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A unified view on lifetime distributions arising from selection mechanisms

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ABSTRACT

In this paper, we formulate a flexible density function from the selection mechanism viewpoint (see, for example, Bayarri and DeGroot (1992) and Arellano-Valle et al. (2006)) which possesses nice biological and physical interpretations. The new density function contains as special cases many models that have been proposed recently in the literature. In constructing this model, we assume that the number of competing causes of the event of interest has a general discrete distribution characterized by its probability generating function. This function has an important role in the selection procedure as well as in computing the conditional personal cure rate. Finally, we illustrate how various models can be deduced as special cases of the proposed model.

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1. Introduction

Recently, there has been a great interest among statisticians and applied researchers in constructing flexible families of distributions to facilitate better modeling of data. Consequently, a significant progress has been made in developing the generalizations of some well-known lifetime distributions and their successful application to problems in areas such as engineering, environmetrics, economics and biomedical sciences. The purpose of this work is to formulate a unified procedure with a biological and physical interpretation that includes as special cases many of these lifetime distributions. For formulating this procedure, we choose the selection approach discussed by Bayarri and DeGroot (1992) and Arellano-Valle et al. (2006). This selection approach is useful for obtaining flexible distributions from the original model based on the occurrence of some related selection random variables. Moreover, we introduce a new notion, called the conditional personal non-cure rate, for which we give an interpretation in terms of selection or weight function. Another related measure is the conditional personal cure rate which is of interest when, for example, successfully treated cancer patients may die from a cause other than the diagnosed cancer.

The rest of this article is organized as follows. In Section 2, the unified model is developed from the selection mechanism viewpoint and the idea of the conditional personal probability is introduced. In Section 3, many of the recently introduced lifetime distributions are obtained as special cases from the proposed unified model, and some new interpretations from

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a biological viewpoint are given to them. Section 4 deals with some mathematical properties of the unified model. Two applications of some distributions, discussed in Section 3, are given in Section 5. Finally, Section 6 offers some concluding remarks.

2. A unified selection distribution

Selection mechanisms arise when a random sample from the entire population might be too difficult or too expensive to secure and so flexible models must be developed to incorporate this constraint on the observations. We formulate the selection distributions here within a biological context, where the population is restricted to patients not cured from an event of interest such as disease or tumor. In biological context, we mean that the damaged cells are competing to produce detectable tumors. The time for the jth damaged cell (clonogens) to transform into a detectable tumor (promotion time) is denoted by X_j, $j = 1, \ldots, N$, where N denotes the unobservable number of damaged cells that can produce the event of interest. In the sequel, we suppose that N has its probability mass function (pmf) given by

$$p_n = P(N = n), \quad n = 0, 1, \ldots. \tag{1}$$

Let $A_N(s) = \sum_{n=0}^{\infty} p_n s^n$ be the corresponding probability generating function (pgf) for $0 < s < 1$, and p_0 the cure rate. We assume that, conditional on N, that the X_j's are i.i.d. having density function $g(x)$ and survival function $S(x) = 1 - G(x)$. Usually, exponential, piecewise exponential (Chen and Ibrahim, 2001) and Weibull distributions are used to represent $g(x)$.

Given $N = n$ and the lifetime $T = t$, let Z_j, $j = 1, \ldots, n$, be independent random variables, independently of N, following a Bernoulli distribution with success probability $G(t)$ indicating the presence of the jth competing cause (or clonogens) at time t. The discrete variable N_t, representing the total number of competing causes among the N initial competing causes that are present at time t, is then given by

$$N_t = \begin{cases} Z_1 + Z_2 + \cdots + Z_N, & \text{if } N > 0, \\ 0, & \text{if } N = 0. \end{cases} \tag{2}$$

It follows from the fundamental formula for conditional probabilities that

$$P(N_t = j) = \sum_{n=j}^{\infty} p_n \overbrace{P(N_t = j | N = n)}^{\text{Binomial}(n, G(t))},$$

and its corresponding pgf (Feller, 1968) is

$$A_{N_t}(s) = A_N[1 - (1 - s)G(t)]. \tag{3}$$

The long-term survival function (Rodrigues et al., 2008) can be obtained from (3) as

$$S_{\text{Pop}}(t) = P(T \geq t) = P(N_t = 0) = A_{N_t}(0) = A_N[S(t)], \tag{4}$$

where $A_N(.)$ is the pgf of the discrete random variable N.

