## Exercício 1

$$\operatorname{sen}(y)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x, \quad u(x,0) = 0$$

$x'(s,r) = \operatorname{sen}(y(s,r))$, $x(0,r) = r$ $\Rightarrow x'(s) = \operatorname{sen}(s) \Rightarrow x(s,r) = r + 1 - \cos(s)$

$y'(s,r) = 1$, $y(0,r) = 0$ $\Rightarrow y(s,r) = s$

$z'(s,r) = x$, $z(0,r) = 0$ $\Rightarrow z'(s,r) = r + 1 - \cos(s) \Rightarrow z(s,r) = (r+1)s - \operatorname{sen}(s)$

$$\left.\begin{array}{l} s = y \\ r = x - 1 + \cos(y) \end{array}\right\} \boxed{u(x,y) = (x + \cos(y))y - \operatorname{sen}(y)}$$

## Exercício 2

$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^2, \quad u(x,0) = \cos(x)$$

$x'(s,r) = x$, $x(0,r) = r$ $\Rightarrow x(s) = re^{s}$

$y'(s,r) = 1$, $y(0,r) = 0$ $\Rightarrow y(s) = s$

$z'(s,r) = z^2$ ; $z(0,r) = \cos(r)$ $\Rightarrow z(s) = \dfrac{1}{\frac{1}{\cos(r)} - s}$

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$$\left.\begin{array}{l} s = y \\ r = xe^{-s} = xe^{-y} \end{array}\right\} \boxed{u(x,y) = \frac{\cos(xe^{-y})}{1 - y\cos(xe^{-y})}}$$

## Exercício 3

$$x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial x} = 0, \quad u(x,0) = e^{-x^2}$$

$x'(s,r) = y(s,r)$, $x(0,r) = r$ $\Rightarrow x(s,r) = r\cosh(s)$

$y'(s,r) = x(s,r)$, $y(0,r) = 0$ $\Rightarrow y(s,r) = r\operatorname{senh}(s)$

$z'(s,r) = 0$, $z(0,r) = e^{-r^2}$ $\Rightarrow z(s,r) = e^{-r^2}$

$$x^2 - y^2 = r^2(\cosh^2(s) - \operatorname{senh}^2(s)) = r^2 \Rightarrow u(x,y) = e^{-(x^2-y^2)}$$

$|x| = |y|$

$e^{-(y^2-x^2)} + 2(y^2-x^2)$

$2(y^2-x^2)$ $\quad -2ye^{-(y^2-x^2)} + 2ye^{y^2-x^2}$

## Exercício 8

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u \quad , \quad u(x,0) = \cos(x)$$

$$x'(s,\pi) = 1 \ , \ x(0,\pi) = \pi \qquad x(s,\pi) = \pi + s$$
$$y'(s,\pi) = 1 \ , \ y(0,\pi) = 0 \qquad y(s,\pi) = s$$
$$z'(s,\pi) = z \ , \ z(0,\pi) = \cos(\pi) \qquad z(s,\pi) = \cos(\pi)e^{s}$$

$$\left.\begin{array}{l} \pi = x - y \\ s = y \end{array}\right.$$

$$\Rightarrow \boxed{u(x,y) = \cos(x-y)e^{y}.}$$

## Exercício 9

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = u^2 , \quad u(x, 2x) = 1 .$$

$$x'(s,\pi) = x^2 \ , \ x(0,\pi) = \pi \qquad x(s,\pi) = \frac{1}{\frac{1}{\pi} - s} = \frac{\pi}{1 - \pi s}$$
$$y'(s,\pi) = y^2 \ , \ y(0,\pi) = 2\pi \qquad y(s,\pi) = \frac{1}{\frac{1}{2\pi} - s} = \frac{2\pi}{1 - 2\pi s}$$
$$z'(s,\pi) = z^2 \ , \ z(0,\pi) = 1 \qquad z(s,\pi) = \frac{1}{1-s}$$

$$f' = f^2 \Leftrightarrow \frac{f'}{f^2} = 1 \Leftrightarrow -\left(\frac{1}{f}\right)' = 1 \Leftrightarrow -\frac{1}{f(t)} + \frac{1}{f(0)} = t \Leftrightarrow (c - t) = \frac{1}{f(t)} \Leftrightarrow f(t) = \frac{1}{c-t}$$

