Horseshoes for a generalized Markus-Yamabe example

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Dedicated to the memory of Prof. Carlos Gutierrez

Abstract

In this note we present an example of a planar diffeomorphism satisfying the generalized Markus-Yamabe conditions, which has a horseshoe. This answers negatively a belief that generically they should be Morse-Smale.

Key words: elliptic points, dissipative perturbations, regions of instability

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2000 Mathematics Subject Classification:

addas-zanata is partially supported by CNPq, grant: 301485/03-8.

1 Introduction

In this paper we study diffeomorphisms of the plane, satisfying some special conditions. In order to present the history of this problem we have to go back to its continuous version, which is known as the Markus-Yamabe Conjecture. Despite its name, the Markus-Yamabe Conjecture is due to Aizerman [1], who stated it as follows:

M-Y Conjecture: *Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field, such that 0 is a singular point, and for any $p \in \mathbb{R}^n$, the Jacobian of X at p has all eigenvalues with negative real part. Then every orbit in $\mathbb{R}^n$ is asymptotically stable to the origin.*

The above problem was cited by Markus and Yamabe in [11], where they prove that the conjecture is true under some additional assumptions. It is clearly true for $n = 1$ and it remained open for $n = 2$ until 1993, when Gutierrez [7] and Fessler [6], independently answered it affirmatively. In 1997, Cima, Gasull, Hubbers, Mañosas and Van den Essen [4] presented polynomial counterexamples to the Markus-Yamabe Conjecture in any dimension greater than two. The conjecture fails for those counterexamples, since they exhibit orbits tending to the point at infinity.

Therefore, the Markus-Yamabe Conjecture is completely answered, being true only in the one and two dimensional cases. Inspired by the Markus-Yamabe Conjecture, La Salle [10] formulated the following conjecture, which is known as the Discrete Conjecture of Markus-Yamabe, or simply the DM-Y Conjecture:

DM-Y Conjecture: *Let f be a C^1-map from R^n to itself such that $f(0) = 0$, and for any $p \in R^n$, the Jacobian of f at p has all its eigenvalues with modulus less than one. Then every orbit in R^n is asymptotically stable to the origin.*

The DM-Y Conjecture is the natural equivalent of the Markus-Yamabe Conjecture for discrete dynamical systems and was motivated by the Hartman-Grobman theorem.

In 1999, A. Cima, A. Gasull and F. Mañosas [5], based on a example of W. Szlenk, proved that the DM-Y Conjecture is false even in the two-dimensional setting. Since the known counter examples to the DM-Y Conjecture have unbounded orbits, Alarcon, Guiñez e Gutierrez [3] strengthened the hypotheses of the DM-Y, in an attempt to guarantee the global stability of the origin, adding the new hypothesis of repulsion at infinity. Even with this new hypothesis, the conjecture is not true, see for instance Theorem 4.4 of [3]. In this article we consider this new setting, which is summarized below:

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^r $(r \geq 1)$ diffeomorphism of the plane satisfying the following conditions:

$$\begin{array}{l} 1)\ \text{for all } p \in \mathbb{R}^2,\ 0 < \det(Df\mid_p) < 1; \\ 2)\ \text{for all } p \in \mathbb{R}^2,\ spec(Df\mid_p) \overset{def.}{=} eigenvalues(Df\mid_p) \subset B_1(0); \\ 3)\ \text{the point at infinity is a repeller;} \\ 4)\ f \text{ has only one fixed point;} \end{array} \tag{1}$$

The above are the so called generalized Markus-Yamabe conditions. The second condition already appeared in the **DM-Y Conjecture** (the first follows from the second plus the assumption that f is orientation preserving) and the third is equivalent to saying that there are simple closed curves γ surrounding the origin, arbitrarily far from it, such that $f(\gamma)$ is contained in the open disk D bounded by γ. In many contexts, this condition is simply known as f is dissipative, see for instance [3]. The fourth is not really a condition, it is a consequence of the previous ones, as shown by Corollary 2 of [2].

Under these hypotheses, the interesting dynamics f may present is restricted to the attractor

$$\Sigma \overset{def.}{=} \bigcap_{n=0}^{\infty} f^n(D). \tag{2}$$

Remember that D is an open disk bounded by a Jordan curve γ such that

$f(\gamma) \subset D$. As f dissipates area, Σ is a compact connected subset of the plane with empty interior.

