x \in X$ is a subgroup of G conjugate to H. When H is the trivial group $\{1\}$ the above construction provides the *left regular representation* of G: this is the permutation representation used to prove Cayley's theorem, since it is *faithful*, that is, π is an injective homomorphism, and so G can be identified as its image in S_n.

The performance of algorithms for computation with permutation groups depends on the degree of the representation and on the order of G. A permutation in n symbols is stored as the list of n images of $\{1, \ldots, n\}$. If $n < 65\,536$ then the permutations can be stored as a 16 bit positive integer. A permutation on more than 65 536 points requires 32 bits per point for storing. For example, permutations on 256 000 points require roughly 1 MB of storage per permutation. Therefore, it is desirable to construct faithful transitive permutation representations of degree as small as possible.

The *normal core* of a subgroup H is the largest normal subgroup of G that is contained in H (or equivalently, the intersection of the conjugates of H). A *core free subgroup* is a subgroup whose normal core is the trivial subgroup. Equivalently, it is a subgroup that occurs as the isotropy subgroup of a transitive, faithful group action. Let $p(G)$ denote the smallest degree of a faithful transitive permutation representation of a finite group G; then

$$p(G) = \max\big\{[G : H] \mid H \text{ is a core free subgroup of } G\big\}.$$

In the case where G is a simple group all transitive permutation representations are faithful, or equivalently, all subgroups are core free. Thus, in this case, $p(G)$ is the index of the largest subgroup of G. In the case of a proper covering group of a finite simple group, this prescription does not work. Let $\hat{G}$ be a proper covering group of a finite simple group G. Then every proper normal subgroup of $\hat{G}$ is contained in the center $Z(\hat{G})$ of $\hat{G}$. Therefore, a core free subgroup of $\hat{G}$ must intersect the center of $\hat{G}$ trivially. It is not difficult to prove the following (see [19]): *Every maximal subgroup of a group G lifts as a maximal subgroup of any proper covering of G, and conversely, every maximal subgroup of a proper covering group is obtained in this way*. In particular, every maximal subgroup of $\hat{G}$ contains its center and so it is not core free. Thus, given a proper covering group G of a finite group we may find the largest core free subgroup by testing whether the lift of a non-maximal subgroup H of the simple group $G/Z(G)$ is the direct product $Z(G) \times H$.

Now in order to construct faithful permutation representations of the remaining proper covering groups $\mathbb{Z}_2.A_{10}$, $\mathbb{Z}_2.A_{13}$, $\mathbb{Z}_2.A_{14}$, $\mathbb{Z}_2.A_{15}$ we use the matrix representations of these groups obtained from [33] and construct the covering map onto the corresponding alternating group $\pi : \mathbb{Z}_2.A_n \to A_n$ by mapping the generators of $\mathbb{Z}_2.A_n$ onto the corresponding generator of A_n – indeed, the generators of the groups in [33] are chosen so that they correspond under covering maps. Now one can use this map to lift a subgroup H of A_n to $\mathbb{Z}_2.A_n$ and test whether it splits a direct product. Unfortunately, in order to obtain suitable the candidates H we must compute all subgroups of A_n and this could be done only for $n = 10, 11$ (see the discussion about maximal subgroups). In Table 3 we present some (bounds on) the minimal degrees of faithful transitive permutation representations of the covering groups of some alternating groups. Note that when $n \geqslant 13$ we have that $p(G) > 65\,536$.

Passing to the outer automorphisms extensions, we start by observing that all finite simple groups and proper covering groups presented in Tables 1 and 2 are such that their automorphism groups are semi-direct products, that is,

$$\mathrm{Aut}(G) = \mathrm{Inn}(G) \rtimes \mathrm{Out}(G).$$

This can be verified explicitly using GAP to compute the automorphism groups and check the conditions: there is a subgroup A of $\mathrm{Aut}(G)$ such that $\mathrm{Inn}(G) \cap A = \{1\}$ and $\mathrm{Inn}(G)A = \mathrm{Aut}(G)$. Using this decomposition it is easy to construct faithful

Table 3
Bounds on the minimal degrees of faithful transitive permutation representations of the covering groups of some simple alternating groups.

G	$\lvert G\rvert$	$p(G)$
$\mathbb{Z}_2.A_5$	120	$= 24$
$\mathbb{Z}_2.A_6$	720	$= 80$
$\mathbb{Z}_2.A_7$	5.040	$= 240$
$\mathbb{Z}_2.A_8$	40.320	$= 240$
$\mathbb{Z}_2.A_9$	362.880	$= 240$
$\mathbb{Z}_2.A_{10}$	3.628.800	$= 2.400$
$\mathbb{Z}_2.A_{11}$	19.958.400	$= 5.040$
$\mathbb{Z}_2.A_{12}$	479.001.600	$\leqslant 60.480$
$\mathbb{Z}_2.A_{13}$	6.227.020.800	$\leqslant 786.240$
$\mathbb{Z}_2.A_{14}$	87.178.291.200	$\leqslant 11.007.360$
$\mathbb{Z}_2.A_{15}$	1.307.674.368.000	$\leqslant 165.110.400$

Table 4
Groups with permutation representations which violate the limitation of the algorithm for computing the lattice of subgroups. The third column lists the size of the largest maximal subgroup.