Logo $\boxed{f(t) = \frac{1}{\frac{1}{f(0)} - t}}$

$$x(1 - \pi s) = \pi \Rightarrow x = \pi + \pi s x = \pi(1 + sx) \Rightarrow \pi = \frac{x}{1+sx}$$

$$y(1 - 2\pi s) = 2\pi \Rightarrow y\left(1 - \frac{2xs}{1+sx}\right) = \frac{2x}{1+sx} \Rightarrow y(1 + sx - 2xs) = 2x$$

$$\Rightarrow y(1 - xs) = 2x \Rightarrow y - yxs = 2x \Rightarrow y - 2x = yxs$$

$$\Rightarrow s = \frac{y - 2x}{yx}$$

Logo $$\boxed{u(x,y) = \frac{1}{1 - \frac{y-2x}{yx}} = \frac{yx}{yx - y + 2x}}$$

# Exercício 14

$$\begin{cases} \Delta u = f \\ u(x',0) = g_1(x') \\ \dfrac{\partial u}{\partial x_n}(x',0) = g_2(x') \end{cases}$$

$u = u(x_1, \dots, x_n) = u(x', x_n)$

$x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$

$$U_0 = u, \quad U_1 = \frac{\partial u}{\partial x_1}, \dots, U_n = \frac{\partial u}{\partial x_n}.$$

Logo $\dfrac{\partial U_0}{\partial x_j} = U_j, \quad j = 1, \dots, n-1$

$$\frac{\partial U_i}{\partial x_n} = \frac{\partial U_n}{\partial x_i}, \quad i = 1, \dots, n-1$$

$$\frac{\partial^2 u}{\partial x_n^2} = -\sum_{j=1}^{n-1} \frac{\partial^2 u}{\partial x_j^2} + f \Rightarrow \frac{\partial U_n}{\partial x_n} = -\sum_{j=1}^{n-1} \frac{\partial U_j}{\partial x_j} + f$$

Logo obtemos

$$\left.\begin{cases} \dfrac{\partial U_0}{\partial x_n} = U_n \\ \dfrac{\partial U_i}{\partial x_n} = \dfrac{\partial U_n}{\partial x_i} \\ \dfrac{\partial U_n}{\partial x_n} = -\displaystyle\sum_{j=1}^{n-1} \dfrac{\partial U_j}{\partial x_j} + f \end{cases}\right\} \quad \frac{\partial U}{\partial x_n} = F\left(U_n, \frac{\partial U_n}{\partial x_i}, i = 1, \dots, n-1, \frac{\partial U_j}{\partial x_j}\right)$$

$$\begin{cases} U_0(x',0) = g_1(x') \\ U_1(x',0) = \dfrac{\partial g_1}{\partial x_1}(x') \\ \vdots \\ U_n(x',0) = g_2(x') \end{cases}$$

# Exercício 16

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u = 0$$

$u(x,0) = x$, $\frac{\partial u}{\partial t}(x,0) = -x$

Seja $u(x,t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}\, t^j x^k$

$$u(x,0) = \sum_{k=0}^{\infty} a_{0k} x^k = x \Rightarrow \begin{cases} a_{00} = 0 \\ a_{01} = 1 \\ a_{0k} = 0,\ k > 1 \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{j,k} j a_{jk} t^{j-1} x^k \Big|_{t=0} = \sum_{k=0}^{\infty} a_{1k} x^k = -x \Rightarrow \begin{cases} a_{10} = 0 \\ a_{11} = -1 \\ a_{1k} = 0,\ k > 1 \end{cases}$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u = 0 \Rightarrow \sum_{j,k} \left[ j(j-1) a_{jk} t^{j-2} x^k - k(k-1) a_{jk} t^j x^{k-2} - a_{jk} t^j x^k \right] = 0$$

$$\Rightarrow \sum_{j,k} \left[ (j+2)(j+1) a_{j+2\,k} - (k+2)(k+1) a_{j\,k+2} - a_{jk} \right] t^j x^k = 0$$

$$(j+2)(j+1) a_{j+2\,k} - (k+2)(k+1) a_{j\,k+2} - a_{jk} = 0$$

$j = 0$: $2 a_{2k} - (k+2)(k+1) a_{0\,k+2} - a_{0k} = 0$

- $k = 0$: $2 a_{20} - 2 a_{02} - a_{00} = 0 \Rightarrow a_{20} = 0$
- $k = 1$: $2 a_{21} - 6 a_{03} - a_{01} = 0 \Rightarrow a_{21} = \frac{1}{2}$
- $k > 1$: $2 a_{2k} - (k+2)(k+1) a_{0\,k+2} - a_{0k} = 0 \Rightarrow a_{2k} = 0$

