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Gauss M. Cordeiro [a] , Giovana O. Silva [b] & Edwin M.M. Ortega [c]

[a] Departamento de Estatística, Universidade Federal de Pernambuco, 50740-540, Recife, PE, Brazil

[b] Departamento de Estatística, Universidade Federal da Bahia, 50740-540, Salvador, BA, Brazil

[c] Departamento de Ciências Exatas, Universidade de São Paulo, 13418-900, Piracicaba, SP, Brazil

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Gauss M. Cordeiro[a], Giovana O. Silva[b] and Edwin M.M. Ortega[c]*

[a] *Departamento de Estatística, Universidade Federal de Pernambuco, 50740-540, Recife, PE, Brazil;*
[b] *Departamento de Estatística, Universidade Federal da Bahia, 50740-540, Salvador, BA, Brazil;*
[c] *Departamento de Ciências Exatas, Universidade de São Paulo, 13418-900, Piracicaba, SP, Brazil*

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We propose a new distribution, the so-called beta-Weibull geometric distribution, whose failure rate function can be decreasing, increasing or an upside-down bathtub. This distribution contains special sub-models the exponential geometric [K. Adamidis and S. Loukas, *A lifetime distribution with decreasing failure rate*, Statist. Probab. Lett. 39 (1998), pp. 35–42], beta exponential [S. Nadarajah and S. Kotz, *The exponentiated type distributions*, Acta Appl. Math. 92 (2006), pp. 97–111; *The beta exponential distribution*, Reliab. Eng. Syst. Saf. 91 (2006), pp. 689–697], Weibull geometric [W. Barreto-Souza, A.L. de Morais, and G.M. Cordeiro, *The Weibull-geometric distribution*, J. Stat. Comput. Simul. 81 (2011), pp. 645–657], generalized exponential geometric [R.B. Silva, W. Barreto-Souza, and G.M. Cordeiro, *A new distribution with decreasing, increasing and upside-down bathtub failure rate*, Comput. Statist. Data Anal. 54 (2010), pp. 935–944; G.O. Silva, E.M.M. Ortega, and G.M. Cordeiro, *The beta modified Weibull distribution*, Lifetime Data Anal. 16 (2010), pp. 409–430] and beta Weibull [S. Nadarajah, G.M. Cordeiro, and E.M.M. Ortega, *General results for the Kumaraswamy-G distribution*, J. Stat. Comput. Simul. (2011). DOI: 10.1080/00949655.2011.562504] distributions, among others. The density function can be expressed as a mixture of Weibull density functions. We derive expansions for the moments, generating function, mean deviations and Rénvy entropy. The parameters of the proposed model are estimated by maximum likelihood. The model fitting using envelops was conducted. The proposed distribution gives a good fit to the ozone level data in New York.

Keywords: beta Weibull; exponential geometric distribution; generalized exponential distribution; maximum-likelihood estimation; observed information matrix; Weibull distribution; Weibull geometric distribution

1. Introduction

Over the last two decades several new models have been proposed that are either derived from or in some way related to the Weibull distribution. They provide a richness that makes them appropriate to model complex data sets. The literature on Weibull models is vast, disjointed, and scattered across many different journals. When modelling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, it does not provide a reasonable parametric fit for modelling phenomenon with non-monotone failure rates such as the bathtub shaped and the unimodal failure rates that are

*Corresponding author. Email: edwin@esalq.usp.br

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common in reliability and biological studies. Such bathtub hazard curves have nearly flat middle portions and the corresponding density functions have a positive anti-mode. An example of the bathtub-shaped failure rate is the human mortality experience with a high infant mortality rate which reduces rapidly to reach a low level. It then remains at that level for quite a few years before picking up again. Unimodal failure rates can be observed in course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

Alternatively, various authors introduced more flexible distributions to model monotone or unimodal failure rates but they are not useful for modelling the bathtub-shaped failure rates. Adamidis and Loukas [1] proposed the exponential geometric (EG) distribution to model lifetime data with the decreasing failure rate function and Gupta and Kundu [2–4] defined another lifetime distribution, referred to as the generalized exponential (GE) (also called the exponentiated exponential) distribution, and investigated some of its mathematical properties. This distribution has only increasing or decreasing failure rate function. Following the same idea of the GE distribution, Silva *et al.* [5,6] defined the generalized exponential geometric (GEG) distribution and demonstrated that its failure rate function can be increasing, decreasing or unimodal. One generalization of the GE distribution was proposed by Barreto-Souza *et al.* [7], refered to as the Weibull geometric (WG) distribution, for modelling monotone or unimodal failure rates. The beta-exponential (BE) distribution studied by Nadarajah and Kotz [8] has also only increasing or decreasing failure rate function.

In this article, we introduce the beta-Weibull geometric (BWG) distribution that generalizes the WG distribution, and study some of its properties. The new distribution due to its flexibility in accommodating unimodal failure rate functions seems to be an important distribution to be used in a variety of problems in modelling survival data. The article is organized as follows. In Section 2, we define the BWG model. In Section 3, we demonstrate that the probability density function of the BWG distribution is a mixture of beta-Weibull (BW) density functions. In Sections 4 and 5, we derive the moments and the moment generating function (mgf), respectively. Section 6 is devoted to the quantile function. The mean deviations and Bonferroni and Lorenz curves are determined in Section 7. The Rényi entropy is calculated in Section 8. In Section 9, we show that the density function of the BWG order statistics is a linear combination of Weibull density functions. Maximum-likelihood estimation of the model parameters and the observed information matrix are discussed in Section 10. In Section 11, we perform a model check based on Martingale-type residuals and generated envelopes In Section 12, we provide an application of the BWG model to the ozone level data in New York. Concluding remarks are given in Section 13.

2. BWG distribution

Suppose that $\{Y_i\}_{i=1}^{Z}$ are independent and identically distributed random variables having a Weibull density function defined by $g_{\lambda,c}(y) = c\lambda^c y^{c-1}\mathrm{e}^{-(\lambda y)^c}$ for $y > 0$, $\lambda > 0$ and $c > 0$ and that Z is a geometric random variable with probability mass function given by $P(z; p) = (1-p)\,p^{z-1}$ for $Z \in N$ and $p \in (0, 1)$. Let $X = \min(\{Y_i\}_{i=1}^{Z})$. The conditional density function of X given $Z = z$ is $g(x|z; \lambda, c) = cz\lambda^c x^{c-1} \exp\{-(\lambda c)^x\}$ and then the WG density function becomes

$$g(x; p, \lambda, c) = c(1-p\lambda^c x^{c-1} \exp\{-(\lambda x)^c\}[1-p\exp\{-(\lambda x)^c\}]^{-2}, \quad x > 0. \tag{1}$$

The cumulative distribution function (cdf) corresponding to Equation (1) is

$$G(x; p, \lambda, c) = \frac{1-\mathrm{e}^{-(\lambda x)^c}}{1-p\,\mathrm{e}^{-(\lambda x)^c}}, \quad x > 0. \tag{2}$$

The idea of the BWG distribution stems as follows: if $G(x)$ denotes the cdf of a random variable, then a beta-G distribution [9,10] is defined (for $a > 0$ and $b > 0$) by

$$F(x) = I_{G(x)}(a, b) = \frac{1}{B(a, b)} \int_0^{G(x)} w^{a-1}(1-w)^{b-1}\,\mathrm{d}w, \tag{3}$$

where $I_y(a, b) = B_y(a, b)/B(a, b)$ is the incomplete beta function ratio, $B_y(a, b) = \int_0^y w^{a-1}(1-w)^{b-1}\,\mathrm{d}w$ the incomplete beta function and $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ the beta function. This class of generalized distributions has been receiving considerable attention over the last years. Eugene *et al.* [9], Nadarajah and Kotz [11], Nadarajah and Gupta [12] and Nadarajah and Kotz [8,13] proposed the beta normal, beta Gumbel, beta Fréchet and BE distributions by taking $G(x)$ in Equation (3) to be the cdf of the normal, Gumbel, Fréchet and exponential distributions, respectively. More recently, Silva *et al.* [5,6] defined the beta modified Weibull (BMW) distribution by taking $G(x)$ in Equation (3) to be the cdf of the modified Weibull (MW) [14] distribution and discussed the maximum-likelihood estimation of its parameters. One of the advantages of the BMW distribution is that it includes as special sub-models several well-known distributions such as the Weibull, Rayleigh, exponentiated Weibull (EW) [15,16], MW, BW and generalized modified Weibull [17] distributions. In a similar manner, we are motivated to study the BWG model because it includes some important distributions that can be useful in analyzing failure data such as the Weibull, EG, GEG and BW distributions.