This paper is organized as follows. In the second section we state and prove our main theorem. In the third section we present a three parameter family of diffeomorphisms of the plane which was the inspiration for our main theorem. For certain parameter values it is conservative, has an elliptic fixed point and the typical complications around it. If we change the parameters in an appropriate way, we get a diffeomorphism satisfying the generalized Markus-Yamabe conditions, which still has transversal homoclinic points (that were already there in the conservative setting). We present some figures to illustrate these behaviors. As the computations for this family are not easy, we prove our main theorem by a different method. We just want to point out that this family, for certain parameters, is an example as in the main theorem.

2 Main Result and Proofs

The question we answer here is about how Σ can be. To be more precise, it was believed by some people in the field that a generic f (in the C^1 topology) satisfying (1) had to be Morse-Smale and thus Σ had to be a Morse-Smale graph (see [9]). Our result proves that this is not the case.

Theorem 1 *: There exists a planar C^r (for all $r \geq 1$) diffeomorphism satisfying (1) which has a horseshoe.*

So, as homoclinic intersections are stable under C^1 perturbations, generic diffeomorphisms satisfying the generalized Markus-Yamabe conditions are not Morse-Smale, that is, there are open sets of such diffeomorphisms which have horseshoes.

Our argument starts with a conservative diffeomorphism of the plane having complicated dynamics in a neighborhood of an elliptic fixed point and after an appropriate perturbation we obtain one satisfying the generalized Markus-Yamabe conditions, in a way that part of the original complication (a horseshoe) is preserved.

Proof of theorem 1:

Let us consider an integrable area preserving C^r (for any $r \geq 1$) twist diffeomorphism of the plane, denoted f_0, which has an elliptic fixed point at the origin and fixes every circle $C_R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = R^2\}$, rotating its points by an angle which increases uniformly with respect to R ($f_0(C_R) = C_R$ and the dynamics of f_0 on each circle is that of a rigid rotation). Clearly, f_0 can be chosen in a way that $Df_0(0, 0)$ is a rotation of an angle different from $\pi.p/q$, for any rational number p/q. To be more precise, if we identify $\mathbb{R}^2$ with the complex plane, f_0 can be written in complex coordinates as

$$z = x+iy \to z.\exp[i(\theta+|z|^2)], \text{ for some } \theta \text{ which is not a rational multiple of } \pi.$$

It is easy to see that the above expression satisfies all the above assumptions. When $(\theta + |z|^2)$ is a rational multiple of π, we have circles of periodic points. The result of Zehnder we apply below tells us that f_0 can be perturbed in a way that some of these circles of periodic points disappear and hyperbolic periodic points with homoclinic intersections come to place.

As we said, we can apply a result of Zehnder from [12], and perturb f_0 in order to obtain an area preserving C^r diffeomorphism f_1, arbitrarily C^r-close to f_0 (for any $r \geq 1$), such that:

- $f_1(x, y) \equiv f_0(x, y)$ for all $(x, y) \notin B_\rho(0)$, for some $\rho > 0$ such that for all $(x, y) \in B_\rho(0)$, $spec(Df_0(x, y))$ is not real.
- f_1 has a periodic hyperbolic point with a transversal homoclinic intersection

The main theorem of [12] is proved for analytic diffeomorphisms of the plane, in an adequate topology. As is pointed out in that paper, a C^r result (for any $r \geq 1$) as we stated above follows from this analytic result, see corollary 1 of theorem 1 of [12].

Now, if for $(x, y) \in \mathbb{R}^2$, we denote the eigenvalues of $Df_1(x, y)$ by $\lambda(x, y)$ and $1/\lambda(x, y)$, we get that they both belong to the unitary circle. This trivially happens for $(x, y) \notin B_\rho(0)$. If $(x, y) \in B_\rho(0)$, as the eigenvalues of $Df_1(x, y)$ are C^0 close to the eigenvalues of $Df_0(x, y)$, which are not real, a simple continuity argument, using the fact that $\det(Df_1(x, y)) = \det(Df_0(x, y)) = 1$ implies that $\|\lambda(x, y)\| = 1$.

So, for any given $0 < \epsilon < 1$ consider the mapping

$$f_2(x, y) = \frac{1}{1 + \epsilon} f_1(x, y).$$

Simple computations show that the point at infinity is a repeller for f_2 because if $R > 0$ is sufficiently large, $f_1(C_R) = f_0(C_R) = C_R$. Thus f_2 satisfies the generalized Markus-Yamabe conditions. Moreover, if $\epsilon > 0$ is sufficiently small, f_2 still has a transversal homoclinic intersection and this proves our theorem. $\square$

3 Inspiring example

The discussion we present now could be considered a "constructive proof" for our theorem if we presented all the necessary computations.