G	$\lvert G\rvert$	$\max\{\lvert H\rvert \,:\, H < G\}$
$\mathbb{Z}_2.J_2$	1.209.600	12.096
$\mathbb{Z}_2.A_1(127)$	2.048.256	16.002
$\mathbb{Z}_2.C_3(2)$	2.903.040	103.680
$\mathbb{Z}_2.A_{10}$	3.628.800	362.880
$G_2(3)$	4.245.696	12.096
$G_2(3).\mathbb{Z}_2$	8.491.392	4.245.696

permutation representations of the extensions by outer automorphism of all quasi-simple groups determined in Section 4.

- If G is a finite simple group given by a transitive permutation representation, we can construct a faithful permutation representation of $G.A = G \rtimes A$ by embedding the stabilizer H group of G into $G.A$ and considering the action of $G.A$ on the set of left cosets $X = G.A/H$.
- If $\hat{G}$ is a covering group of a finite simple group G, given by a transitive permutation representation, we can construct a faithful permutation representation of $\hat{G}.A$ by constructing the semi-direct product $\hat{G} \rtimes A$ using the action of $A \subset \mathrm{Aut}(G)$ by automorphisms on $\hat{G}$ and then proceeding as in the previous item.

Note that if the permutation representation of G has degree k the degree of the permutation representation of $G.A$ obtained above is $n|A|$.

Summarizing the results described above, we have the following. *For all finite simple groups and their satellites that have codon representations we were able to compute a transitive faithful permutation representation with (near) smallest degree, with the exception of three groups*: $\mathbb{Z}_2.A_{13}$, $\mathbb{Z}_2.A_{14}$, $\mathbb{Z}_2.A_{15}$.

The next task is the computation of the subgroups of the groups obtained in the first step. In fact, we need only the set of conjugacy classes of subgroups and the relation of subconjugacy between these classes – this structure is called the *lattice of conjugacy classes of subgroups* of a group. Given two subgroups H and K of a subgroup G we say that H is *subconjugate* to K in G if H is conjugate in G to a subgroup of K. This relation is invariant under conjugacy in G and so it induces a partial order on the set of conjugacy classes of subgroups of G: $[H] \preceq [K]$ if and only if H is subconjugate to K in G.

The computation of the lattice of conjugacy classes of subgroups is by far the hardest part of our endeavour and it will take great advantage of our efforts to obtain permutation representations with degree as small as possible. In GAP the computation of the lattice of conjugacy classes of subgroups is performed by the *method of cyclic extensions* developed by Joachim Neubüser and implemented by Alexander Hulpke. This approach consists in calculating the subgroups "layer by layer" starting at the "bottom" and "going up" at each step. The k-th *layer of subgroups* of G consists of all the subgroups H of G such that the composition series of H has length k. The first layer consists of all cyclic subgroups of G of prime order of G. The second layer is obtained from the members of the first layer by cyclic extension with cyclic groups of prime order and so on. At the end of the process one obtains the set of all solvable subgroups of G. In order to include the non-solvable subgroups one has to insert "by hand" the perfect subgroups of G. In GAP this is done with the help of a library of finite perfect groups and provides, up to isomorphism, a list of all perfect groups whose sizes are less than 10^6, with a few exceptions (see the GAP manual [22]).

Among the finite simple groups that we were able to construct permutation representations of, there are six groups that violate this constraint: they are listed in the first column of Table 4.

In order to overcome this technical limitation, we observe that the largest maximal subgroups of the first five groups are at most ten times smaller, and thus all subgroups of these are within the range of the algorithm. Now we take advantage of the fact that the maximal subgroups of the (satellites) of alternating groups and the Chevalley groups have been classified and, more importantly, in our case their generators can be downloaded from the Internet database *Atlas of Finite Group*

Table 5
Finite simple groups and their satellites of order $\leqslant 10^{10}$ that have codon representations — minimal degree faithful permutation representations (Degree), number of conjugacy classes of subgroups (Subgroups), number of conjugacy classes of non-abelian subgroups (Non-abelian) and number of conjugacy classes of maximal subgroups (Maximal).