From now on, let $u = \exp\{-(\lambda x)^c\}$. The cdf of the BWG distribution is defined from Equation (2) by

$$\begin{aligned} F(x; p, \lambda, c, a, b) &= I_{(1-u)/(1-pu)}(a, b) \\ &= \frac{1}{B(a, b)} \int_0^{(1-u)/(1-pu)} w^{a-1}(1-w)^{b-1}\,\mathrm{d}w, \quad x > 0, \end{aligned} \tag{4}$$

where $p \in (0, 1)$, $\lambda > 0$ is a scale parameter and $a > 0$, $b > 0$ and $c > 0$ are shape parameters. The BWG density function corresponding to Equation (4) is

$$f(x; p, \lambda, c, a, b) = \frac{c(1-p)^b \lambda^c x^{c-1} u^b (1-u)^{a-1}(1-pu)^{-(a+b)}}{B(a, b)}, \quad x > 0. \tag{5}$$

A random variable X having density function (5) is denoted by $X \sim$BWG(p, λ, c, a, b). The failure rate function corresponding to Equation (5) reduces to

$$h(x; p, \lambda, c, a, b) = \frac{c(1-p)^b \lambda^c x^{c-1} u^b (1-u)^{a-1}(1-pu)^{-(a+b)}}{B(a, b) I_{(1-u)/(1-pu)}(a, b)}, \quad x > 0. \tag{6}$$

Clearly, the beta exponential geometric (BEG) distribution is obtained from Equation (5) for $c = 1$. When $b = 1$, in addition to $c = 1$, we obtain the GEG distribution. For $b = c = 1$, the GE distribution follows as the limiting distribution (the limit is defined in terms of the convergence in distribution) of the BWG distribution when $p \to 0^+$. On the other hand, if $p \to 1^-$, we obtain the distribution of a random variable Y such that $P(Y = 0) = 1$. Hence, the parameter p can be interpreted as a degeneration parameter, because the GE distribution converges to a distribution degenerated in zero, when p varies from zero to one. For $a = b = 1$, Equation (5) becomes the WG density function. In addition, if $c = 1$, we obtain the EG distribution. When p approaches zero (and $a = b = 1$), it leads to the Weibull distribution. If $c = 1$ in addition to $p \to 0^+$, the BWG distribution reduces to the BE distribution. The following distributions are new sub-models: the beta Rayleigh geometric (BRG), exponentiated Weibull geometric (EWG), BEG, exponentiated Rayleigh geometric, beta Rayleigh (BR) and Rayleigh geometric. Other sub-models are: BE, EW,

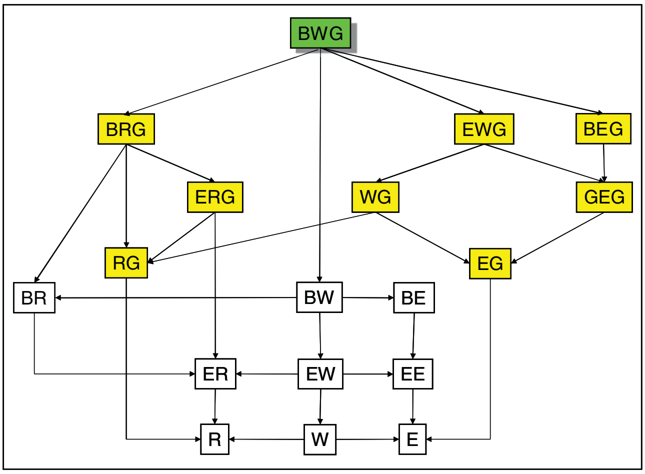

Figure 1. Relationships of the BWG sub-models.

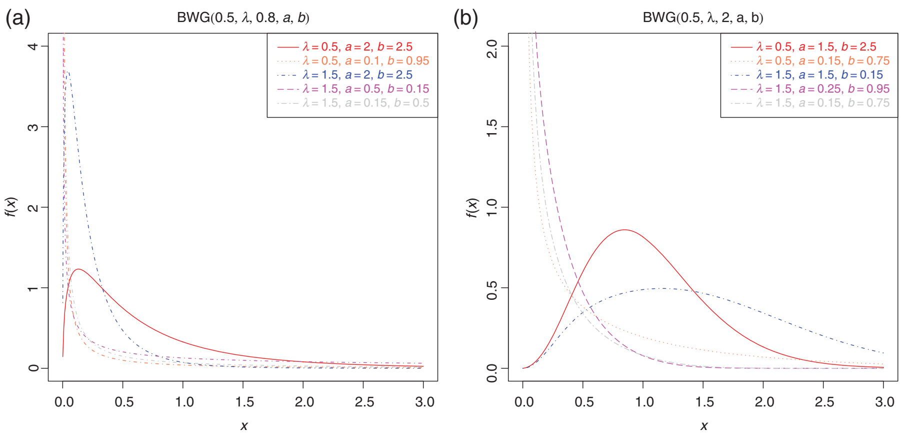

Figure 2. Plots of the BWG density function for some parameter values.

exponentiated Rayleigh (ER), exponentiated exponential (EE), Rayleigh (R), Weibull (W) and exponential (E). Several special sub-models of the BWG model are illustrated in Figure 1.

Plots of the BWG density function for selected parameter values are given in Figure 2. An important characteristic of the BWG distribution is that its density function can be bimodal for certain parameter values as shown in Figure 3. According to Barreto-Souza *et al.* [7], the WG failure rate function can only be increasing, decreasing or unimodal. However, the BWG failure rate function can also be bathtub shaped; see the plots of this function in Figure 4.

3. Expansion of the density function

For $|z| < 1$ and $\rho > 0$, we readily have

$$(1-z)^{-\rho} = \sum_{j=0}^{\infty} \frac{\Gamma(\rho+j)}{\Gamma(\rho)\, j!}\, z^j. \qquad (7)$$

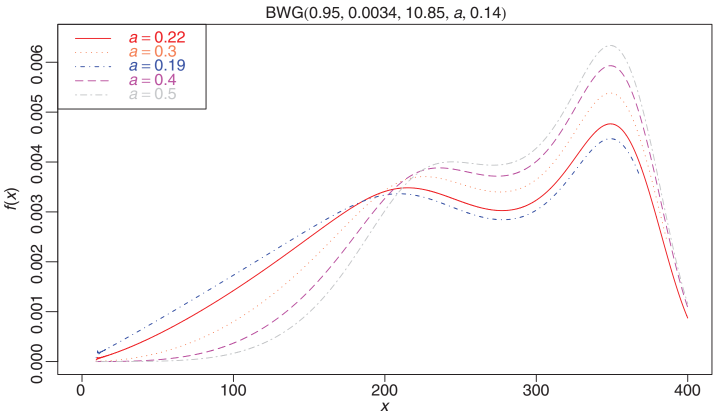

Figure 3. Plots of the BWG density function for some parameter values.

