Teorema de Renovação

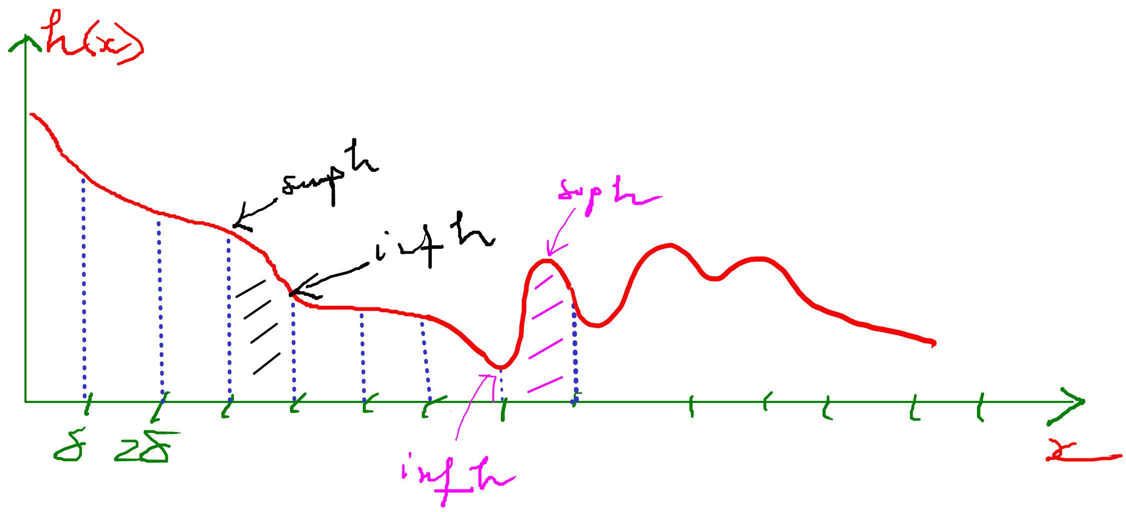
Def: Funções diretamente Riemann integráveis (dRi).

Dada $h:[0,\infty)\to\mathbb{R}$, sejam

$$I^\delta = I^\delta(h) = \sum_{k=0}^{\infty} \delta \underbrace{\sup\{h(x),\ x\in[k\delta,(k+1)\delta)\}}_{\bar{h}(k,\delta)}$$

$$I_\delta = I_\delta(h) = \sum_{k=0}^{\infty} \delta \underbrace{\inf\{ \text{— — — — —} \}}_{\underline{h}(k,\delta)}$$

(se estiverem bem definidas).



1) $h:[0,\infty)\to\mathbb{R}^+$ será dita dRi se

$$\lim_{\delta\to 0} I_\delta = \lim_{\delta\to 0} I^\delta < \infty.$$

(Nesse caso, o limite $= \int_0^\infty h(s)\,ds$.)

integral de Lebesgue

2) $h : [0,\infty) \to \mathbb{R}$ será dita dRi se h^+ e h^- forem dRi.

Nesse caso, temos que $\bar{I}^{\delta}(h)$ e $\underline{I}_{\delta}(h)$ estão bem definidos

e

$$\lim_{\delta \to 0} \underline{I}_{\delta} = \lim_{\delta \to 0} \bar{I}^{\delta} = \int_0^\infty h(s)\,ds \,. \qquad (*)$$

Obs 1) h dRi $\Rightarrow$ h limitada $\left(\|h\|_\infty = \sup_{0 \le s < \infty} |h(s)| < \infty\right)$

e $\quad h(s) \xrightarrow[s \to \infty]{} 0$

2) Se $h : [0,\infty) \to [0,\infty)$, não crescente e

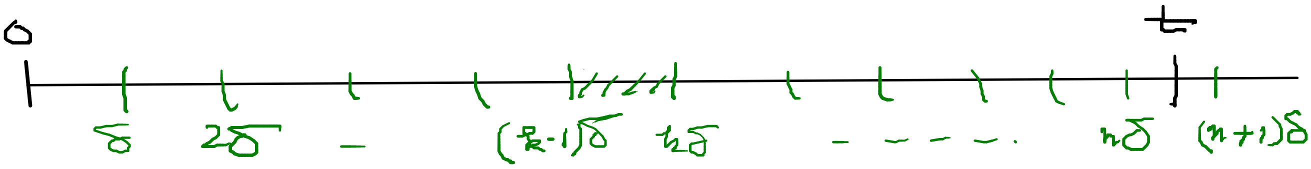
$$\int_0^\infty h(s)\,ds < \infty,$$

então h é dRi. (Verifique.)

Teorema de Renovação: Se F for não aritmética e h for dRi, então

$$H(t) = h * U(t) \xrightarrow[t \to \infty]{} \frac{1}{\mu} \int_0^\infty h(s)\,ds.$$

Dem.



$$H(t) = \int_0^t h(t-s)\,dU(s) = I + II :=$$

$$\sum_{k=1}^{n} \int_{(k-1)\delta}^{k\delta} h(t-s)\,dU(s) + \int_{n\delta}^{t} h(t-s)\,dU(s) \quad (0)$$

$$s \in [(k-1)\delta, k\delta] \Rightarrow t-s \in [(n-k)\delta, (n-k+1)\delta)$$

$$\underline{h}(n-k,\delta) \leq h(t-s) \leq \overline{h}(n-k,\delta)$$

$$\therefore \ \underline{III} := \sum_{j=0}^{n-1} \underline{h}(j,\delta)\, U\{[(n-j-1)\delta, (n-j)\delta)\}$$

$$\overset{(1)}{\leq} \ I$$

$$\overset{(2)}{\leq} \sum_{j=0}^{n-1} \overline{h}(k,\delta) \ \text{— — —} =: \overline{IV}$$

$$\underline{III} = \sum_{j=0}^{\infty} \underline{h}(j,\delta)\, \overbrace{U\{[(n-j-1)\delta, (n-j)\delta)\}}^{\leq U(\delta)} \xrightarrow[n\to\infty]{} \delta/\mu$$

$$\lim_{t\to\infty} \underline{III} = \frac{1}{\mu}\, \underline{I}_\delta \text{, pelo Teo de Blackwell}$$

(usando também o Teo da Conv. Dominada).

$$\text{Similar/e: } \lim_{t\to\infty} \overline{IV} = \frac{1}{\mu}\, I^\delta$$

$$\frac{1}{\mu}\, \underline{I}_\delta \leq \underline{\lim_{t\to\infty}} \ I \leq \overline{\lim_{t\to\infty}} \ I \leq \frac{1}{\mu}\, I^\delta \quad (3)$$

$$|\underline{II}| \leq \|h\|_\infty\, U(n\delta, n\delta + \delta)$$

$$\therefore \ \overline{\lim_{t\to\infty}} \ |\underline{II}| \leq \overset{c}{\frac{\|h\|_\infty}{\mu}}\, \delta \qquad (4)$$

Concluímos de $(\theta - 4)$ que

$$\frac{1}{\mu}\bar{I}_\delta - c\delta \le \underline{\lim}\, H(t) \le \overline{\lim}\, H(t) \le \frac{1}{\mu}\bar{I}^\delta + c\delta .$$

Como δ é arbitrário e h é dRi, segue de $(*)$ que

$$\lim_{t \to \infty} H(t) = \int_0^\infty h(s)\, ds .$$

$\square$