The additive coloring problem on graphs

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Definition

Names: Additive Coloring Problem
Lucky Labeling Problem
Vertex Coloring by Sums

Let $G = (V, E)$ be an undirected, simple graph, and consider a *labeling* $f: V \rightarrow \{1, \ldots, k\}$ of its vertices.
For a given set $S \subset V$, let $f(S)$ be the sum of labels over S.

f is an *additive k-coloring* $\Leftrightarrow f(N(u)) \neq f(N(v)) \quad \forall (u, v) \in E$
where $N(v)$ = set of neighbors of v

$\eta(G)$ = *additive chromatic number* (minimum k such that G has an additive k-coloring)

Finding $\eta(G)$ is an **NP**-Hard problem.

Motivation

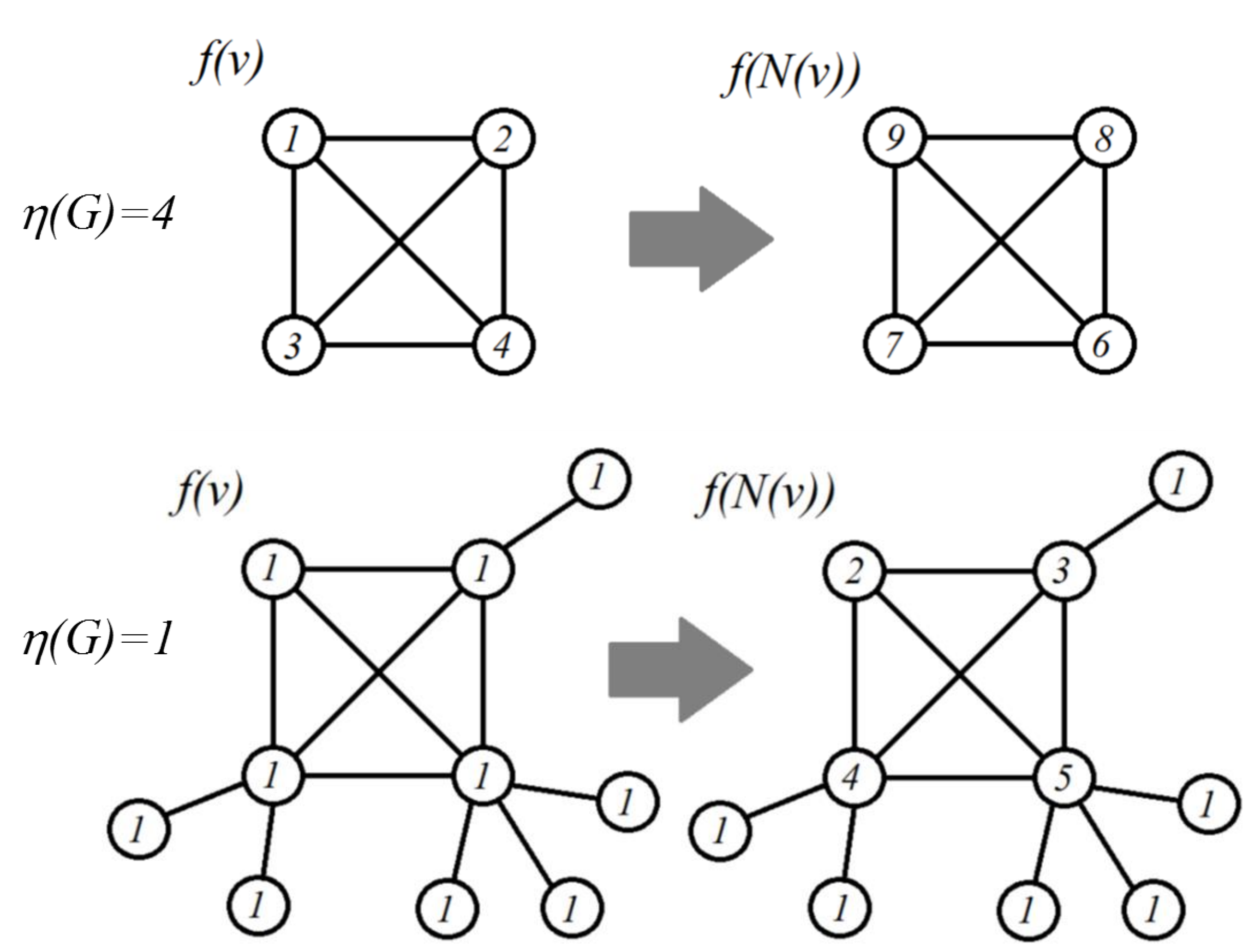
It was first presented in 2009 by Czerwinski, Grytczuk and Zelazny who proposed a **conjecture** that for every graph G, $\eta(G) \leq \chi(G)$, where $\chi(G)$ is the chromatic number of G.