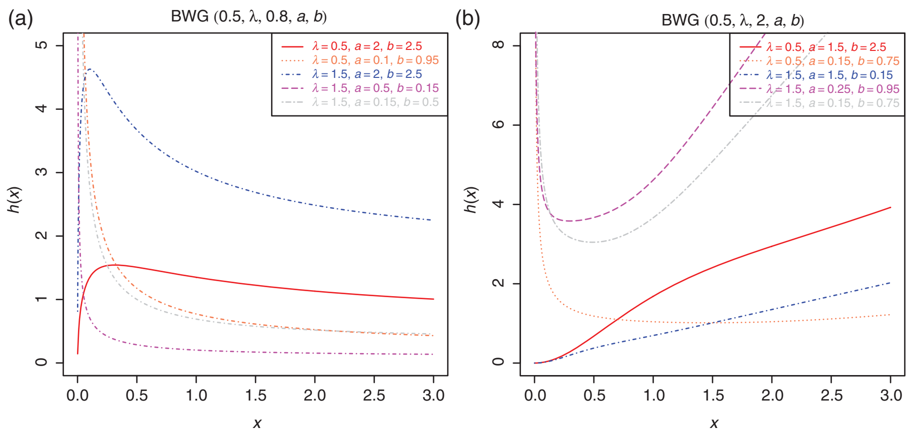

Figure 4. Plots of the BWG hazard rate function for some parameter values.

Applying Equation (7) into Equation (5) gives

$$f(x;\, p, \lambda, c, a, b) = \sum_{j=0}^{\infty} w_j \pi(x;\, \lambda, c, a, b + j). \tag{8}$$

Here, $\pi(x;\, \lambda, c, a, b)$ represents the BW(λ, c, a, b) density function [18] (with scale parameter $\lambda > 0$ and positive shape parameters c, a and b) given by

$$\pi(x;\, \lambda, c, a, b) = \frac{c\lambda^c x^{c-1}}{B(a, b)} \exp\{-b(\lambda x)^c\}\{1 - \mathrm{e}^{-(\lambda x)^c}\}^{a-1}, \tag{9}$$

whose coefficients w_j are

$$w_j = \frac{(1-p)^b\, p^j \Gamma(b + j)}{\Gamma(b)\, j!}.$$

We can verify that $\sum_{j=0}^{\infty} w_j = 1$. Equation (8) reveals that the BWG density function can be expressed as a mixture of BW densities. So, we can obtain some mathematical properties of the BWG distribution directly from those properties of the BW distribution. It is evident from Equation (8) that the BWG reduces to the BW distribution when $p = 0$.

4. Moments

Suppose $Y_j \sim \mathrm{BW}(\lambda, c, a, b + j)$, i.e. Y_j has the BW distribution with parameters λ, c, a and $b + j$. Cordeiro *et al.* [18] obtained the sth moment of Y_j for a real non-integer as

$$E(Y_j^s) = \frac{\Gamma(s/c+1)}{\lambda^s B(a, b+j)} \sum_{m=0}^{\infty} \frac{(-1)^m \binom{a-1}{m}}{(b+j+m)^{s/c+1}}. \tag{10}$$

If $a > 0$ is an integer, the index m in the above sum stops at $a - 1$. For $a = b = 1$, Equation (10) yields precisely the sth moment of the Weibull distribution. Let $X \sim \mathrm{BWG}(p, \lambda, c, a, b)$. By combining Equations (8) and (10), the sth moment of X can be expressed as

$$\mu_s' = E(X^s) = \lambda^{-s}\, \Gamma\left(\frac{s}{c}+1\right) \sum_{j,m=0}^{\infty} \frac{(-1)^m \binom{a-1}{m} w_j}{B(a, b+j)\,(b+j+m)^{s/c+1}}. \tag{11}$$

The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. Graphical representation of these quantities for some choices of parameter b as function of a, and for some choices of parameter a as function of b, by fixing $\lambda = 2$, $c = 3$ and $p = 0.3$, are given in Figures 5 and 6, respectively. These figures reveal that the skewness and kurtosis curves increase when b decreases for fixed a and decrease when a decreases for fixed b.

The central moments (μ_p) and cumulants (κ_p) of X can be obtained from Equation (11) by

$$\mu_p = \sum_{k=0}^{p} \binom{p}{k} (-1)^k\, \mu_1'^{p}\, \mu_{p-k}' \quad \text{and} \quad \kappa_p = \mu_p' - \sum_{k=1}^{p-1} \binom{p-1}{k-1} \kappa_k\, \mu_{p-k}',$$

respectively, where $\kappa_1 = \mu_1'$. Thus, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$, etc. The pth descending factorial moment of X is

$$\mu_{(p)}' = E[X^{(p)}] = E[X(X-1) \times \ldots \times (X-p+1)] = \sum_{m=0}^{p} s(p, m)\, \mu_m',$$

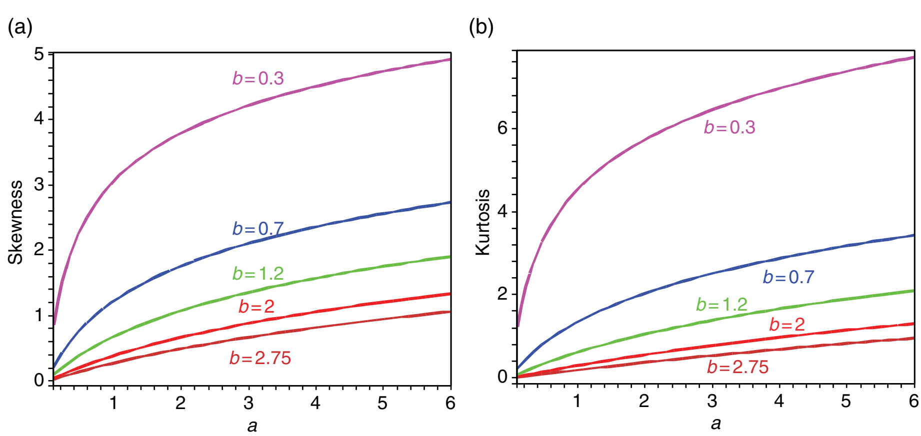

Figure 5. Skewness and kurtosis of the BWG distribution as a function of a for some values of b.

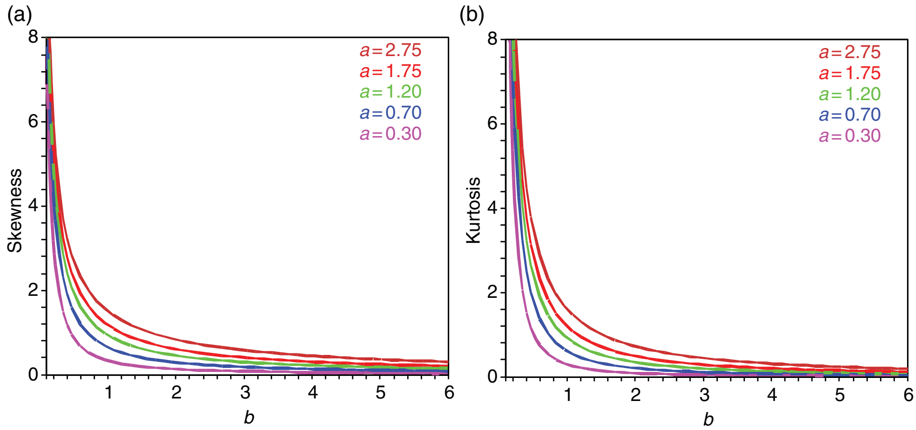

Figure 6. Skewness and kurtosis of the BWG distribution as a function of b for some values of a.

where $s(r,m) = (m!)^{-1}[d^m m^{(r)}/dx^m]_{x=0}$ is the Stirling number of the first kind. The factorial moments of X are given by

$$\mu'_{(p)} = \sum_{j,k=0}^{\infty} \sum_{m=0}^{p} \frac{(-1)^k \binom{a-1}{k} w_j\, s(p,m)\, \Gamma\left(\frac{m}{c}+1\right)}{\lambda^m\, B(a, b+j)\,(b+j+k)^{m/c+1}}.$$

5. Moment generating function

An expression for the mgf of a random variable X having the BWG distribution can be obtained from Equation (8) and the mgf of the BW distribution. If Y is a random variable having the BW(λ, c, a, b) density function (9), the mgf of Y, say $M_Y(t) = E[\exp(tY)]$, was determined by Cordeiro *et al.* [18] using the the Wright generalized hypergeometric function defined by