$\Rightarrow a_{2k} = \frac{1}{2} \delta_{k1}$

$j = 1$: $6 a_{3k} - (k+2)(k+1) a_{j\,k+2} - a_{jk} = 0$

- $k = 0$: $6 a_{30} - 2 a_{12} - a_{00} = 0 \Rightarrow a_{30} = 0$
- $k = 1$: $6 a_{31} - 6 a_{13} - a_{11} = 0 \Rightarrow a_{31} = -\frac{1}{6}$
- $k > 1$: $6 a_{3k} - (k+2)(k+1) a_{1\,k+2} - a_{1k} = 0 \Rightarrow a_{3k} = 0$

$\Rightarrow a_{3k} = -\frac{1}{6} \delta_{k1}$

$j = 2$: $12 a_{4k} - (k+2)(k+1) a_{2\,k+2} - a_{2k} = 0$

$a_{4k} = \frac{1}{12} a_{21} \delta_{k1} = \frac{1}{24} \delta_{k1}$

Por indução $\boxed{a_{jk} = (-1)^j \frac{1}{j!} \delta_{k1}}$

Concluímos que

$$u(x,t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^j \frac{1}{j!} \delta_{k1} t^j x^k = x \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} t^j = x e^{-t}$$

$$\boxed{u(x,t) = x e^{-t}}$$

## EXERCÍCIO 18

Considere $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$, $S = \{(x,t);\ t = 0\}$.

A normal de S é $(0,1)$. Mas $-\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t}$ tem símbolo principal $P(\xi_1, \xi_2) = -\xi_1^2$.

Logo $P(n_x, n_y) = P(0,1) = 0 \Rightarrow S$ é característico.

Seja $u(x,t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} t^j x^k$. Suponha que $u(x,0) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k$

Logo $\sum_{k=0}^{\infty} a_{0k} x^k = \sum_{k=0}^{\infty} (-1)^k x^{2k} \Rightarrow a_{02k} = (-1)^k$, $a_{0\,2(k+1)} = 0$.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \sum_{j,k} j a_{jk} t^{j-1} x^k + k(k-1) a_{jk} t^j x^{k-2} = 0$$

$$\Rightarrow \sum_{j,k} \left[ (j+1) a_{j+1\,k} + (k+2)k\, a_{j\,k+2} \right] t^j x^k = 0 \Rightarrow \boxed{a_{j+1\,k} = \frac{-k(k+2)}{(j+1)} a_{j\,k+2}}$$

$$a_{jk} = -\frac{k(k+2)}{j} a_{j-1\,k+2} = (-1)^2 \frac{k(k+2)^2(k+4)}{j(j-1)} a_{j-2\,k+4} = (-1)^3 \frac{k(k+2)^2(k+4)^2(k+6)}{j(j-1)(j-2)} a_{j-3\,k+6}$$

$$= \dots = (-1)^j \frac{k(k+2)^2(k+4)^2 \dots (k+2(j-1))^2 (k+2j)}{j!} a_{0\,k+2j}$$

$k$ $\quad k+2j$ $\quad (k+j-1)^{j/2}$

$k(k+2)(k+4)(k+6)$ $\quad (k+j)^{j/2}$ $\quad \frac{(k+2j)!}{k!\,j!}$ $\quad \frac{(2k+2j)!}{(2k)!\,j!}$

Note que $|a_{jk}| \geq \frac{(k+j)^{j/2}}{j!}$ se $k$ é par

$$\sum_{j,k} |a_{jk} t^j x^k| = \sum_{j,k} |a_{j2k} t^j x^{2k}| \geq \sum_{j,k} \frac{(2k)^j}{j!} t^j x^{2k} = \sum_k (e^{2t})^k x^{2k}$$

$$\frac{(2k+2j)!}{(2k)!\,j!} t^j x^k \qquad \frac{(e^{2t})^{k+1}}{(e^{2t})^k} = e^{2t}$$

$k(k+2)^2(k+4)$

$k(k+4) \geq (k+2)^2$

$k^2 + 4k \geq k^2 + 4k + 4$