Motivated by the work of Arellano-Valle et al. (2006), we start with a definition of a selection distribution and its association with the pgf $A_{N_t}(s)$ and density function $g(x)$ of the promotion time random variable X. First, we assume that the population is divided into two sub-populations of cured and non-cured patients defined by the following binary random variable for any time t:

$$U_t = \begin{cases} 1, & \text{if } N_t \geq 1, \\ 0, & \text{if } N_t = 0, \end{cases} \tag{5}$$

where $P(U_t = 1) = 1 - P(N_t = 0) = 1 - p_0$.

Definition 2.1 (*Selection Distribution*)**.** Let T be a non-negative lifetime random variable and X the promotion time with probability density function (pdf) $g(x)$. We define the selection distribution of T as the conditional distribution of X, given $U_t = 1$.

This definition simply states that the selection probability distribution of T is the probability distribution of X, truncated by non-cured patients. We show that this viewpoint is quite useful to obtain new classes of flexible lifetime distributions and also to unify many models proposed recently in the literature.

Indeed, if X in Definition 2.1 has pdf $g(x)$, then T has a pdf $f_T(t)$ given by

$$f_T(t) = \frac{g(t)\, P(U_t = 1 \mid X \leq t)}{P(U_t = 1)} = \frac{g(t)\, P(U_t = 1 \mid X \leq t)}{1 - p_0}. \tag{6}$$

In fact, (6) can be expressed as a weighted distribution (Bayarri and DeGroot, 1992)

$$f_T(t) = \frac{w(t)\, g(t)}{E[w(X)]}, \tag{7}$$

where the weight function $w(t)$ is precisely

$$w(t) = P(U_t = 1 \mid X \leq t), \tag{8}$$

and $E[w(X)]$ is the mean of $w(X)$ with respect to $g(t)$.

Definition 2.2. The lifetime T is under the first-activation at time t if $N_t = 1$ or $T = \min\{X_1, \ldots, X_N\}$.

Definition 2.3. The lifetime T is under the last-activation at time t if $N_t = N$ or $T = \max\{X_1, \ldots, X_N\}$.

The first-activation at time t means that the cancer patient died from a specific clonogen in the presence of other clonogens and $P(N_t = 1)$ is called the crude cumulative probability or cumulative incidence function (CIF) (Yu et al., 2010). On the other hand, the last-activation at time t means that all clonogens are activate at time t and $P(N_t = N)$ is the so-called net survival at time t (Yu et al., 2010) and it is a measure of survival if all causes of death other than the cancer of interest were to be eliminated. As mentioned by Yu et al. (2010), the net survival is a desirable measure for evaluating the progress of cancer treatment and control efforts since the interpretation of excess mortality due to cancer is not affected by changes in mortality due to other diseases.

Theorem 2.4. *The crude cumulative distribution and the net survival at time t are given by*

$$P(N_t = 1) = \left.\frac{G(t)\mathrm{d}A_N(s)}{\mathrm{d}s}\right|_{s=S(t)}, \tag{9}$$
$$P(N_t = N) = A_N[G(t)],$$

respectively.

Proof. The crude cumulative distribution simply follows from (3) and the net survival is obtained from the following result:

$$P(N_t = N) = \sum_{n=0}^{\infty} p_n P(N_t = n \mid N = n) = \sum_{n=0}^{\infty} p_n G(t)^n. \quad \square$$

Definition 2.5 (*Conditional Personal Non-Cure Rate Under the First-Activation*)**.** Let T be the lifetime of some treated cancer patient under the first-activation process and X the promotion time with pdf $g(x)$. The conditional probability of the patient dying from the damaged or initiated cells (clonogens), given that $X \leq t$, called the "conditional personal non-cure rate", is defined as

$$\gamma_{\mathrm{np}}(t) = P(U_t = 1 \mid X \leq t). \tag{10}$$

Indeed, we can show from (9) that

$$\gamma_{\mathrm{np}}(t) = \frac{P(N_t = 1)}{G(t)} = \left.\frac{\mathrm{d}A_N(s)}{\mathrm{d}s}\right|_{s=S(t)}, \tag{11}$$

and from (6) the selection distribution of T is then given by

$$f_T(t) = \frac{g(t)}{1 - p_0}\left\{\left.\frac{\mathrm{d}A_N(s)}{\mathrm{d}s}\right|_{s=S(t)}\right\}. \tag{12}$$

The corresponding proportion of patients dying from causes other than the diagnosed cancer $\gamma_p = 1 - \gamma_{\mathrm{np}}$, given that $X \leq t$, is defined as the conditional personal cure rate. This measure will be of natural interest since it corresponds to successfully treated cancer patients who may not die from cancer during the time t. Analogously, the selection distribution of T under the last-activation at time t is given by

$$f_T(t) = \frac{g(t)}{1 - p_0}\left\{\left.\frac{\mathrm{d}A_N(s)}{\mathrm{d}s}\right|_{s=G(t)}\right\}. \tag{13}$$

We had not chosen any r-activation that is between the first-activation and last-activation, since from Cooner et al. (2007), $r \mid N \sim \mathrm{DiscreteUnif}(1, N)$ and $P(N = 0) = 0$ jointly imply $w(t) = 1$, i.e., we do not select any distribution, or simply $f_T(y) = g(t)$.