Consider the following 3-parameter family of diffeomorphisms $f_{a,b,c} : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$f_{a,b,c}(x, y) = (a \exp(-x^2) - by, cx), \text{ for positive parameters } a, b, c$$

Clearly,

$$Df_{a,b,c}(x,y) = \begin{pmatrix} -2ax\exp(-x^2) & -b \\ c & 0 \end{pmatrix}$$

So, if $b, c < 1$ and $a^2/2e < bc$, it is easy to see that $f_{a,b,c}$ satisfies the generalized conditions of Markus-Yamabe. Suppose $a = 2$ and $b = c = 1$. This is a conservative mapping, whose dynamics can be seen at figure 1. There is an elliptic fixed point, surrounded by an elliptic island, with all the expected complications, transversal homoclinic intersections, "islands around islands" and so on. If we now choose b and c smaller, but very close to one, we obtain a generalized Markus-Yamabe example, which for sufficiently small $|b-1|+|c-1|$ must have a horseshoe by the transversality of the homoclinic intersection that existed when $b = c = 1$. See figure 2 for a picture of the stable and unstable manifolds of a 3-periodic saddle for such dissipative parameters. Clearly, the computations which prove the above assertions are far from being simple.

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Figure captions.

Figure 1. Diagram showing the dynamics of $f_{a,b,c}$ for conservative parameters, $a = 2$ and $b = c = 1$.

Figure 2. Diagram showing homoclinic intersections for the stable and unstable manifolds of a hyperbolic 3-periodic saddle for parameters which correspond to a mapping satisfying the generalized conditions of Markus-Yamabe, $a = 2$ and $b = c = 0.997$.

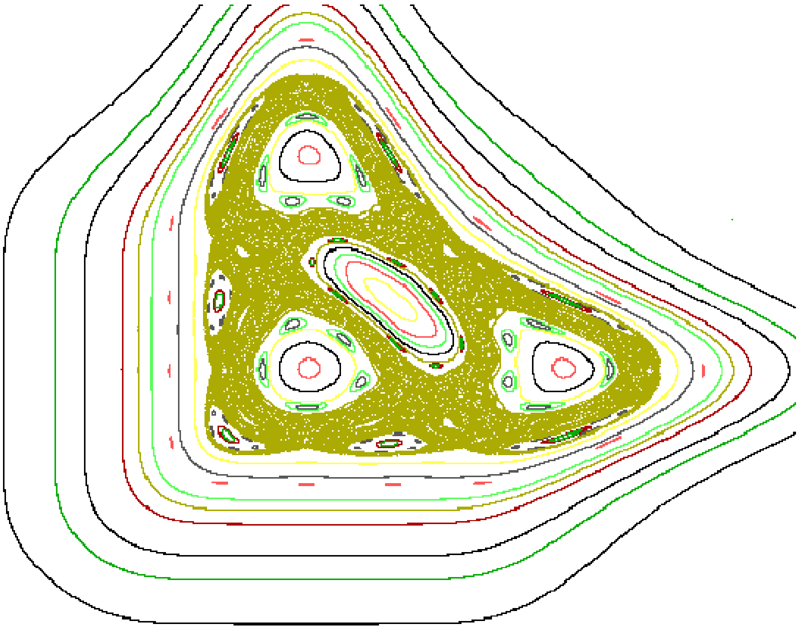

Figure 1

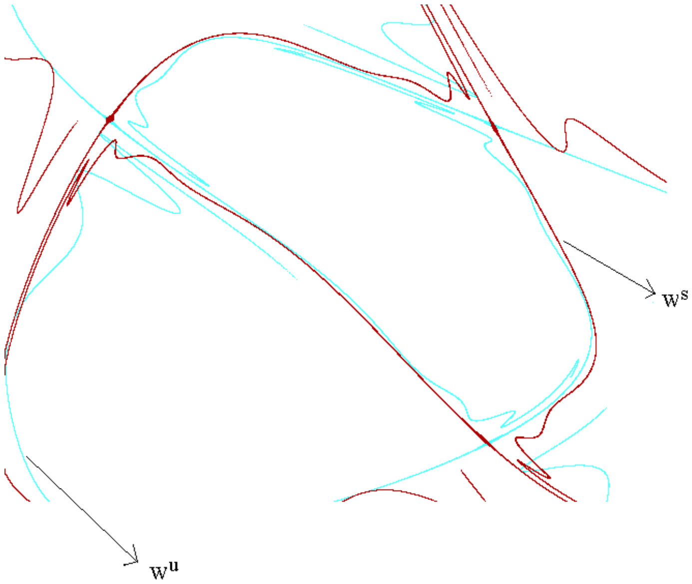

Figure 2