G	$\|G\|$	$p(G)$	Subgroups	Non-abelian	Maximal
$G_2(2)$	12.096	63	100	72	5
$\mathbb{Z}_2.A_8$	40.320	240	168	135	6
$\mathbb{Z}_2.A_8.\mathbb{Z}_2$	80.640	480	329	279	7
$\mathbb{Z}_{4_1}.A_2(4)$	80.640	224	279	234	9
$\mathbb{Z}_{4_2}.A_2(4)$	80.640	224	284	233	9
$\mathbb{Z}_2.A_2(4).\mathbb{Z}_{2_1}$	80.640	224	330	257	10
$\mathbb{Z}_{4_1}.A_2(4).\mathbb{Z}_{2_3}$	161.280	224	360	286	6
$\mathbb{Z}_{4_2}.A_2(4).\mathbb{Z}_{2_2}$	161.280	224	609	508	6
$A_2(4).\mathbb{Z}_3$	60.480	42	100	76	5
$A_2(4).\mathbb{Z}_6$	120.960	42	143	109	6
$\mathbb{Z}_2.B_2(3)$	51.840	80	162	120	5
$\mathbb{Z}_2.B_2(3).\mathbb{Z}_2$	103.680	240	492	430	6
$\mathbb{Z}_2.Sz(8)$	58.240	1.040	42	24	4
$Sz(8).\mathbb{Z}_3$	87.360	195	39	25	5
${}^2A_2(4)$	62.400	65	34	20	4
${}^2A_2(4).\mathbb{Z}_2$	124.800	260	80	59	5
${}^2A_2(4).\mathbb{Z}_4$	249.600	260	120	94	5
$A_1(64)$	262.080	65	76	19	5
$A_1(64).\mathbb{Z}_2$	524.160	390	127	72	6
$A_1(64).\mathbb{Z}_3$	786.240	390	102	63	6
$A_1(64).\mathbb{Z}_6$	1.572.480	390	182	134	7
$\mathbb{Z}_2.J_2$	1.209.600	200	244	192	8
$\mathbb{Z}_2.A_1(127)$	2.048.256	256	51	31	5
$\mathbb{Z}_2.C_3(2)$	2.903.040	240	1.685	1.572	8
$2.A_{10}$	3.628.800	2.400	552	491	7
$G_2(3)$	4.245.696	351	433	378	10
$G_2(3).\mathbb{Z}_2$	8.491.392	702	399	342	6

Representations [33]. Thus we can treat the remaining cases by including the maximal subgroups by hand and computing their lattices of subgroups. Since any subgroup is contained in at least one maximal subgroup, we were able to construct all conjugacy classes of subgroups. Once we have eliminated the possible repetitions we end up with the lattice of conjugacy classes of subgroups of these five groups. The last group of Table 4 can also be treated by the same method since in the previous step we have obtained all the subgroups of its largest maximal subgroup.

Unfortunately, this *ad hoc* procedure only works when the group is already given by a permutation representation, because we still have to apply the lattice subgroup algorithm to fairly large groups. Therefore we could not employ this strategy to find maximal core free subgroups and build small degree faithful permutation representations of the three largest alternating groups with codon representations.

Summarizing the results described above, we have the following. *It was possible to compute the lattice of conjugacy classes of subgroups of all finite simple groups and their satellites that we were able to construct a small degree faithful permutation representation of.*

Some intermediate data obtained during these computations are collected in Table 5. We list 27 groups whose minimal degree faithful permutation representation was constructed. For each group G in Table 5 we list its order $|G|$, its minimal degree of faithful permutation representation $p(G)$, the number of conjugacy classes of subgroups, the number of conjugacy classes of non-abelian subgroups (these are the only subgroups with the potential to reproduce the multiplet structure of the genetic code, since the abelian groups have all irreducible representations one-dimensional) and the number of conjugacy of maximal subgroups. It should be noted that we construct only the extensions necessary to lift all the codon representations. For example, we need two covering groups of $A_2(4)$, namely $\mathbb{Z}_{4_1}.A_2(4)$ and $\mathbb{Z}_{4_2}.A_2(4)$, in order to lift all the ten codon representations (one linear and nine projective) listed in the first entry of Table 2. In fact, the most complicated case is $A_2(4)$, which has non-cyclic Schur multiplier and non-abelian (solvable) outer automorphism group. This is the only case where the outer automorphism group is non-cyclic, but there is one more group in our list with non-cyclic Schur multiplier: ${}^2B_2(8)$. However, only in the case of $A_2(4)$ may one have several non-isomorphic maximal covering groups.

Restriction of characters and branching rules

In this last stage, we bring in the codon representations, which are studied through their characters. Let us start by explaining how GAP computes and handles characters and character tables.

