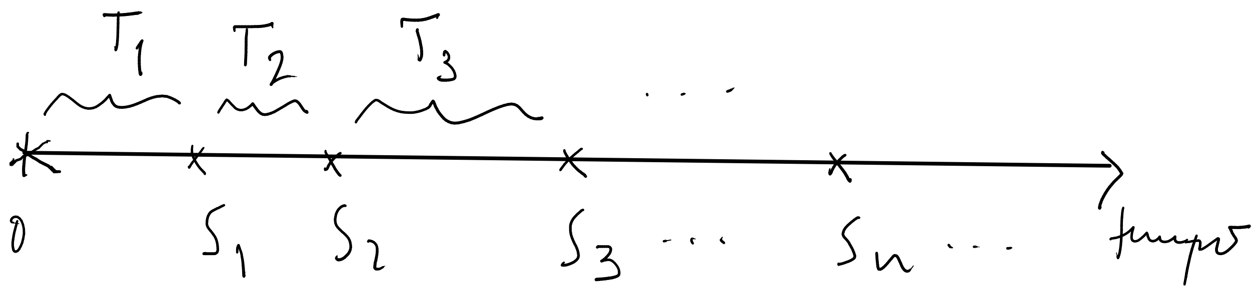
Teoria da Renovação

$T, T_1, T_2, \ldots$ v.a.'s iid positivas (>0).

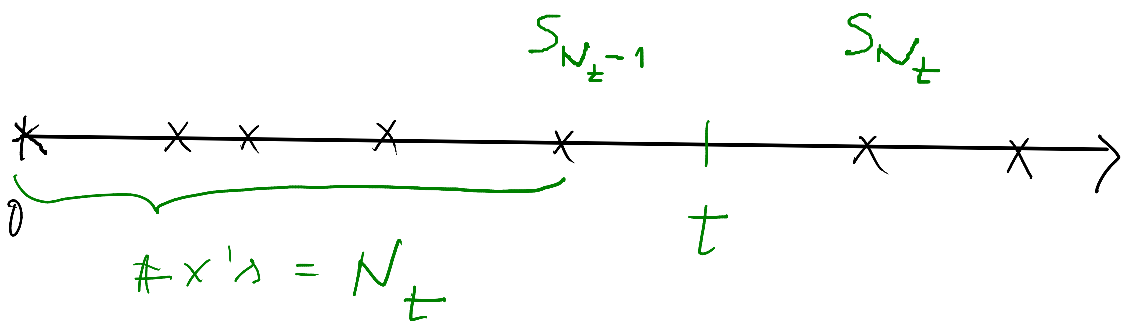
tempos entre renovações de um sistema



$$S_n = \sum_{i=1}^{n} T_i, \quad i \geq 1; \quad S_0 = 0$$

Dado $t \geq 0$, seja N_t o nº de renovações no sistema até o tempo t:

$$N_t = \inf\{n \geq 0 : S_n > t\}, \quad t \geq 0$$



<u>Obs</u>: $N_t \geq k \iff S_k \leq t$, $k, t \geq 0$

Seja $\mu = \mathbb{E}(T) \in (0, \infty]$.

<u>Teo 1</u> a) $\dfrac{N_t}{t} \xrightarrow[t\to\infty]{qc} \dfrac{1}{\mu}$

b) $\dfrac{\mathbb{E}(N_t)}{t} \xrightarrow[t\to\infty]{} \dfrac{1}{\mu}$

<u>Dem</u>: (a) Não é difícil verificar que

$$N_t \xrightarrow[t\nearrow\infty]{qc} \infty. \qquad (1)$$

Verifica-se também prontamente que

$$S_{N_t - 1} \leq t \leq S_{N_t}$$

$$\therefore \quad \overset{\nearrow 1}{\frac{N_t - 1}{N_t}}\ \overset{\nearrow \mu}{\frac{S_{N_t-1}}{N_t - 1}} \leq \frac{t}{N_t} \leq \overset{\nearrow \mu}{\frac{S_{N_t}}{N_t}}$$

qc qdo $t \nearrow \infty$ (pela LFGN)

Invertendo: $\square$(a)

(b)

$$\mathbb{E}(N_t) = \mathbb{E}(N_t;\, N_t \le \overbrace{\sqrt{M+T}}^{K = K_t})$$

$$+\, \mathbb{E}(N_t;\, N_t > K)$$

$$\mathbb{E}(N_t;\, N_t > K) = \overbrace{K\,\mathbb{P}(N_t > K)}^{II}$$

$$+\, \underbrace{\mathbb{E}(N_t - K;\, N_t - K > 0)}$$

$$I := \mathbb{E}(N_t - K)^+ = \sum_{i \ge 0} \mathbb{P}(N_t - K > i)$$

$$= \sum_{i \ge 0} \mathbb{P}(N_t \ge K + i)$$

$$= \sum_{j \ge K} \underbrace{\mathbb{P}(N_t \ge j)}_{\mathbb{P}(S_j \le t)}$$

$$\mathbb{P}(S_j \le t) \overset{\theta > 0}{=} \mathbb{P}(-\theta S_j \ge -\theta t)$$

