On Periodic points of area preserving torus homeomorphisms

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Abstract

In this note we prove that if an area preserving orientation preserving homeomorphism of the torus has a periodic point then, it has infinitely many periodic points, unless it is homotopic to the identity and its rotation set is a segment with irrational slope and one rational point. Moreover, we present an example of such a mapping with only one fixed point.

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1 Main result

The result proved in this note is inspired by Franks theorem on area preserving homeomorphisms of the annulus [3]. In that paper it is proved that an area preserving homeomorphism of the closed annulus which has one periodic point, has infinitely many periodic points. This result is used to prove that for every metric of positive Gaussian curvature on S^2, there is an infinite number of closed geodesics.

Here we are concerned with periodic points of area preserving orientation preserving homeomorphisms h of the torus, $\mathrm{T}^2 = \mathbb{R}^2/\mathbb{Z}$. Lifts of h to the cylinder $S^1 \times \mathbb{R}$ and to the plane are denoted, respectively, by $\widehat{h}$ and $\widetilde{h}$.

Before stating our result, we need a definition.

Definition: We say that an area preserving orientation preserving homeomorphism $h : \mathrm{T}^2 \to \mathrm{T}^2$ belongs to *case **, if h is homotopic to the identity and its rotation set (see [8]) is a segment with irrational slope containing a rational point.

Our theorem is the following:

Theorem 1 *: An area and orientation preserving homeomorphism $h : \mathrm{T}^2 \to \mathrm{T}^2$ that does not belong to case * and has a periodic point, has infinitely many periodic points.*

Remarks:

1) In case some iterate of h is homotopic to the conjugate of a Dehn twist by an element of $GL(2,\mathbb{Z})$, then there are periodic points with arbitrarily large periods (see the proof below).

2) In case h belongs to *case **, it is possible that the number of periodic points is finite, as the example after the proof shows.

Proof of theorem 1:

As h is area and orientation preserving, the map induced by h on $\pi_1(\mathrm{T}^2)$, denoted by h^* can be characterized by a 2×2 matrix A with integer coefficients and determinant equal to 1. So one of the following possibilities holds:

i) A is the identity

ii) a power of A, let us say, $A^n = \underbrace{A.A...A}_{n\ times} = M \times D \times M^{-1}$, where $M \in GL(2,\mathbb{Z})$ and

$$D = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \text{ for some integer } k \neq 0.$$

iii) A has real eigenvalues which have modulus different from 1

The result in case iii), and in case i) whenever h does not belong to *case **, is already known in the literature, and is compiled here for completeness.

In case i), there are several possibilities:

a) the rotation set of $\widetilde{h}$, $\rho(\widetilde{h})$, defined by Misiurewicz-Ziemian as

$$\rho(\widetilde{h}) = \overset{\infty}{\underset{i=1}{\cap}} \overline{\underset{n\geq i}{\cup} \left\{ \frac{\widetilde{h}^n(\widetilde{z}) - p_2(\widetilde{z})}{n} : \widetilde{z} \in \mathbb{R}^2 \right\}}$$

has non-empty interior. In this case, the main theorem of [4] concludes the proof.

b) $\rho(\widetilde{h})$ is a single point, which is rational because we are assuming that h has a periodic point. The proof is concluded by one of the main results of Le Calvez [7].

c) $\rho(\widetilde{h})$ is a line segment, which contains a rational point by hypothesis. This case divides in 2 sub cases:

– c1) $\rho(\widetilde{h})$ contains 2 rational points, which implies that it contains infinitely many rational points. The proof ends by the main result of [5].

– c2) $\rho(\widetilde{h})$ contains only one rational point, that is the segment has irrational slope. This is *case **.

In case iii) the theorem is true because of Nielsen theory. Moreover, in this case h is isotopic to a linear Anosov mapping of the torus, so its dynamics is quite complex, for instance it has positive topological entropy.

This leave us with case ii) where, by an appropriate linear coordinate change, $M^{-1}h^nM \overset{def.}{=} g$ is isotopic to a Dehn twist. So, as in [1], g has a vertical rotation interval associated to it, that is, given a lift $\widehat{g} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ of g, we can define

$$\rho_V(\widehat{g}) = \overset{\infty}{\underset{i=1}{\cap}} \overline{\underset{n\geq i}{\cup} \left\{ \frac{p_2 \circ \widehat{g}^n(\widehat{z}) - p_2(\widehat{z})}{n} : \widehat{z} \in S^1 \times \mathbb{R} \right\}}, \tag{1}$$

where p_2 is the projection on the vertical coordinate. In [1] it was proved that $\rho_V(\widehat{g})$ is a closed interval and to every rational r/s in the interior of $\rho_V(\widehat{g})$, there corresponds at least 2 fixed points for $\widehat{g}^s - (0, r)$.