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p)\\ (\beta_1, B_1), \ldots, (\beta_q, B_q)\end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{x^n}{n!}.$$

They demonstrated that

$$M_Y(t) = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \binom{a-1}{j}}{(b+j)}\, {}_1\Psi_0\left[\begin{matrix}(1, 1/c)\\ -\end{matrix}; \frac{t}{\lambda(b+j)^{1/c}}\right], \tag{12}$$

provided that $c > 1$. Clearly, special formulas for the mgf of the Weibull, BE, EW, EE and GE can be obtained from Equation (12) by substitution of known parameters. So, we can express the mgf of X from Equations (8) and (12) as

$$M_X(t) = \sum_{j,m=0}^{\infty} \frac{(-1)^j \binom{a-1}{j} w_m}{(b+m+j) B(a, b+m)}\, {}_1\Psi_0\left[\begin{matrix}(1, 1/c)\\ -\end{matrix}; \frac{t}{\lambda(b+m+j)^{1/c}}\right]. \tag{13}$$

An alternative expansion for the mgf of X follows from the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x\,\middle|\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right)=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+t)\prod_{j=1}^{n}\Gamma(1-a_j-t)}{\prod_{j=n+1}^{p}\Gamma(a_j+t)\prod_{j=m+1}^{p}\Gamma(1-b_j-t)}x^{-t}\,\mathrm{d}t,$$

where $i=\sqrt{-1}$ is the complex unit and L denotes an integration path ([19, Section 9.3] for a description of this path). The Meijer G-function contains many integrals with elementary and special functions. Some of these integrals are included in Prudnikov *et al.* [20].

We assume the condition $c=r/s$, where $r\geq 1$ and $s\geq 1$ are co-prime integers, which is not too restrictive since every real number can be approximated by a rational number. Using the formula $\exp\{-g(x)\}=G_{0,1}^{1,0}(g(x)\mid {}^{-}_{0})$ for $g(\cdot)$ an arbitrary function and a result in Prudnikov *et al.* [20, Volume 3, Equation (2.24.1.1)], Cordeiro *et al.* [18] showed that

$$M_Y(t)=\frac{c\lambda^c r^{c-1/2}(-t)^{-c}}{(2\pi)^{(r+s)/2-1}B(a,b)}\sum_{j=0}^{\infty}(-1)^j\binom{a-1}{j}G_{s,r}^{r,s}\times\left(\frac{\lambda^s c^s(b+j)^{s/c}r^r}{(-t)^r s^s}\,\middle|\,\begin{matrix}\frac{1-c}{r},\frac{2-c}{r},\ldots,\frac{r-c}{r}\\ 0,\frac{1}{s},\ldots,\frac{s-1}{s}\end{matrix}\right).\qquad(14)$$

The mgf of X can be obtained from Equations (8) and (14) as

$$M_X(t)=K(t)\sum_{j,m=0}^{\infty}\frac{(-1)^j\binom{a-1}{j}w_m}{B(a,b+m)}G_{s,r}^{r,s}\times\left(\frac{\lambda^s c^s(b+m+j)^{s/c}r^r}{(-t)^r s^s}\,\middle|\,\begin{matrix}\frac{1-c}{r},\frac{2-c}{r},\ldots,\frac{r-c}{r}\\ 0,\frac{1}{s},\ldots,\frac{s-1}{s}\end{matrix}\right),\qquad(15)$$

where $K(t)=c\lambda^c r^{c-1/2}(-t)^{-c}/(2\pi)^{(r+s)/2-1}$. Equations (13) and (15) are the main results of this section.

6. Quantile function

The quantile function, say $x=Q(z;p,\lambda,c,a,b)=F^{-1}(z;p,\lambda,c,a,b)$, of the BWG distribution follows by inverting Equation (4) as

$$x=Q(z;p,\lambda,c,a,b)=\lambda^{-1}\left[\log\left(\frac{1-pw}{1-w}\right)\right]^{1/c},\qquad(16)$$

where $w=Q_{a,b}(z)$ denotes the quantile function of the beta distribution with parameters a and b.

We can obtain some expansions for $Q_{a,b}(z)$ in the Wolfram website[1] such as

$$\begin{aligned} Q_{a,b}(z) &= v + \frac{(b-1)}{(a+1)}v^2 + \frac{(b-1)(a^2+3ba-a+5b-4)}{2(a+1)^2(a+2)}v^3 \\ &\quad + \frac{\begin{array}{c}(b-1)[a^4+(6b-1)a^3+(b+2)(8b-5)a^2\\ +(33b^2-30b+4)a+b(31b-47)+18]\end{array}}{3(a+1)^3(a+2)(a+3)}v^4 \\ &\quad + O(v^5), \end{aligned} \tag{17}$$

where $v = [a\,B(a,b)\,z]^{1/a}$ for $a > 0$.

The simulation of the BWG distribution is very easy: if T is a random variable having a beta distribution with parameters a and b, then the variate $X = \lambda^{-1}[\log\{(1-pT)(1-T)^{-1}\}]^{1/c}$ defined from Equation (16) follows the BWG distribution (4).

7. Mean deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. If X has the BWG distribution (5), we can derive the mean deviations about the mean $\mu_1' = E(X)$ and about the median M from

$$\delta_1 = \int_0^\infty |\, x - \mu_1' \,|\, f(x; p, \lambda, c, a, b)\,\mathrm{d}x \quad \text{and} \quad \delta_2 = \int_0^\infty |\, x - M \,|\, f(x; p, \lambda, c, a, b)\,\mathrm{d}x,$$

respectively. The median M is the solution of the nonlinear equation

$$I_{\left[\frac{1-\exp\{-(\lambda x)^c\}}{1-p\exp\{-(\lambda x)^c\}}\right]}(a, b) = \frac{1}{2}.$$

These measures can be determined from the relationships

$$\delta_1 = 2[\mu_1' F(\mu_1') - J(\mu_1')] \quad \text{and} \quad \delta_2 = \mu_1' - 2J(M), \tag{18}$$

where $F(\mu_1') = F(\mu_1'; p, \lambda, c, a, b)$ is easily calculated from Equation (4) and $J(q) = \int_0^q x\, f(x; p, \lambda, c, a, b)\,\mathrm{d}x$. From Equation (8) and using the incomplete gamma function $\gamma(\alpha, z) = \int_0^z u^{\alpha-1}\mathrm{e}^{-u}\,\mathrm{d}u$, we can obtain

$$J(q) = \sum_{j,m=0}^{\infty} w_j\, \kappa_j\, \gamma(1 + c^{-1}, (b+j+m)(\lambda q)^c), \tag{19}$$

where

$$\kappa_j = \frac{(-1)^m\, \Gamma(a+b+j)}{\lambda\, m!\, \Gamma(b+j)\Gamma(a-m)(b+j+m)^{(1/c)+1}}.$$

Equation (19) is the basic quantity to calculate the mean deviations δ_1 and δ_2 in Equation (18). It can also be used to determine Bonferroni and Lorenz curves which have applications in fields like economics, reliability, demography, insurance and medicine. They are defined by

$$B(\pi) = \frac{J(q)}{\pi \mu_1'} \quad \text{and} \quad L(\pi) = \frac{J(q)}{\mu_1'},$$

respectively, where $q = Q(\pi; p, \lambda, c, a, b)$ is computed from Equation (16) for a given probability π.