The *character table* of a group G is a square array indexed by the conjugacy classes and the irreducible characters of G. A *character* of a group G is a special type of *class function*, that is, a complex function $\chi : G \to \mathbb{C}$ invariant under conjugation, or equivalently, constant on the conjugacy classes of G. A representation ρ of a group G on a complex vector space V affords a character of G by

$$\chi_V(g) = \mathrm{Tr}(\rho(g)).$$

When the representation ρ is irreducible we say the character afforded by ρ is an *irreducible character* of G. The main result about characters is that they completely capture the structure of the representations of the group G. More precisely, if V is a finite dimensional representation of G with decomposition into irreducible components $\{V_1, \ldots, V_k\}$ and multiplicities $\{m_1, \ldots, m_k\}$ then

$$\chi_V = \sum_{i=1}^{k} m_i \chi_{V_i}.$$

Moreover, the number of distinct irreducible characters is equal to the number of conjugacy classes of G.

If the group G is given as a set of permutations that generate the group, the set of conjugacy classes of G can be computed very quickly and the result is a list of representatives of each conjugacy class, together with their *basic conjugacy invariants*: the order of the elements in the class and the number of elements in the class.

The next step is the computation of the irreducible characters. The basic idea is due to Burnside, starting with some observations about the group algebra of G. Let $\mathbb{C}[G]$ be the *group algebra* of the group G. It is a finite dimensional associative algebra over $\mathbb{C}$ isomorphic to a direct product of full matrix algebras. Its center $Z(\mathbb{C}[G])$ has a natural basis given by the *class sums* C_i – the sum of all elements in the conjugacy class $\mathfrak{C}_i$ of G. The *structure constants* with respect to this basis, defined by

$$C_i\,C_j = \sum_k m_{ij}^k C_k,$$

are non-negative integers given by

$$m_{ij}^k = \#\big\{(x, y) \in G \times G \mid x \in \mathfrak{C}_i,\ y \in \mathfrak{C}_j,\ xy = z \in \mathfrak{C}_k,\ \text{for fixed } z\big\}.$$

Now we can consider the structure constants as elements of a matrix $M_i = \left(m_j^k\right)$ associated with the conjugacy class $\mathfrak{C}_i$. Since $Z(\mathbb{C}[G])$ is a commutative algebra, it follows that the matrices commute with each other and thus they can be simultaneously diagonalized. Let λ_i^l be the eigenvalues of the matrix M_i and λ_{i*}^l be the eigenvalues of the matrix M_i^t. Then

$$\chi^l(\mathfrak{C}_i) = \frac{d_l \lambda_i^l}{|\mathfrak{C}_i|}, \quad \text{where } d_l = \sqrt{|G| \Big/ \left(\sum_i \frac{\lambda_i^l}{|\mathfrak{C}_i|} \lambda_{i*}^l\right)}.$$

In order to compute these values exactly, there is the *Dixon–Schneider method*, where the linear algebra computations are done over a finite field $\mathbb{Z}_p$, where p is a prime number satisfying certain conditions. At the end the results are lifted back to $\mathbb{C}$. The implementation of the Dixon–Schneider algorithm has several optimizations rendering it very efficient for groups of size $< 10^9$.

Once we have calculated the character table of G, GAP offers a large arsenal of operations on characters that we can perform. The operations necessary for calculating the branching rules are the restriction of a character to that of a subgroup and the decomposition of a character into its irreducible components. In order to perform the operation of restriction of the character of a group G to one of its subgroups H it is necessary to compute the character tables of G and H and to calculate the *fusion map* from H to G. This map describes the relation between the conjugacy classes of G and H. Assuming that we have fixed orderings of the conjugacy classes of G and H and recalling that each conjugacy class of H is contained in a unique conjugacy class of G, but several conjugacy classes of H can be contained in the same conjugacy class of G one may describe the fusion map as follows: it is a list of the same size as the number of conjugacy classes of G such that its i-th position is the number of the position of the conjugacy class of G (in the order fixed above) that contains the i-th conjugacy class of H (in the order fixed above).

In possession of the fusion map, we can restrict any class function from G to H. If we start with a character of G then its restriction to a subgroup H is a character of G. Using the character table of H we can compute the dimensions of the irreducible components and their multiplicities. This information completely determines the *branching rules* of the corresponding representation when the symmetry is broken from G to H. The branching rules can be arranged in a table of positive numbers where the second column contains the dimension of the irreducible components – the *multiplets* – and the first column contains its multiplicity – the *number of multiplets of a fixed size*. For example, the branching rules of the genetic code are represented by (compare with the list in the beginning of the introduction)

$$\begin{pmatrix} 3 & 6 \\ 5 & 4 \\ 2 & 3 \\ 9 & 2 \\ 2 & 1 \end{pmatrix} \tag{15}$$

Now we can implement the search for symmetries in the evolution of the genetic code. Pick one of the groups of Table 5 and proceed as follows:

(1) Compute the lattice of conjugacy classes of subgroups $\mathcal{L}(G)$ of G (including G).
(2) Compute the character table of a representative subgroup H for every class $[H] \in \mathcal{L}(G)$.
(3) Compute the fusion map of a representative subgroup H for every class $[H] \in \mathcal{L}(G)$.
(4) For every character of a codon representations of G, compute its restriction to a representative subgroup H for every class $[H] \in \mathcal{L}(G)$.
(5) For every restricted character obtained in the previous step, compute the dimensions of the irreducible constituents and their multiplicities, and store this information as a table of branching rules, with an appropriate re-ordering of the rows, if necessary.
(6) Compare with the branching rules of the genetic code equation (15).

In this way we would obtain the groups with codon representations that reproduce the multiplet distribution of the genetic code through a *complete symmetry breaking*, that is, *without freezing in the last step*. In fact, we performed this computation with all the groups from Table 5 and the final result is the following: *None of the groups of* Table 5 *can reproduce the multiplet distribution of the genetic code without freezing in the last step*.

Therefore we should modify the program in order to include the *partial symmetry breaking*, allowing the freezing of some multiplets. Now we consider *pairs of subgroups* (H, K) of G, where K is a maximal subgroup of H. We want to find those pairs of subgroups such that the restriction of a codon representation of G to H and then to K reproduces the distribution of multiplets of the genetic code if we allow some of the multiplets of H to remain unbroken after the restriction to K. In order to accomplish this task we need to understand the branching of the multiplets from H to K. This information cannot be obtained from the rule of branching from G to H and from G to K alone. However, it is possible to couple this information with some simple criteria in such a way as to allow the discarding of a large amount of pairs of subgroups that could not reproduce the distribution of multiplets of the genetic code through a partial symmetry breaking. One hopes that the number of remaining cases will be small enough that they can be inspected individually, without much effort.

Hence it is necessary to include just one additional step in the program outlined before: after the computation of the lattice of subgroups of G one must compute the maximal subconjugacies among the conjugacy classes of subgroups of G, that is, for each $[H] \in \mathcal{L}(G)$ one should find the elements $[K] \in \mathcal{L}(G)$ such that K is conjugate to a maximal subgroup of H.

There is a set of conditions formulated in Ref. [6] and used extensively in Ref. [18], with the purpose of treating the medium rank Lie algebras and they were very effective in drastically reducing the number of potential subalgebras that would reproduce the distribution of multiplets of the genetic code. Aiming at the adapt these conditions to our situation we introduce the following terminology. A pair of subgroups (H, K) of G, with K maximal in H, is called *admissible* for a certain codon representation of G if:

(a) H has *less* than 21 multiplets and K has *more* than 21 multiplets,
(b) H has *at least* three multiplets of dimension $\geqslant 6$ and *at most* two multiplets of dimension 1, and four multiplets of *odd* dimension,
(c) K has *at least* two multiplets of dimension 1 and *no* multiplet of dimension 5 or $\geqslant 7$,
(d) if K does not have a multiplet of dimension 3 then H must have *at least* two multiplets of dimension 3,
(e) if K does not have a multiplet of dimension 4 then H must have *at least* five multiplets of dimension 4,
(f) if K does not have a multiplet of dimension 6 then H must have *at least* three multiplets of dimension 6.

These conditions are simple enough that they can be easily programmed in GAP into the last step in the program outlined before. It should be noted that these conditions are necessary but not sufficient for ensuring that an admissible pair of subgroups will reproduce the distribution of multiplets of the genetic code. Therefore, after one eliminates the non-admissible pairs one still has to decide which of the remaining admissible pairs are really capable of reproducing the distribution of multiplets of the genetic code.

The case by case analysis of the admissible pairs of subgroups can be done in the following way. First we choose a representative of K that is a subgroup of H and not just subconjugate to a subgroup of H. Then we restrict the character χ of the codon representation to H and decompose it into irreducible components $\{\chi_1, \ldots, \chi_r\}$. Finally, we restrict each irreducible component χ_i obtained above to K and decompose it into irreducible components $\{\chi_{i,1}, \ldots, \chi_{i,s}\}$. Then we can check whether by leaving some of the characters of H unbroken we can obtain the distribution of multiplets of the genetic code. Now, recall that the irreducible characters determine the isotypic components of the restricted representation: two multiplets are in the same isotypic component if and only if the have the same irreducible character. Hence, the following property should be preserved when one freezes some multiplets in the case by case analysis described above:

- the multiplets of H that are in the same isotypic component, i.e., have the same character, should all be broken or be frozen.