As we are supposing that h has a periodic point, this means that g also has one, for instance with vertical rotation number p/q. As h preserves area, so does g. This means that it makes sense to compute the vertical rotation number of the Lebesgue measure, given by:

$$\rho_V(Leb) = \int_{T^2} \phi(z)dz, \text{ where } \phi(z) = p_2 \circ \widehat{g}(\widehat{z}) - p_2(\widehat{z}),$$

for any $\widehat{z}$ that projects on z. If $\rho_V(Leb) = p/q$, then by theorem 5 of [1], we are done. That theorem says that $\widehat{g}^q - (0, p)$, which is a homeomorphism of the cylinder that satisfies the "infinity twist condition" (see below), has periodic points with all possible rotation numbers (in the cylinder), because it has the following intersection property:

- the image under $\widehat{g}^q - (0, p)$ of any homotopically non trivial simple closed curve in the cylinder intersects itself

Definition: By "infinity twist condition" we mean that $p_1 \circ (\widetilde{g}^q(\widetilde{z}) - (0, p))$ goes to $+(-)\infty$ as $p_2(\widetilde{z})$ goes to $+(-)\infty$, where p_1 is the projection on the horizontal coordinate and $\widetilde{g} : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of g to the plane.

So suppose that $\rho_V(Leb) \neq p/q$. In this case, by the Birkhoff ergodic theorem, we get that there exists a point $z' \in \mathrm{T}^2$ such that

$$\rho_V(\widehat{z}') = \lim_{n\to\infty} \frac{p_2 \circ \widehat{g}^n(\widehat{z}') - p_2(\widehat{z}')}{n} \text{ exists and is different from } p/q.$$

So, by theorem 6 of [1], for every r/s between $\rho_V(\widehat{z}')$ and p/q there exists at least 2 fixed points of $\widehat{g}^s - (0, r)$. This also follows from the same type of result above, because the mapping $\widehat{g}^r - (0, s)$ satisfies the same intersection property as above (note that the fact that r/s is between $\rho_V(\widehat{z}')$ and p/q implies that there is an orbit for $\widehat{g}^r - (0, s)$ going down the cylinder and one going up; this gives the required intersection property for homotopically non trivial curves). So the theorem is proved. ∎

2 An example in *case **

To conclude, we present an example of a mapping belonging to *case ** which has only one periodic point.

Start with a C^∞ function $a : \mathbb{R}^2 \to \mathbb{R}$ which satisfies the following properties:

1. a is 2-biperiodic and $a(\widetilde{x}, \widetilde{y}) \geq 0$
2. $a(\widetilde{x}, \widetilde{y}) = 0$ if and only if $\widetilde{x} = 2.n$ and $\widetilde{y} = 2.m$ for integers n, m
3. $a(\widetilde{x}, \widetilde{y}) = a(\widetilde{x}, -\widetilde{y})$
4. $\int_{-1}^{1} a(\widetilde{x}, \widetilde{y}) d\widetilde{y} = 2\int_0^1 a(\widetilde{x}, \widetilde{y}) d\widetilde{y} =$const, which implies that $\int_0^1 \frac{\partial a(\widetilde{x},\widetilde{y})}{\partial \widetilde{x}} d\widetilde{y} = 0$

Now, we want to construct a system of differential equations in the plane

$$\begin{aligned} \tfrac{d\widetilde{x}}{dt} &= a(\widetilde{x}, \widetilde{y}) \\ \tfrac{d\widetilde{y}}{dt} &= b(\widetilde{x}, \widetilde{y}) \end{aligned} \qquad (2)$$

for a certain function $b(\widetilde{x}, \widetilde{y})$ so that:

- the system (2) is area preserving
- $b(\widetilde{x}, 2n + 1) = 0$ for all integer n
- b is a 2-biperiodic C^∞ function

In order for system (2) to be area preserving, we must impose that

$$\frac{\partial a}{\partial \widetilde{x}} + \frac{\partial b}{\partial \widetilde{y}} \equiv 0 \text{ in the whole plane.}$$

But this gives $b(\widetilde{x},\widetilde{y}) = \int_0^{\widetilde{y}} \left(-\frac{\partial a(\widetilde{x},\widetilde{s})}{\partial \widetilde{x}}\right) d\widetilde{s} + c(\widetilde{x})$. If $c(\widetilde{x})$ is a 2-periodic C^∞ function, then condition 4 above implies that b is also 2-biperiodic. The function $c(\widetilde{x})$ is chosen by imposing that

$$b(\widetilde{x}, 2n+1) = 0 \ \Leftrightarrow c(\widetilde{x}) = \int_0^{2n+1} \left(\frac{\partial a(\widetilde{x},\widetilde{s})}{\partial \widetilde{x}}\right) d\widetilde{s} = 0.$$

So $b(\widetilde{x},\widetilde{y}) = -\int_0^{\widetilde{y}} \left(\frac{\partial a(\widetilde{x},\widetilde{s})}{\partial \widetilde{x}}\right) d\widetilde{s}$ and thus $b(\widetilde{x}, n) = 0$, for all integer n. The above construction implies that the vector field $\widetilde{X}(\widetilde{x},\widetilde{y}) = (a(\widetilde{x},\widetilde{y}), b(\widetilde{x},\widetilde{y}))$ is C^∞, area preserving, 2-biperiodic and has singularities at all points with both coordinates even. Moreover, the function $a(\widetilde{x},\widetilde{y})$ could have been chosen in a way such that its Taylor expansion is not reduced to 0 at points $(\widetilde{x},\widetilde{y})$ with both coordinates even. Therefore, $\widetilde{X}$ is of Lojasiewicz type at its singularities.