8. Rényi entropy

The entropy of a random variable X is a measure of the uncertainty variation. The Rényi entropy is defined as $I_R(\gamma) = 1/(1-\gamma)\log\{\int_{\mathbb{R}} f^{\gamma}(x)\,\mathrm{d}x\}$, where $\gamma > 0$ and $\gamma \neq 1$. We obtain from Equation (5)

$$f^{\gamma}(x) = \frac{c^{\gamma}(1-p)^{b\gamma}\lambda^{c\gamma}x^{(c-1)\gamma}u^{b\gamma}(1-u)^{(a-1)\gamma}(1-pu)^{-(a+b)\gamma}}{B(a,b)^{\gamma}}.$$

By expanding $(1-u)^{(a-1)\gamma}$ and $(1-pu)^{-(a+b)\gamma}$ (as in Equation (7)), we obtain

$$f^{\gamma}(x) = x^{(c-1)\gamma}\sum_{i,j=0}^{\infty} v_{i,j}\,\mathrm{e}^{-(b\gamma+i+j)(\lambda x)^{c}},$$

where

$$v_{i,j} = \frac{(-1)^{i}\,p^{j}\,c^{\gamma}\lambda^{c\gamma}\Gamma((a-1)\gamma+1)\Gamma((a+b)\gamma+j)}{\Gamma((a+b)\gamma)B(a,b)^{\gamma}\Gamma((a-1)\gamma+1-i)i!\,j!}.$$

But, for $a > 0$,

$$\int_0^{\infty} x^{a-1}\,\exp\left(-\delta x^{c}\right)\,\mathrm{d}x = c^{-1}\,\delta^{-a/c}\,\Gamma\left(\frac{a}{c}\right)$$

and then

$$\int_0^{\infty} f^{\gamma}(x)\,\mathrm{d}x = c^{-1}\,\lambda^{-[(c-1)\gamma+1]}\,\Gamma\left(\frac{(c-1)\gamma+1}{c}\right)\sum_{i,j=0}^{\infty} v_{i,j}\,(b\gamma+i+j)^{-[(c-1)\gamma+1]/c}.$$

The Rényi entropy $I_R(\gamma)$ follows immediately from this equation.

9. Order statistics

We obtain an explicit expression for the density of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$, in a random sample of size n from the BWG distribution. It is well-known that

$$f_{i:n}(x) = \frac{f(x)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^{j}\binom{n-i}{j}F(x)^{i+j-1},$$

for $i = 1,\ldots,n$. For a beta-G model defined from the parent functions $g(x)$ and $G(x)$, $f_{i:n}(x)$ can be written as

$$f_{i:n}(x) = \frac{g(x)G(x)^{a-1}\{1-G(x)\}^{b-1}}{B(a,b)B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^{j}\binom{n-i}{j}F(x)^{i+j-1}. \tag{20}$$

The incomplete beta function expansion for b real non-integer gives

$$F(x) = I_{G(x)}(a,b) = \frac{G(x)^{a}}{B(a,b)}\sum_{m=0}^{\infty}\frac{(1-b)_m G(x)^{m}}{(a+m)m!} = G(x)^{a}\sum_{m=0}^{\infty} a_m\,G(x)^{m},$$

where

$$a_m = \frac{(1-b)_m}{B(a,b)(a+m)m!}$$

and $(f)_k = f(f+1)\ldots(f+k-1) = \Gamma(f+k)/\Gamma(f) = (-1)^k\Gamma(1-f)/\Gamma(1-f-k)$ is the Pochhammer symbol.

We use an equation of Gradshteyn and Ryzhik [19, Section 0.314] for a power series raised to a positive integer j

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)^j = \sum_{i=0}^{\infty} c_{j,i}\, x^i, \tag{21}$$

whose coefficients $c_{j,i}$ (for $i = 1, 2, \ldots$) are easily obtained from the recurrence equation

$$c_{j,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} (jm - i + m) a_m c_{j,i-m} \tag{22}$$

and $c_{j,0} = a_0^j$. Hence, $c_{j,i}$ can be calculated from $c_{j,1}, \ldots, c_{j,i-1}$ and then from $a_0, \ldots, a_i$. The coefficients $c_{j,i}$ can be given explicitly in terms of the a_i's although it is not necessary for programming numerically our expansions in any algebraic or numerical software. By Equation (21), we can express

$$F(x)^{i+j-1} = \left(\sum_{m=0}^{\infty} a_m\, G(x)^m\right)^{i+j-1} = \sum_{m=0}^{\infty} c_{i+j-1,m}\, G(x)^m,$$

where the coefficients $c_{i+j-1,m}$ follow from Equation (22), and then

$$f_{i:n}(x) = \frac{g(x)G(x)^{a-1}\{1-G(x)\}^{b-1}}{B(a,b)B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \sum_{m=0}^{\infty} c_{i+j-1,m}\, G(x)^m.$$

From Equation (5), we have

$$f_{i:n}(x) = \frac{c(1-p)^b \lambda^c x^{c-1} u^b}{B(a,b)B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j}$$
$$\times \sum_{m=0}^{\infty} c_{i+j-1,m}\, (1-u)^{a+m-1} (1-pu)^{-(a+b+m)}.$$

Using two binomial expansion twice, we can write

$$f_{i:n}(x) = \sum_{r,s,m=0}^{\infty} \frac{(-1)^s p^r v_{i,n,m} \Gamma(a+m)\Gamma(a+b+m+r)}{\Gamma(a+b+m)\Gamma(a+m-s)(b+r+s)r!\,s!}\, g_{(b+r+s)^{1/c}\lambda,c}(x), \tag{23}$$

where

$$v_{i,n,m} = \frac{(1-p)^b}{B(a,b)B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} c_{i+j-1,m}$$

and $g_{\lambda,c}(x)$ denotes (as before) the Weibull density function with scale parameter λ and shape parameter c. Equation (23) shows that the density function of the BWG order statistics can be expressed as a linear combination of Weibull density functions.

Moments of order statistics play an important role in quality control testing and reliability, where a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictors are often based on moments of order statistics. The ordinary, inverse and factorial moments of the BWG order statistics can be calculated from a weighted infinite linear combination of those quantities for Weibull distributions. For example, the hth moment about zero of a Weibull random variable T with parameters λ and c is $E(T^h) = \lambda^{-h}\Gamma(h/c + 1)$ and then we immediately obtain the hth generalized moment of $X_{i:n}$ as

$$E(X_{i:n}^h) = \lambda^{-h}\Gamma\left(\frac{h}{c} + 1\right) \sum_{r,s,m=0}^{\infty} \frac{(-1)^s p^r (b+r+s)^{h/c-1} v_{i,n,m} \Gamma(a+m)\Gamma(a+b+m+r)}{\Gamma(a+b+m)\Gamma(a+m-s)\, r!\, s!}. \tag{24}$$

10. Maximum-likelihood estimation

We determine the maximum-likelihood estimates (MLEs) of the parameters of the BWG distribution from complete samples only. Let $x_1, \ldots, x_n$ be a random sample of size n from the BWG(p, λ, c, a, b) distribution. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (p, \lambda, c, a, b)^{\mathrm{T}}$ can be written as

$$\begin{aligned} l(\boldsymbol{\theta}) &= n[\log(c) + b\log(1-p) + c\log(\lambda)] - n\log[B(a,b)] + (c-1)\sum_{i=1}^{n}\log(x_i) \\ &\quad + b\sum_{i=1}^{n}\log(u_i) + (a-1)\sum_{i=1}^{n}\log(1-u_i) - (a+b)\sum_{i=1}^{n}\log(1-p\,u_i), \end{aligned} \tag{25}$$

where $u_i = \exp\{-(\lambda x_i)^c\}$ is a transformed observation. The log-likelihood can be maximized either directly by using the SAS (PROC NLMIXED) or the MaxBFGS routine in the matrix programming language Ox (see, [21]) or by solving the nonlinear likelihood equations obtained by differentiating Equation (25). The components of the score vector $U(\boldsymbol{\theta})$ are given by

$$U_p(\boldsymbol{\theta}) = -\frac{nb}{(1-p)} + (a+b)\sum_{i=1}^{n}\frac{u_i}{(1-pu_i)},$$

$$U_\lambda(\boldsymbol{\theta}) = \frac{nc}{\lambda} - \frac{bc}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^c + \frac{(a-1)c}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_i}{(1-u_i)} - \frac{(a+b)pc}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_i}{(1-pu_i)},$$

$$\begin{aligned} U_c(\boldsymbol{\theta}) &= n[c^{-1} + \log(\lambda)] + \sum_{i=1}^{n}\log(x_i) - b\sum_{i=1}^{n}(\lambda x_i)^c\log(\lambda x_i) + (a-1)\sum_{i=1}^{n}\frac{(\lambda x_i)^c\log(\lambda x_i)u_i}{(1-u_i)} \\ &\quad - (a+b)p\sum_{i=1}^{n}\frac{(\lambda x_i)^c\log(\lambda x_i)u_i}{(1-pu_i)}, \end{aligned}$$

$$U_a(\boldsymbol{\theta}) = -n[\psi(a) + \psi(a+b)] + \sum_{i=1}^{n}\log(1-u_i) - \sum_{i=1}^{n}\log(1-pu_i),$$

$$U_b(\boldsymbol{\theta}) = -n[\psi(b) + \psi(a+b)] + \sum_{i=1}^{n}\log(u_i) - \sum_{i=1}^{n}\log(1-pu_i),$$

where $\psi(\cdot)$ is the digamma function.