The problem as well as the conjecture has recently gained interest from the scientific community:

A. Ahadi, A. Dehghan and E. Mollaahmadi, *On the Lucky Labeling of Graphs*, Manuscript. `http://arxiv.org/abs/1007.2480`
Orlow N.: Advances in Lucky Labelling graphs, Fibsum graphs, and 3-regular graph decompositions. Research Experiences for Graduate Students, August 2009.
Grappe R., Grippo L. N., Valencia-Pabon M.: Lucky number of bounded-treewidth graphs. Tech. Rep. LIPN 2012, submitted to the WG 2012 conference.
Ahadi A., Dehghan A., Kazemi M., Mollaahmadi E.: Computation of lucky number of planar graphs is NP-hard. Inform. Process. Lett. **112**, 109–112 (2012)
Akbari S., Ghanbari M., Manaviyat R., Zare S.: On the Lucky Choice Number of Graphs. Graphs and Combinatorics **29**, 157–163 (2013)
Grytczuk J., Bartnicki T., Czerwiński S., Bosek B., Matecki G., Zelazny W.: Additive colorings of planar graphs. Graphs and Combinatorics **30**, 1087–1098 (2014)
Brandt A., Diemunsch J., Jahanbekam S.: Lucky Choice Number of Planar Graphs with Given Girth. Manuscript.
`http://math.ucdenver.edu/~sjahanbekam/Lucky.pdf`
Miller M., Rajasingh I., Emilet D. A., Jemilet D. A.: d-Lucky Labeling of Graphs. Procedia Computer Science (ICRTC-2015) **57**, 766–771 (2015)

However, the additive chromatic number is known for very few families of graphs.

Example



Some known results

Cliques: $\eta(K_n) = n$
Trees: $\eta(T) = 1$ or 2
Circuits: $\eta(C_n) = 2$ if n is even, 3 if n is odd

Characterization of 1-additive colorings:
$\eta(G) = 1 \Leftrightarrow |N(u)| \neq |N(v)| \quad \forall (u, v) \in E$

Upper bound: $\eta(G) \leq \Delta^2 - \Delta + 1$ (Δ = máximum degree in G)

Lower bounds: **1)** $\eta(G) \geq |T|$
where $T \subset V$ satisfies: $u, v \in T \Rightarrow u, v$ are true twins.

2) $\eta(G) \geq \lceil |Q| / (|V| - |Q| + 1) \rceil$, where Q is a clique of G

New Results

Regular Bipartite: $\eta(G) = 2$
Complete r-partite: $\eta(G) = \max\{\lceil s_i / |V_i| \rceil : i = 1, \ldots, r\}$
where: $|V_i| \geq |V_{i+1}| \quad \forall\, i = 1, \ldots, r\text{-}1$
$s_r = |V_r|, \quad s_i = \max\{1 + s_{i+1}, |V_i|\} \quad \forall\, i = 1, \ldots, r\text{-}1$
Fans: $\eta(F_n) = 2$ ($F_n = P_{n+1}$ + *universal vertex*)
Windmill graphs: $\eta(W^m{}_n) = n - 1$
($W^m{}_n = m$ copies of K_n which share a single vertex)
Wheels: $\eta(W_n) = 2$ if n is even, 3 if n is odd
($W_n = C_n$ + *universal vertex*)
Split graphs: $\eta(G) \leq |Q|$, where Q is a maximal clique of G

Thin/thick headless spiders with q legs: $\eta(G) = \lceil (q+1)/2 \rceil$
Complete suns of m rays: $\eta(G) = \lceil (m+2)/3 \rceil$

All these families of graphs satisfy the conjecture!

Computational experiment

A tool for solving the additive coloring problem was implemented based on CPLEX and this formulation:

Let $G = (V, E)$ be a graph, $E_2 = \{(u, v), (v, u) : (u, v) \in E\}$ (edges occur in both directions), integer variables k and $f(v)$ for all $v \in V$, and binary variables $z(u, v)$ for all $(u, v) \in E_2$, where $z(u, v) = 1$ if and only if $f(N(u)) < f(N(v))$.

min k ⬅ computes $\eta(G)$
subject to

$$\begin{aligned}
& f(N(u)) - f(N(v)) + M_{uv} z(u, v) \leq M_{uv} - 1, && \forall\ (u, v) \in E_2 \\
& z(u, v) + z(v, u) = 1, && \forall\ (u, v) \in E \\
& 1 \leq f(v) \leq UB, && \forall\ v \in V \\
& f(v) \leq k, && \forall\ v \in V \\
& z(u, v) \in \{0, 1\}, && \forall\ (u, v) \in E_2 \\
& k, f(v) \in \mathbb{Z}_+, && \forall\ v \in V
\end{aligned}$$

where $M_{uv} = 1 + |N(u) \backslash N(v)| UB - |N(v) \backslash N(u)|$ for all $(u, v) \in E_2$
UB = upper bound of $\eta(G)$

Then, the conjecture was tested over all connected graphs up to 10 vertices (~12000000 instances).

Our tool: `http://www.fceia.unr.edu.ar/~daniel/stuff/acp.zip`
Instances: `http://users.cecs.anu.edu.au/~bdm/data/graphs.html`
Tool for obtaining $\chi(G)$: `http://rhydlewis.eu/resources/gCol.zip`