Finally, we should point out that once we have found a pair of subgroups that reproduces the distribution of multiplets of the genetic code we can find all sequences of subgroups of G, each maximal in the previous subgroup, that ends in H. Nevertheless, it is very common to find more than one such sequence and there are no criteria for selecting a preferable one.

The final result of our computations in this framework is: *We found ten pairs of subgroups that can reproduce the multiplet distribution of the genetic code with freezing in the last step*.

Table 6
List of groups G that provide symmetry breaking schemes reproducing the degeneracies of the genetic code together with the pairs of subgroups (H, K) that comprise the two last steps in the process. The symbol ":" denotes a semi-direct product and the symbol "." denotes a non-split extension.

#	Group G	$\|G\|$	Subgroup H	$\|H\|$	Subgroup K	$\|K\|$
1	$\mathbb{Z}_2.B_2(3):\mathbb{Z}_2$	103.608	$Q_8:(\mathbb{Z}_3^2:\mathbb{Z}_2^2)$	288	$Q_8:(\mathbb{Z}_3^2\times\mathbb{Z}_2)$	144
2					$Q_8:D_{12}$	96
3					$\mathbb{Z}_3^2:\mathbb{Z}_2^3$	72
4			$Q_8:(\mathbb{Z}_3^2:\mathbb{Z}_2)$	144	$Q_8:\mathbb{Z}_3^2$	72
5					$Q_8:S_3$	48
6					$\mathbb{Z}_3^2:\mathbb{Z}_2^2$	36
7	$\mathbb{Z}_2.C_3(2)$	2.903.040	$(\mathbb{Z}_2^2.\mathbb{Z}_2^4):\mathbb{Z}_3^2$	576	$Q_8:(\mathbb{Z}_3^2\times\mathbb{Z}_2)$	144
8	$G_2(3)$	4.245.696	$Q_8:(\mathbb{Z}_3^2:\mathbb{Z}_2)$	144	$Q_8:\mathbb{Z}_3^2$	72
9					$Q_8:S_3$	48
10					$\mathbb{Z}_3^2:\mathbb{Z}_2^2$	36

6. Conclusion and outlook

In the last section we shall present our findings with more detail. Table 6 summarizes the results. Among the 27 groups analyzed there are three groups possessing pairs of subgroups which reproduce the degeneracies of the genetic code.

The group $B_2(3)$

The simple group $B_2(3)$ can be realized as a (projective) matrix group in four different ways. By definition it is the derived special orthogonal group $SO_5(3)'$ and alternatively, using the Lie algebra canonical isomorphism $B_2\cong C_2$, is the projective symplectic group $PSp_4(3)$. Now recall from Section 2 the exceptional isomorphism of Chevalley groups $B_2(3)\cong {}^2A_3(2)$. Thus one can also realize ${}^2A_3(2)$ as the special unitary group $SU_4(2)$ and alternatively, using the Lie algebra canonical isomorphism $A_3\cong D_3$, as the derived special orthogonal group $SO_6^-(2)'$. It is also the Weyl group of the E_6 exceptional Lie algebra.

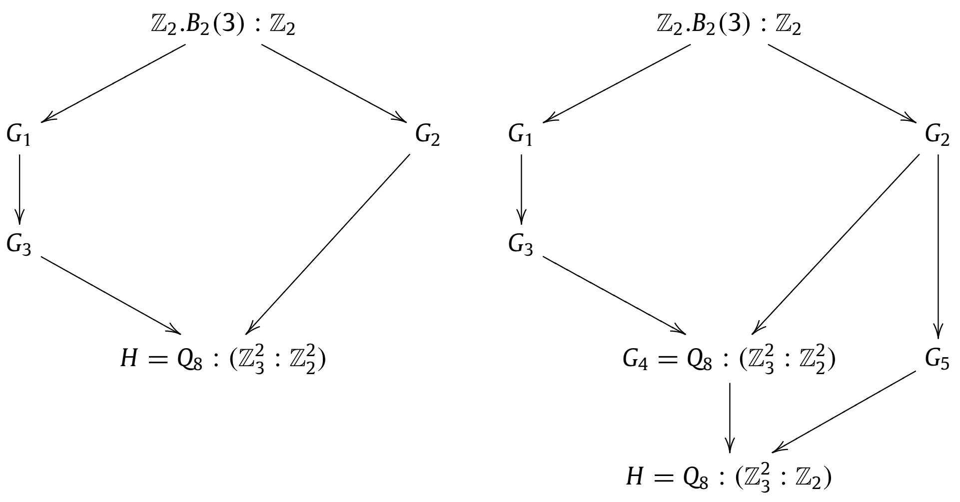
The Schur multiplier of $B_2(3)$ is $\mathbb{Z}_2$ and its outer automorphism group is $\mathbb{Z}_2$. Thus there are three satellites, some of them which also may be realized as matrix groups:

$$\mathbb{Z}_2.B_2(3)\cong Sp_4(3),\qquad B_2(3):\mathbb{Z}_2\cong SO_3(5)\cong SO_6^-(2),$$
$$\mathbb{Z}_2.B_2(3):\mathbb{Z}_2\quad\text{(two isoclinic forms)}.$$

It is more convenient to work with the double extension $\mathbb{Z}_2.B_2(3):\mathbb{Z}_2$ as we can consider all codon representations at once. In fact, $\mathbb{Z}_2.B_2(3):\mathbb{Z}_2$ has four codon representations: two faithful complex conjugate representations and two real representations that descend to $B_2(3):\mathbb{Z}_2$.