For interval estimation and hypothesis tests on the model parameters, we require the 5×5 observed information matrix $J = J(\boldsymbol{\theta})$ given in the appendix. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $N_5(0, I(\boldsymbol{\theta})^{-1})$, where $I(\boldsymbol{\theta})$ is the expected information matrix. In practice, we can replace $I(\boldsymbol{\theta})$ by the observed information matrix evaluated at $\hat{\boldsymbol{\theta}}$ (say $J(\hat{\boldsymbol{\theta}})$). We can construct approximate confidence intervals and confidence regions for the individual parameters and for the hazard and survival functions based on the multivariate normal $N_5(0, J(\hat{\boldsymbol{\theta}})^{-1})$ distribution.

Further, the likelihood ratio (LR) statistic can be used for comparing this distribution with some of its special sub-models. We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct the LR statistics for testing some sub-models of the BWG distribution. For example, the test of $H_0 : a = b = 1$ versus H : H_0 *is not true* is equivalent to compare the BWG and WG distributions and the LR statistic reduces to

$$w = 2\{\ell(\hat{p}, \hat{\lambda}, \hat{c}, \hat{a}, \hat{b}) - \ell(\tilde{p}, \tilde{\lambda}, \tilde{c}, 1, 1)\},$$

where $\hat{p}$, $\hat{\lambda}$, $\hat{c}$, $\hat{a}$ and $\hat{b}$ are the MLEs under H and $\tilde{p}$, $\tilde{\lambda}$ and $\tilde{c}$ are the estimates under H_0.

11. Residual analysis

In order to study departures from the error assumption and the presence of outliers, there are various residuals proposed in the literature (see [22]). Here, we consider the Martingale-type residual.

11.1. *Martingale-type residual*

The Martingale residual introduced in the counting process [23] can be expressed as

$$r_{M_i} = \delta_i + \log[1 - F(x_i; \hat{\boldsymbol{\theta}})],$$

where $\delta_i = 0$ denotes the censored observation, $\delta_i = 1$ uncensored and $\hat{\boldsymbol{\theta}}$ is the MLE of the parameter vector $\boldsymbol{\theta}$. For the BWG model, $F(x_i; \hat{\boldsymbol{\theta}})$ is defined by Equation (4). It takes values between $+1$ and $-\infty$, and then its distribution is skewed. Hence, another possibility is to use a transformation of the Martingale residual in order to have a new residual symmetrically distributed around zero. Therneau *et al.* [24] proposed the deviance residuals, which, for the Cox model with no time-dependent explanatory variables, can be written as

$$r_{D_i} = \text{sign}(r_{M_i})[-2\{r_{M_i} + \delta_i \log(\delta_i - r_{M_i})\}]^{1/2}, \tag{26}$$

where r_{M_i} is the Martingale residual. In the BWG model, the residual given in Equation (26) is not a component of the deviance, but we will use it in order to have a transformation of the Martingale residual.

We recommend the use of normal probability plots for r_{D_i} with the simulated envelope as suggested by Atkinson [25] that can be produced as follows: (i) fit the model and generate a sample of n independent observations from the fitted model as if it were the true model; (ii) fit the model to the generated sample and compute the values of the residuals; (iii) repeat steps (i) and (ii) m times; (iv) obtain ordered values of the residuals, $r^*_{(i)v}$, $i = 1, 2, \ldots, n$ and $v = 1, 2, \ldots, m$; (iv) consider n sets of the m ordered statistics; for each set compute its average, 5th percentile and 95th percentile; (v) plot these values and the ordered residuals of the original sample against the normal scores. The minimum and maximum values of the m ordered statistics yield the envelope. Observations corresponding to residuals outside the limits provided by the simulated envelope are worthy of further investigation. Additionally, if a considerable proportion of points falls outside the envelope, then we have evidence against the adequacy of the fitted model.

12. Application: pollution data

The data from the New York State Department of Conservation correspond to the daily ozone level measurements in New York in May–September, 1973. Recently, Nadarajah [26] and Leiva *et al.* [27] analysed these data using a truncated inverted beta and an extended Birnbaum–Saunders (BS) distribution, respectively. We fit a BWG distribution to these data. Table 1 lists the MLEs of the model parameters. Since the value of the Akaike information criterion (AIC) is smaller for the BWG distribution compared with those values of the other sub-models, the new distribution seems to be a very competitive model to these data.

The inverted beta and truncated inverted beta distributions are very popular models in hydrology. Nadarajah [26] fitted a truncated version of the inverted beta distribution to air pollution data from New York. Its density function is given by

- *Inverted beta distribution*

$$g(x) = \frac{x^{\alpha-1}}{B(\alpha,\beta)(1+x)^{\alpha+\beta}}, \quad x > 0,$$

for $\alpha > 0$ and $\beta > 0$. The corresponding cdf is

$$G(x) = \frac{B_{x/(1+x)}(\alpha,\beta)}{B(\alpha,\beta)}.$$

- *Truncated inverted beta distribution*

The pdf of the truncated version (suggested by Nadarajah [26]) is given by

$$f(x) = \frac{x^{\alpha-1}}{d\, B(\alpha,\beta)(1+x)^{\alpha+\beta}}, \tag{27}$$

for $0 \le B \le x \le A < \infty$, where $d = G(A) - G(B)$. The cdf corresponding to Equation (27) becomes $F(x) = [G(x) - G(B)]/d$. Nadarajah [26] refers to Equation (27) as the truncated inverted beta distribution.

Table 1. MLEs of the model parameters for the daily ozone level data, the corresponding SEs (given in parentheses) and the AIC statistics.

Model	p	λ	c	a	b	AIC
BWG	0.9580	0.0175	2.6166	0.8619	0.3367	1086.6
	(0.0482)	(0.0040)	(0.7082)	(0.3788)	(0.1064)	
WG	0.7555	0.0128	1.7366	1	1	1087.4
	(0.1744)	(0.0033)	(0.2076)	–	–	
BW	–	0.1655	1.1084	3.122	0.1276	1088.8
	–	(0.095)	(0.1743)	(1.0061)	(0.0127)	
Weibull	–	0.0217	1.3402	1	1	1089.2
	–	(0.00159)	(0.0954)	–	–	

An alternative approach for modelling these data can be provided by the BS distribution. There are various extensions of this lifetime distribution; see, for example, the BS-t-Student [28] and extended BS (EBS) [27] distributions, respectively, among others. The BS density function is given by

- *BS distribution*

$$f(x) = \frac{x^{-3/2}(x+\beta)}{2\alpha\sqrt{2\beta\pi}} \exp\left\{-\frac{1}{2\alpha^2}\left(\frac{x}{\beta}+\frac{\beta}{x}-2\right)\right\}, \quad x > 0,$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter.

- *BS-t-Student distribution*

$$f(x) = \frac{\Gamma((\nu+1)/2)\,(x+\beta)}{2\alpha\sqrt{\beta\nu\pi}\,\Gamma(\nu/2)\sqrt{x^3}}\left[1+\frac{1}{\nu\alpha^2}\left(\frac{x}{\beta}+\frac{\beta}{x}-2\right)\right]^{-(\nu+1)/2}, \quad x > 0,$$

where ν is the number of degrees of freedom of the t distribution. See, for example, Díaz-García and Leiva [28] and Barros *et al.* [29].

