Item 1) [10 pontos]

Considere o experimento de lançar dois dados honestos de forma independente. O evento certo é dado por $\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$ e a medida de probabilidade satisfaz: $\mathrm{P}(\{(i,j)\}) = 1/36$ para $(i,j) \in \Omega$. Sejam X e Y variáveis aleatórias definidas sobre os elementos de Ω tais que $X(i,j) = \min\{i,j\}$ e $Y(i,j) = \max\{i,j\}$.

(a) [2 pontos] Encontre os suportes de X e Y;

Observe que $X(\Omega) = \{1,2,3,4,5,6\}$ e $Y(\Omega) = \{1,2,3,4,5,6\}$. Observamos que estes também são os suportes de X e Y, uma vez que cada elemento tem probabilidade positiva.

(b) [2 pontos] Encontre as funções de probabilidades de X e de Y: $f_X(x) = \mathrm{P}(X = x)$ e $f_Y(y) = \mathrm{P}(Y = y)$;

Primeiramente, para $X(i,j)$, temos

$$\begin{aligned}
\mathrm{P}(X=1) &= \mathrm{P}(\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(3,1),(4,1),(5,1),(6,1)\}) = \frac{11}{36};\\
\mathrm{P}(X=2) &= \mathrm{P}(\{(2,2),(2,3),(2,4),(2,5),(2,6),(3,2),(4,2),(5,2),(6,2)\}) = \frac{9}{36};\\
\mathrm{P}(X=3) &= \mathrm{P}(\{(3,3),(3,4),(3,5),(3,6),(4,3),(5,3),(6,3)\}) = \frac{7}{36};\\
\mathrm{P}(X=4) &= \mathrm{P}(\{(4,4),(4,5),(4,6),(5,4),(6,4)\}) = \frac{5}{36};\\
\mathrm{P}(X=5) &= \mathrm{P}(\{(5,5),(5,6),(6,5)\}) = \frac{3}{36};\\
\mathrm{P}(X=6) &= \mathrm{P}(\{(6,6)\}) = \frac{1}{36}.
\end{aligned}$$

Para $Y(i,j)$, temos

$$\begin{aligned}
\mathrm{P}(Y=1) &= \mathrm{P}(\{(1,1)\}) = \frac{1}{36};\\
\mathrm{P}(Y=2) &= \mathrm{P}(\{(2,1),(2,2),(1,2)\}) = \frac{3}{36};\\
\mathrm{P}(Y=3) &= \mathrm{P}(\{(3,1),(3,2),(3,3),(1,3),(2,3)\}) = \frac{5}{36};\\
\mathrm{P}(Y=4) &= \mathrm{P}(\{(4,1),(4,2),(4,3),(4,4),(1,4),(2,4),(3,4)\}) = \frac{7}{36};\\
\mathrm{P}(Y=5) &= \mathrm{P}(\{(5,1),(5,2),(5,3),(5,4),(5,5),(1,5),(2,5),(3,5),(4,5)\}) = \frac{9}{36};\\
\mathrm{P}(Y=6) &= \mathrm{P}(\{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),(2,6),(3,6),(4,6),(5,6),(6,6)\}) = \frac{11}{36}.
\end{aligned}$$

(c) [2 pontos] Calcule $\mathbb{E}[X]$ e $\mathbb{E}[Y]$;

Para $X(i,j)$, temos

$$\mathbb{E}[X] = 1 \cdot \frac{11}{36} + 2 \cdot \frac{9}{36} + 3 \cdot \frac{7}{36} + 4 \cdot \frac{5}{36} + 5 \cdot \frac{3}{36} + 6 \cdot \frac{1}{36} = \frac{91}{36} \approx 2,528.$$

Para $Y(i,j)$, temos

$$\mathbb{E}[Y] = 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{9}{36} + 6 \cdot \frac{11}{36} = \frac{161}{36} \approx 4,472.$$

(d) [2 pontos] Seja $Z : \Omega \longrightarrow \mathbb{R}$ tal que

$$Z(i,j) = \begin{cases} 1 & \text{se } i < j,\\ 0 & \text{se } i = j,\\ -1 & \text{se } i > j. \end{cases}$$

Encontre a função de probabilidade de $Z : f_Z(z) = \mathrm{P}(Z = z)$;

$$\begin{aligned}
\mathrm{P}(Z = \ \ 1) &= \mathrm{P}(\{(1,2),(1,3),(1,4),(1,5),(1,6),\\
&\qquad (2,3),(2,4),(2,5),(2,6),(3,4),\\
&\qquad (3,5),(3,6),(4,5),(4,6),(5,6)\}) = {}^{15}/_{36};\\
\mathrm{P}(Z = -1) &= \mathrm{P}(\{(2,1),(3,2),(3,1),(4,3),(4,2),\\
&\qquad (4,1),(5,4),(5,3),(5,2),(5,1),\\
&\qquad (6,5),(6,4),(6,3),(6,2),(6,1)\}) = {}^{15}/_{36};\\
\mathrm{P}(Z = 0) &= \mathrm{P}(\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}) = {}^{6}/_{36}.
\end{aligned}$$

(e) [2 pontos] Calcule $\mathbb{E}[Z]$;

Para Z temos

$$\mathbb{E}[Z] = 1 \cdot \frac{15}{36} + 0 \cdot \frac{6}{36} - 1 \cdot \frac{15}{36} = 0.$$