Eqs. (12) and (13) are important since they show how the pgf works as a selection mechanism and how it unifies in a simple way many of the distributions recently proposed in the literature. It also enables the calculation of the personal cure rate, which is a measure that is of interest in the treatment of cancer patients, for example. The weight function $w(t)$ in (8) is concerned with selected patients at risks, and this assists in obtaining the conditional personal cure rate. These results are summarized in Table 1.

Table 1
Selection mechanisms and personal cure rates.

Selection distribution	First-activation	Last-activation
$f_T(t)$	$\frac{g(t)}{1-p_0}\left\{\left.\frac{dA_N(s)}{ds}\right\|_{s=S(t)}\right\}$	$\frac{g(t)}{1-p_0}\left\{\left.\frac{dA_N(s)}{ds}\right\|_{s=G(t)}\right\}$
$S_T(t)$	$\frac{A_N[S(t)]-p_0}{1-p_0}$	$\frac{1-A_N[G(t)]}{1-p_0}$
$h_T(t)$	$\frac{g(t)\left\{\left.\frac{dA_N(s)}{ds}\right\|_{s=S(t)}\right\}}{A_N[S(t)]-p_0}$	$\frac{g(t)\left\{\left.\frac{dA_N(s)}{ds}\right\|_{s=G(t)}\right\}}{1-A_N[G(t)]}$
$\gamma_p(t)$	$1-\left.\frac{dA_N(s)}{ds}\right\|_{s=S(t)}$	$1-\left.\frac{dA_N(s)}{ds}\right\|_{s=G(t)}$

3. Some special models

In this section, we demonstrate how many existing models can be deduced as special cases of the proposed unified model. In addition, this viewpoint also results in a biological interpretation for these cases.

- Generalized exponential Poisson (GEP) distribution.
  Barreto-Souza and Cribari-Neto (2009) introduced the GEP distribution with two parameters α and λ, and they showed that it has a desirable physical interpretation. That is, if there are n components in a parallel system and the lifetimes of the components are independently and identically distributed as exponential Poisson (EP) (Kuş, 2007), then the system lifetime follows the GEP law. Here, we give a different characterization for the GEP distribution from our unified model. Consider a sequence of independent Bernoulli trials, where the kth trial has probability of success α/k, for $k = 1, 2, \ldots$, $0 < \alpha < 1$. The trial number X for which the first success occurs follows the so-called Sibuya distribution with parameter α, say Sibuya(α) (Christoph and Schreiber, 2000; Devroye, 1993), given by $P(X = r) = (-1)^{r-1}\alpha(\alpha-1)\ldots(\alpha-r+1)/r!$. The pgf of X (Pillai and Jayakumar, 1995) is

$$A_X(s) = 1 - (1-s)^{\alpha}. \tag{14}$$

  Now, define $M \sim$ Sibuya(α) and $X_i \sim$ P(λ), and

$$N = \begin{cases} X_1 + \cdots + X_M : & \text{if } M > 1 \\ 0 : & \text{if } M = 0. \end{cases}$$

  Then, we have

$$A_N(s) = 1 - [1 - \exp\{-\lambda(1-s)\}]^{\alpha}. \tag{15}$$

  From the first-activation mechanism in Eq. (15), by taking $S(x) = \exp(-\beta x)$, we obtain the GEP distribution

$$f_T(t;\boldsymbol{\theta}) = \frac{\alpha\lambda\beta}{(1-e^{-\lambda})^{\alpha}}\{1 - e^{-\lambda+\lambda\exp(-\beta t)}\}^{\alpha-1}e^{-\lambda-\beta t+\lambda\exp(-\beta t)}, \tag{16}$$