The pair of faithful complex representations provides three distinct branching schemes, numbered 1–3 in Table 6. The two real representations are related by the automorphism group of $\mathbb{Z}_2.B_2(3):\mathbb{Z}_2$, which is isomorphic to $\mathbb{Z}_2$, and they also provide three distinct branching schemes, numbered 4–6 in Table 6. Note that the larger group of the pair of subgroups (H, K) is the same in the schemes 1–3 as is the larger group in the schemes 4–6.

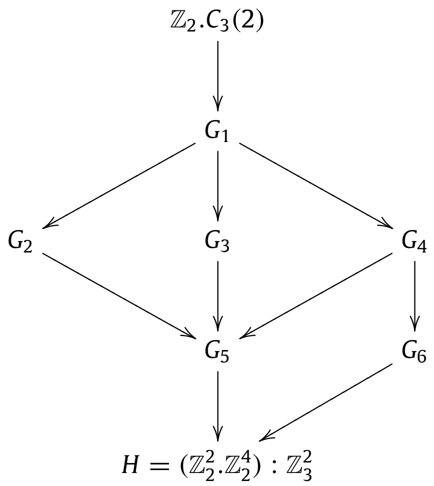
Finally, it is possible to compute all intermediary subgroups between $\mathbb{Z}_2.B_2(3):\mathbb{Z}_2$ and the subgroups $H=Q_8:(\mathbb{Z}_3^2:\mathbb{Z}_2^2)$ and $H=Q_8:(\mathbb{Z}_3^2:\mathbb{Z}_2)$, respectively, leading to the following partial lattices:



The group $C_3(2)$

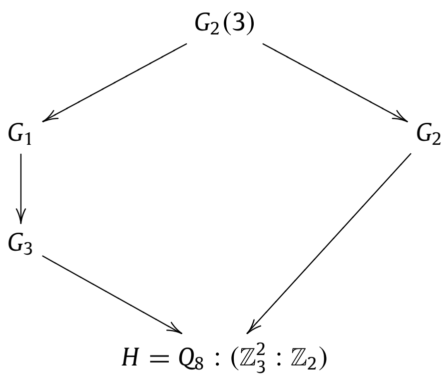
The simple group $C_3(2)$ can be realized as a matrix group in two different ways. By definition it is the symplectic group $Sp_6(2)$ and alternatively, using the special isomorphism of Chevalley groups, $B_n(2^m) \cong C_n(2^m)$, is the orthogonal group $O_7^+(2)$. The Schur multiplier of $C_3(2)$ is $\mathbb{Z}_2$ and its outer automorphism group is trivial. Thus there is only one satellite, the double covering group $\mathbb{Z}_2.C_3(2)$.

As before, it is more convenient to work with the double extension $\mathbb{Z}_2.C_3(2)$ as we can consider all codon representations at once. In fact, $\mathbb{Z}_2.C_3(2)$ has two faithful complex conjugate codon representations which provide one branching scheme, numbered 7 in Table 6. The intermediary subgroups between $\mathbb{Z}_2.C_3(2)$ and the subgroup $H = (\mathbb{Z}_2^2.\mathbb{Z}_2^4) : \mathbb{Z}_3^2$ can be computed, leading to the following partial lattice:



The group $G_2(3)$

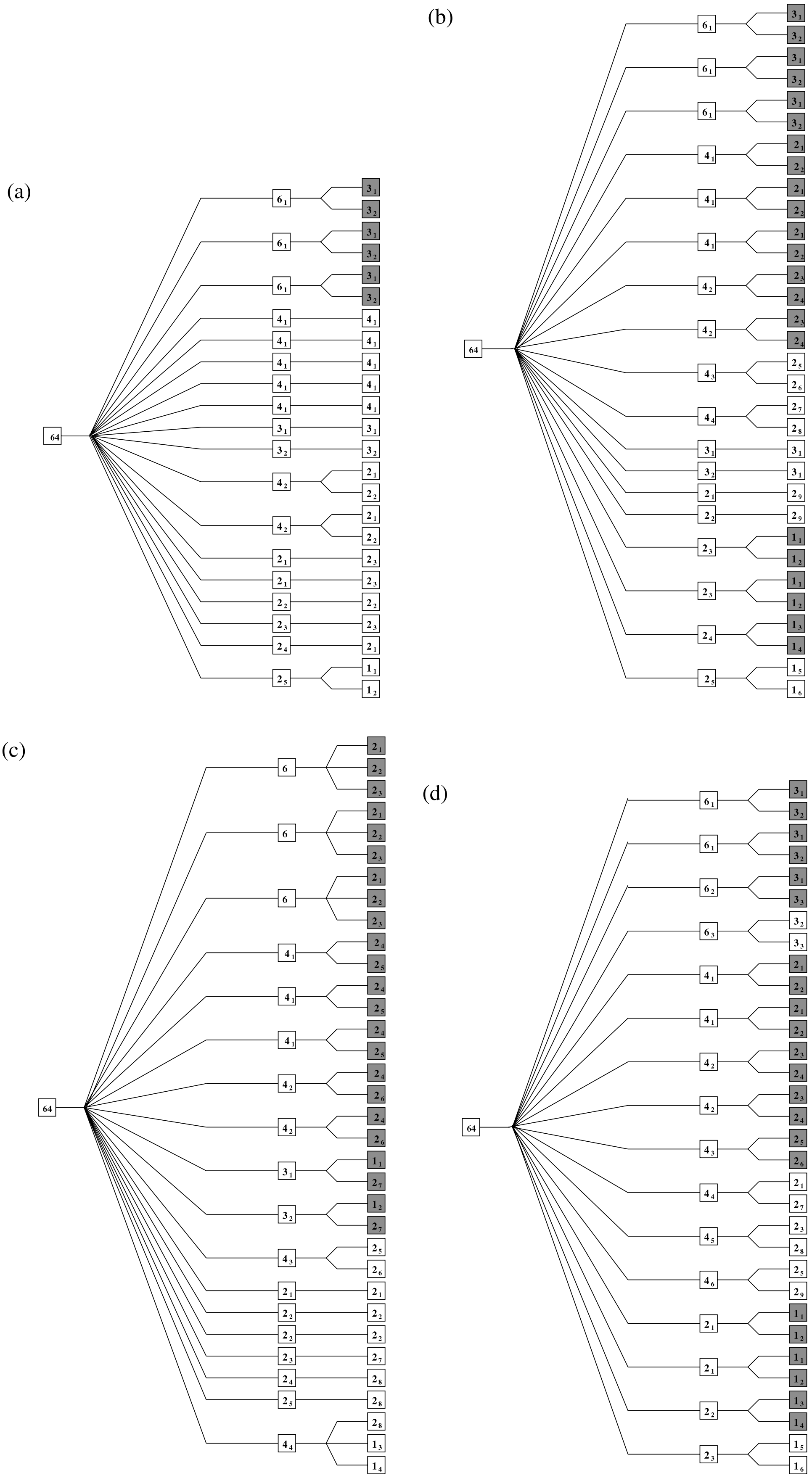
The simple group $G_2(3)$ can be naturally realized as a subgroup of the full matrix group $GL_7(3)$, since there is a natural inclusion of complex Lie algebras of G_2 into $\mathfrak{gl}(7, \mathbb{C})$. The Schur multiplier of $G_2(3)$ is $\mathbb{Z}_3$ and its outer automorphism group is $\mathbb{Z}_2$. Thus there are two satellites, the triple covering group $\mathbb{Z}_3.G_2(3)$ and the full automorphism group $G_2(3) : \mathbb{Z}_2$. Here it is more convenient to work with $G_2(3)$ since all codon representations are linear and the branching schemes are fully contained in $G_2(3)$. There are two complex conjugate codon representations which provide three distinct branching schemes numbered 8–10 in Table 6. The intermediate subgroups between $G_2(3)$ and $H = Q_8 : (\mathbb{Z}_3^2 : \mathbb{Z}_2)$ come from the following partial lattice:



The branching patterns

In the following we present the branching patterns provided by the ten pairs of subgroups of Table 6. There are four distinct branching patterns, represented as trees (a), (b), (c), (d) in the next part. The six pairs of subgroups of $B_2(3)$ and the three pairs of subgroups of $G_2(3)$ give three distinct branching patterns and the pair of subgroups of $C_3(2)$ gives the remaining branching pattern.

Branching pattern (a) corresponds to the pairs of subgroups 2, 5, 9 from Table 6, branching pattern (b) corresponds to pairs of subgroups 1, 4, 8 from Table 6, branching pattern (c) corresponds to pairs of subgroups 3, 6, 10 from Table 6 and branching pattern (c) corresponds to the pairs of subgroups 7 from Table 6.



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