$$= \mathbb{P}(e^{-\theta S_j} \ge e^{-\theta t}) \overset{\text{Markov}}{\le} e^{\theta t}\{\mathbb{E}(e^{-\theta T})\}^j$$

$$= e^{\theta t + j \log \underbrace{\mathbb{E}(e^{-\theta T})}}$$

$$= \int_0^\infty e^{-\theta t}\, dF(t), \quad F \text{ fç distr. de } T$$

$$= \int_0^\infty dF(t) \int_t^\infty \theta e^{-\theta s} ds = \int_0^\infty \theta e^{-\theta s} \underbrace{F(s)}_{1-\overline{F}(s)} ds$$

$\underbrace{\overline{F}(s)}_{P(T>s)}$

$$= 1 - \theta \underbrace{\int_0^\infty e^{-\theta s} \overline{F}(s) ds}_{\nearrow \int_0^\infty \overline{F}(s) ds = \mu \text{ pdo } \theta \searrow 0}$$

$\therefore \exists \theta_0 > 0$: se $0 \leq \theta \leq \theta_0$, então

$$\int_0^\infty e^{-\theta s} \overline{F}(s) ds \geq \begin{cases} \mu/2, & \text{se } \mu < \infty \\ L, & \text{se } \mu = \infty \end{cases}$$

cte arbitrária

$\therefore \log \mathbb{E} e^{-\theta T} \leq \log(1 - \theta \tilde{L})$; $\tilde{L} = L \wedge \mu/2$

$\leq -\theta \frac{\tilde{L}}{2}$, se $0 < \theta \leq \theta_1$

$\therefore P(S_j \leq t) \leq e^{-\theta_1 (\frac{\tilde{L}}{2} j - t)}$

escolhemos M: > 2

$$\therefore I = e^{\theta_1 t} \underbrace{\sum_{j \geq \lceil Mt \rceil} e^{-\theta_1 \frac{\tilde{L}}{2} j}}_{\underbrace{e^{-\theta_1 \frac{\tilde{L}}{2} \lceil Mt \rceil}}_{\leq e^{-\frac{\tilde{L} M}{2} \theta_1 t}} / \underbrace{1 - e^{-\theta_1 \tilde{L}/2}}_{= \text{const} = c^{-1} > 0}} \leq c\, e^{-\theta_1 t \left(\frac{\widetilde{\tilde{L} M}}{2} - 1 \right)} \leq c\, e^{-\theta_1 t} \xrightarrow[t \nearrow \infty]{} 0$$

$$II = K\, \mathbb{P}(N_t > K) = K\, \mathbb{P}(S_t < K)$$

$$\leq c \lceil Mt \rceil e^{-\theta_1 t} \xrightarrow[t \nearrow \infty]{} 0$$

Agora,

$$\frac{1}{t} E\left(N_t \,;\, N_t \le \lceil Mt \rceil\right) = E\Big(\overset{N_t/t}{\overset{\|}{n_t}} \,;\, N_t \le \lceil Mt \rceil\Big)$$

$$\left| E\left(n_t - \frac{1}{\mu} \,;\, N_t \le \lceil Mt \rceil\right)\right|$$

$$\le E\left(\left|n_t - \frac{1}{\mu}\right| \,;\, n_t \le 2M \right)$$

$$\underbrace{\le \varepsilon + E\Big\{ \overbrace{\left|n_t - \frac{1}{\mu}\right|}^{\le 2M + \frac{1}{\mu}} \,;\, \left|n_t - \frac{1}{\mu}\right| > \varepsilon \,,\, n_t \le 2M \Big\}}_{\le \left(2M + \frac{1}{\mu}\right) P\left(\left|n_t - \frac{1}{\mu}\right| > \varepsilon\right) \xrightarrow[t \nearrow \infty]{} 0} \quad (*)$$

Finalmente

$$\left| \frac{1}{t} E(N_t) - \frac{1}{\mu} \right| \le E\left(\left|n_t - \frac{1}{\mu}\right| \,;\, N_t \le \lceil Mt \rceil \right) + \left(\frac{1}{\mu} \cancelto{0}{P\left(N_t > \lceil Mt \rceil\right)} + \cancelto{0}{I} + \cancelto{0}{II} \right) \Big/ t$$

pelo $t \nearrow \infty$

$\therefore \limsup\limits_{t \nearrow \infty} \left| \cdots \right| \overset{(*)}{\le} \varepsilon$ e o resultado segue de $\varepsilon > 0$ ser arbitrário.

$\square$ (b)

Obs: Segue do argumento acima que $U(t) := E(N_t) < \infty \quad \forall t$.