This means that $\widetilde{X}$ induces an area preserving vector field X on the torus obtained by identifying the opposite sides of the square $[-1,1]\times[-1,1]$, which has only one singularity, at $(0,0)$. Clearly all orbits of X are periodic, except for one which is homoclinic to the singularity (the horizontal line with $y = 0$). See figure 1 for a picture of the orbits of X.

Now consider the following one parameter family of perturbed vector fields:

$$\widetilde{X}_\theta = \widetilde{X} + \theta(0, c(\widetilde{x})),$$

where $\theta > 0$ is a real number very close to 0 and $c(\widetilde{x})$ is a 2-periodic C^∞ function such that $c(\widetilde{x}) \geq 0$ and $c(\widetilde{x}) = 0$ if and only if $\widetilde{x} = 2n$, for all integer n.

For an appropriate choice of θ, the perturbed vector field $\widetilde{X}_\theta$ induces a vector field on the torus, X_θ, which has no closed solution, is area preserving and has only one singularity, at $(0,0)$. To see this, let us analyze the following transversal section of the torus flow, $\sum_\theta = \{-1\}\times[-1,1) \equiv \{1\}\times[-1,1)$ (the index θ means that we are considering the section $\{-1\}\times[-1,1) \equiv \{1\}\times[-1,1)$ with the vector field X_θ). First note that all closed orbits of X_θ must cross $\sum_\theta$, since the first coordinate of $\widetilde{X}_\theta$ is $a(\widetilde{x},\widetilde{y})$, which is strictly positive at all regular points.

In the following we will show that the dynamics in a neighborhood of $(0,0)$ is topologically the same for X and X_θ. As X is of Lojasiewicz type, so is X_θ if θ is small enough. So the dynamics of X_θ in the neighborhood of $(0,0)$ can be described by Dumortier's result [2], which says that it can be obtained by gluing a finite number of sectors, which can be of 4 types: elliptic, attracting, expanding and hyperbolic. As the index of $(0,0)$ is zero and X_θ is area preserving there are exactly 2 hyperbolic sectors and the dynamics is topologically the same as for X (in the area preserving setting elliptic, attracting and expanding sectors are not allowed).

So we can define a first return map to $\sum_\theta$. The only problem is with the point $z_0 \in \sum_\theta$ whose forward orbit is asymptotic to $(0,0)$ and does not cross $\sum_\theta$ again. But as there is only one orbit which is backward asymptotic to $(0,0)$, we use the first point of it that hits $\sum_\theta$ to define the return of z_0.

Thus for small enough θ, we have a return map $r_\theta : \sum_\theta \to \sum_\theta$ which is a circle homeomorphism. And by construction of X, $r_0 = id$ and so the rotation number $\rho(r_0) = 0$. For any small $\theta > 0$, $\rho(r_\theta) > 0$ (because the return of every point is above the point) and the function $\theta \to \rho(r_\theta)$ is continuous. So, take a θ^* such that $\rho(r_{\theta^*})$ is irrational. This implies that r_{θ^*} has no periodic point and so X_θ has no closed orbit.

The time one flow of this vector field is a diffeomorphism of the torus, homotopic to the identity, area preserving, which has only one periodic point (which is fixed), so its rotation set has no interior and is not reduced to a point, because in this case the rotation vector of the Lebesgue measure would be zero and there would be infinitely many periodic points, by Le Calvez result [7] (in particular, there would be at least 3 fixed points, by Conley-Zehnder theorem). So the rotation set is a segment and, as there is only one periodic point, it has irrational slope and we are in *case **.

This example was inspired by one which is attributed to Katok, see [6].

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Figure captions.

Figure 1. Diagram showing the dynamics of X

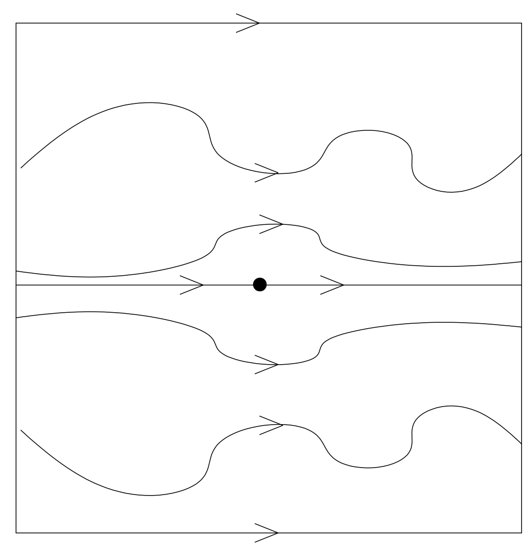

Figure 1