- *EBS distribution*

$$f(x) = 2\,c'_x\,\phi(c_x)\,\Phi(\lambda c_x), \quad x > 0,$$

where

$$c_x = \nu + \frac{(x/\beta)^{1/\sigma} - (\beta/x)^{1/\sigma}}{\alpha}, \quad c'_x = \frac{x^{2/\sigma}+\beta^{2\sigma}}{\sigma\alpha\beta^{1/\sigma}x^{1+1/\sigma}},$$

and $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal pdf and cdf, respectively. Here, the shape parameters are $\alpha > 0$, $\nu \in \mathbf{R}$ and $\lambda \in \mathbf{R}$, β is a scale parameter and $\sigma > 0$ is a power parameter; see, Leiva *et al.* [27].

Table 2 lists the MLEs of the parameters (the standard errors are given in parentheses) for the inverted beta, truncated inverted beta, BS, BS-t-Student and EBS distributions fitted to daily ozone level data and the values of the AIC statistic. These numerical results are obtained using the SAS (PROC NLMIXED). Based on the AIC criterion, we conclude that the BWG distribution provides a superior fit to these data than the other models.

In order to assess if the model is appropriate, we plot in Figure 7(a) and (b) the histogram of the data and the fitted BWG and WG distributions and the empirical and their estimated survival functions, respectively. These plots indicate that the BWG distribution provides a better fit to these data. In addition, the normal probability plot for the modified Martingale-type residual ($r_{\mathrm{MD}i}$) with the generated envelope is presented in Figure 8. This plot provides further evidence that the BWG distribution is adequate to fit these data.

Table 2. MLEs of the model parameters for the daily ozone level data and the AIC statistics.

Model	α	β				AIC
Inverted beta	28.576	1.379				1121.8
Truncated inverted beta	18.539	0.704				1098.1
	α	β	ν	λ	σ	AIC
BS	0.982	28.093	–	–	–	1102.2
BS-t-Student	0.810	30.878	7.241	–	–	1092.8
EBS distribution	0.780	0.610	−3.530	−0.090	3.620	1092.0

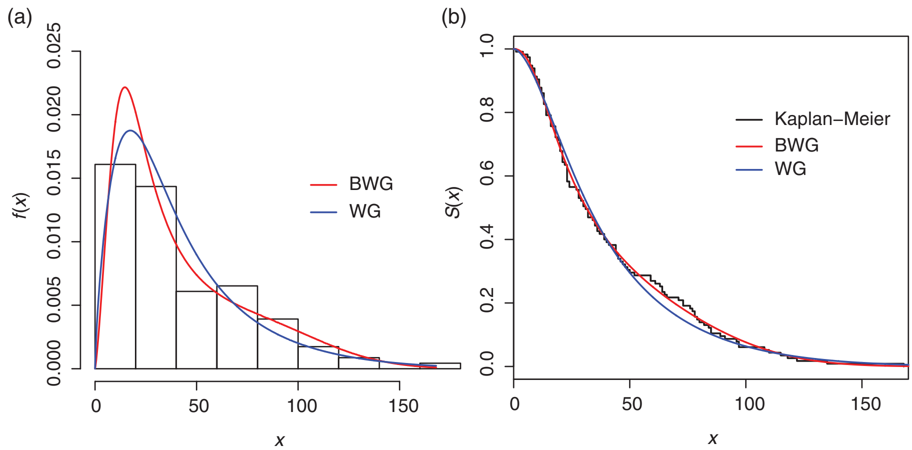

Figure 7. (a) Estimated densities of the BWG and WG models for daily ozone level data. (b) Estimated survival function from the fitted BWG and WG distributions and the empirical survival for daily ozone level data.

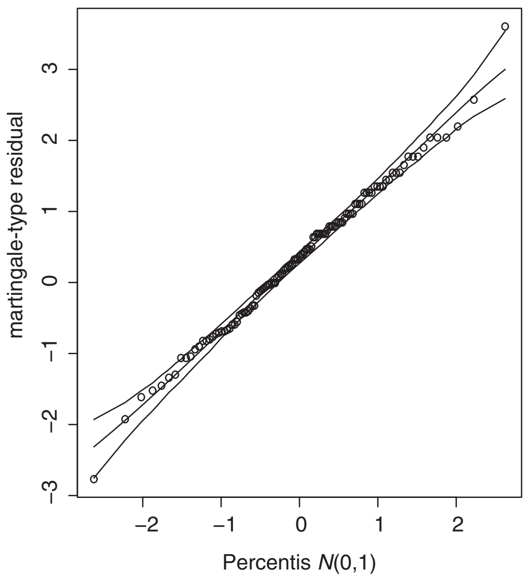

Figure 8. Normal probability plot for the Martingale-type residual from the BWG model fitted to the pollution data.

13. Concluding remarks

The Weibull distribution is commonly used to model the lifetime of a system. However, it does not exhibit a bathtub-shaped failure rate function and thus it cannot be used to model the complete lifetime of the system. We propose a new model called the BWG distribution, whose failure rate function can be increasing, decreasing and upside-down bathtub that extends the WG and BW distributions introduced by Barreto-Souza *et al.* [7] and Cordeiro *et al.* [18], respectively, among other distributions. The BWG distribution is quite flexible in analysing positive data in place of some other special sub-models. Its density function can be expressed as a mixture of Weibull densities. We provide a mathematical treatment of the distribution including expansions for the density function, generating function, mean deviations, Bonferroni and Lorenz curves, order statistics and their moments. The estimation of the parameters is approached by the method of maximum likelihood and the observed information matrix is calculated. An application shows that the BWG distribution could provide a better fit than other well-known lifetime models. We hope that the new model may attract wider applications in statistics.

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Note

1. http://functions.wolfram.com/06.23.06.0004.01

References

[1] K. Adamidis and S. Loukas, *A lifetime distribution with decreasing failure rate*, Statist. Probab. Lett. 39 (1998), pp. 35–42.
[2] R.D. Gupta and D. Kundu, *Generalized exponential distribution*, Aust. N. Z. J. Stat. 41 (1999), pp. 173–188.
[3] R.D. Gupta and D. Kundu, *Exponentiated exponential family: An alternative to gamma and Weibull distributions*, Biom. J. 43 (2001a), pp. 117–130.
[4] R.D. Gupta and D. Kundu, *Generalized exponential distribution: Different method of estimations*, J. Stat. Comput. Simul. 69 (2001b), pp. 315–337.
[5] R.B. Silva, W. Barreto-Souza, and G.M. Cordeiro, *A new distribution with decreasing, increasing and upside-down bathtub failure rate*, Comput. Statist. Data Anal. 54 (2010), pp. 935–944.
[6] G.O. Silva, E.M.M. Ortega, and G.M. Cordeiro, *The beta modified Weibull distribution*, Lifetime Data Anal. 16 (2010), pp. 409–430.
[7] W. Barreto-Souza, A.L. de Morais, and G.M. Cordeiro, *The Weibull-geometric distribution*, J. Stat. Comput. Simul. 81 (2011), pp. 645–657.
[8] S. Nadarajah and S. Kotz, *The beta exponential distribution*, Reliab. Eng. Syst. Saf. 91 (2006), pp. 689–697.
[9] N. Eugene, C. Lee, and F. Famoye, *Beta-normal distribution and its applications*, Comm. Statist. Theory Methods 31 (2002), pp. 497–512.
[10] M.C. Jones, *Family of distributions arising from distribution of order statistics*, Test 13 (2004), pp. 1–43.
[11] S. Nadarajah and S. Kotz, *The beta Gumbel distribution*, Math. Probl. Eng. 10 (2004), pp. 323–332.
[12] S. Nadarajah and A.K. Gupta, *The beta Fréchet distribution*, Far East J. Theor. Stat. 14 (2004), pp. 15–24.
[13] S. Nadarajah and S. Kotz, *The exponentiated type distributions*, Acta Appl. Math. 92 (2006), pp. 97–111.
[14] C.D. Lai, M. Xie, and D.N.P. Murthy, *A modified Weibull distribution*, Trans. Reliab. 52 (2003), pp. 33–37.
[15] G.S. Mudholkar, D.K. Srivastava, and M. Freimer, *The exponentiated Weibull family*, Technometrics 37 (1995), pp. 436–445.
[16] V.G. Cancho, H. Bolfarine, and J.A. Achcar, *A Bayesian analysis of the exponentiated Weibull distribution*, J. Appl. Statist. Sci. 8 (1999), pp. 277–242.
[17] J.M.F. Carrasco, E.M.M. Ortega, and G.M. Cordeiro, *A generalized modified Weibull distribution for lifetime modeling*, Comput. Statist. Data Anal. 53 (2008), pp. 450–462.
[18] S. Nadarajah, G.M. Cordeiro, and E.M.M. Ortega, *General results for the Kumaraswamy-G distribution*, J. Stat. Comput. Simul. (2011). DOI:10.1080/00949655.2011.562504.
[19] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 2000.
[20] A.P. Prudnikov, Y.A. Brychkov, and O.I. Marichev, *Integrals and Series*, Vol. 1, Gordon and Breach Science Publishers, Amsterdam, 1986.
[21] J.A. Doornik, *An Object-Oriented Matrix Language Ox 5*, Timberlake Consultants Press, London, 2007.
[22] D. Collet, *Modelling Survival Data in Medical Research*, Chapman and Hall, London, 2003.
[23] T.R. Fleming and D.P. Harrington, *Counting Process and Survival Analysis*, Wiley, New York, 1991.
[24] T.M. Therneau, P.M. Grambsch, and T.R. Fleming, *Martingale-based residuals for survival models*, Biometrika 77 (1990), pp. 147–60.
[25] A.C. Atkinson, *Plots, Transformations and Regression: An Introduction to Graphical Methods of Diagnostics Regression Analysis*, 2nd ed., Clarendon Press, Oxford, 1985.
[26] S. Nadarajah, *A truncated inverted beta distribution with application to air pollution data*, Stoch. Environ. Res. Risk Assess. 22 (2008), pp. 285–289.
[27] V. Leiva, F. Vilca, N. Balakrishnan, and A. Sanhueza, *A skewed sinh-normal distribution and its properties and application to air pollution*, Comm. Statist. Theory Methods 39 (2010), pp. 426–443.
[28] J.A. Díaz-García and V. Leiva, *A new family of life distributions based on elliptically contoured distributions*, J. Statist. Plann. Inference 128 (2005), pp. 445–457.
[29] M. Barros, G.A. Paula, and V. Leiva, *An R implementation for generalized Birnbaum–Saunders distributions*, Comput. Statist. Data Anal. 53 (2009), pp. 1511–1528.

Appendix

Let $u_i = \exp\{-(\lambda x_i)^c\}$. The elements of the observed information matrix $J(\boldsymbol{\theta})$ for the parameters (p, λ, c, a, b) are

$$J_{pp} = -\frac{nb}{(1-p)^2} + (a+b)\sum_{i=1}^{n}\frac{u_i^2}{(1-pu_i)^2},$$

$$J_{p\lambda} = -\frac{(a+b)c}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_i}{(1-pu_i)} - \frac{(a+b)pc}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_i^2}{(1-p_i)^2},$$

$$J_{pc} = -(a+b)\sum_{i=1}^{n}\frac{(\lambda x_i)^c \log(\lambda x_i) u_i}{(1-pu_i)} - (a+b)p\sum_{i=1}^{n}\frac{(\lambda x_i)^c \log(\lambda x_i) u_i^2}{(1-pu_i)^2},$$

$$J_{pa} = \sum_{i=1}^{n}\frac{u_i}{(1-pu_i)}, \quad J_{pb} = -\frac{n}{(1-p)}\sum_{i=1}^{n}\frac{u_i}{(1-pu_i)},$$

$$\begin{aligned}
J_{\lambda\lambda} = &-\frac{nc}{\lambda^2} + \frac{bc(1-c)}{\lambda^2}\sum_{i=1}^{n}(\lambda x_i)^c + \frac{(c-1)(a-1)c}{\lambda^2}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_i}{(1-u_i)} - \frac{(a-1)c^2}{\lambda^2}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c} u_i}{(1-u_i)} \\
&- \frac{(a-1)c^2}{\lambda^2}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c} u_i^2}{(1-u_i)^2} + \frac{(1-c)(a+b)pc}{\lambda^2}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_i}{(1-pu_i)} + \frac{(a+b)pc^2}{\lambda^2}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c} u_i}{(1-pu_i)} \\
&+ \frac{(a+b)p^2c^2}{\lambda^2}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c} u_i^2}{(1-pu_i)^2},
\end{aligned}$$

$$\begin{aligned}
J_{\lambda c} = &\frac{n}{\lambda} - \frac{bc}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^c[\log(\lambda x_i) + c^{-1}] + \frac{(a-1)c}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c \log(\lambda x_i) u_i}{(1-u_i)} + \frac{(a-1)}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_i}{(1-u_i)} \\
&- \frac{(a-1)c}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c} \log(\lambda x_i) u_i}{(1-u_i)} - \frac{(a-1)c}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c} \log(\lambda x_i) u_i^2}{(1-u_i)^2} \\
&- \frac{(a+b)pc}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c \log(\lambda x_i) u_i}{(1-pu_i)} - \frac{(a+b)p}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_i}{(1-pu_i)} \\
&+ \frac{(a+b)pc}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c} \log(\lambda x_i) u_i}{(1-pu_i)} + \frac{(a+b)p^2c}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c} \log(\lambda x_i) u_i^2}{(1-pu_i)^2},
\end{aligned}$$

$$J_{\lambda a} = \frac{c}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_i}{(1-u_i)} - \frac{pc}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_i}{(1-pu_i)},$$

$$J_{\lambda b} = -\frac{c}{\lambda}\sum_{i=1}^{n}(\lambda x_i)^c - \frac{pc}{\lambda}\sum_{i=1}^{n}\frac{(\lambda x_i)^c u_i}{(1-pu_i)},$$

$$\begin{aligned}
J_{cc} = &-\frac{n}{c^2} - b\sum_{i=1}^{n}(\lambda x_i)^c \log^2(\lambda x_i) + (a-1)\sum_{i=1}^{n}\frac{[1-(\lambda x_i)^c](\lambda x_i)^c \log^2(\lambda x_i) u_i}{(1-u_i)} \\
&+ (a+b)p\sum_{i=1}^{n}\frac{[(\lambda x_i)^c - 1](\lambda x_i)^c \log^2(\lambda x_i) u_i}{(1-pu_i)} - (a-1)\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c} \log^2(\lambda x_i) u_i^2}{(1-u_i)^2} \\
&+ (a+b)p^2\sum_{i=1}^{n}\frac{(\lambda x_i)^{2c} \log^2(\lambda x_i) u_i^2}{(1-pu_i)^2},
\end{aligned}$$

$$J_{ca} = \sum_{i=1}^{n}\frac{(\lambda x_i)^c \log(\lambda x_i) u_i}{(1-u_i)} - p\sum_{i=1}^{n}\frac{(\lambda x_i)^c \log(\lambda x_i) u_i}{(1-pu_i)},$$

$$J_{cb} = -\sum_{i=1}^{n}(\lambda x_i)^c \log(\lambda x_i) - p\sum_{i=1}^{n}\frac{(\lambda x_i)^c \log(\lambda x_i) u_i}{(1-pu_i)},$$

$$J_{aa} = -n[\psi'(a) - \psi'(a+b)], \quad J_{ab} = n\psi'(a+b) \quad \text{and } J_{bb} = -n[\psi'(b) - \psi'(a+b)].$$