  where $\boldsymbol{\theta} = (\alpha, \beta, \lambda)$. Further, if $\alpha = 1$, we have the EP distribution (Kuş, 2007). Various properties and inferential methods for this two-parameter distribution with decreasing failure rate are discussed by Kuş (2007).
- Classical Lehmann alternative distributions.
  There has been several attempts at modeling failure time data by the classical Lehmann type I and II alternatives given by $F_T(t) = [G(t)]^{\alpha}$ and $F_T(t) = 1 - [1 - G(t)]^{\alpha}$, respectively, where $G(t)$ is the parent cumulative function and α is a positive real number. Recently, the first form has also been refereed to as the exponentiated-G (Exp-G) distributions. Some examples, discussed by Nadarajah and Kotz (2006), are the exponentiated exponential (EE), exponentiated gamma, exponentiated Weibull, exponentiated Gumbel and exponentiated Fréchet distributions, which extend the exponential, gamma, Weibull, Gumbel and Fréchet distributions, respectively. The advantage of this approach lies in its flexibility to model both monotonic as well as non-monotonic failure rates even though the baseline failure rate may be monotonic. Lehmann type I and II models are easily obtained from the Sibuya generating function (14) under the first-activation and last-activation mechanisms in Table 1 by setting $p_0 = 0$.
  We give a simple example. Assuming $S(x) = e^{-\beta x}$, from (14), we obtain, under the first-activation mechanism in Table 1, the EE (also called generalized exponential) distribution (Gupta and Kundu, 1999). Its density function is

$$f_T(t,\boldsymbol{\theta}) = \alpha\beta e^{-\beta t}(1 - e^{-\beta t})^{\alpha-1}, \tag{17}$$

  where $\boldsymbol{\theta} = (\alpha, \beta)$. The EE cumulative function has closed form and so its inference based on censored data can be handled more easily than with the gamma distribution (Gupta and Kundu, 1999). Here, we have provided above a nice biological interpretation for it through the first-activation selection.
- The Weibull-geometric (WG) distribution.
  Barreto-Souza et al. (2010) proposed the WG distribution (with decreasing failure rate), which generalizes the exponential geometric (EG) distribution due to Adamidis and Loukas (1998). Taking the pgf as $A(s) = \frac{1-p}{1-ps}$, corresponding

to a geometric distribution with parameter p, and the Weibull survival function $S(t) = \exp\{-(\beta t)^{\alpha}\}$, we obtain, under first-activation mechanism in Table 1, the WG density function

$$f_T(t, \boldsymbol{\theta}) = \alpha\beta^{\alpha}(1-p)t^{\alpha-1}\exp\{-(\beta t)^{\alpha}\}[1-p\exp\{-(\beta t)^{\alpha}\}]^{-2},$$

where $\boldsymbol{\theta} = (\alpha, \beta)$.

- Exponential Conway–Maxwell Poisson (ECOMP) distribution.
The Conway–Maxwell Poisson distribution (COM-Poisson), first introduced by Conway and Maxwell (1961), was revived recently by Shmueli et al. (2005). The COM-Poisson distribution generalizes the Poisson distribution in an elegant and flexible way, allowing for under-dispersion as well as over-dispersion. This distribution was also discussed by Kadane et al. (2006) from a Bayesian viewpoint, and an elicitation program to find the hyper-parameters from the predictive distribution was discussed there as well; see also Kokonendji et al. (2008) for more details on the COM-Poisson distribution. This distribution can be expressed in the exponential form and can then be viewed as a weighted Poisson distribution with weight function $\mathrm{w}(m; \phi) = (m!)^{1-\phi}$ (Kokonendji et al., 2008; Rodrigues et al., 2009). The pmf of the COM-Poisson distribution for a discrete variable M is given by

$$P(M = m; \eta, \phi) = \frac{1}{Z(\eta, \phi)}\frac{\eta^m}{(m!)^{\phi}}, \quad m = 0, 1, 2, \ldots, \tag{18}$$

where $Z(\eta, \phi) = \sum_{j=0}^{\infty}\frac{\eta^j}{(j!)^{\phi}}$. Therefore, the cure fraction turns out to be

$$p_0 = P(M = 0; \eta, \phi) = \frac{1}{Z(\eta, \phi)}. \tag{19}$$

The corresponding pgf is

$$A(s) = \frac{Z(\eta s, \phi)}{Z(\eta, \phi)}. \tag{20}$$

Now, by applying the first-activation mechanism, we obtain the ECOMP distribution (Cordeiro et al., in press) with pdf

$$f_T(t; \boldsymbol{\theta}) = \frac{\beta}{Z(\lambda, \phi) - 1}\sum_{j=1}^{\infty}\frac{j\lambda^j}{(j!)^{\phi}}\exp(-j\beta t), \quad y > 0, \tag{21}$$

where $\boldsymbol{\theta} = (\beta, \lambda, \phi)^T$.

- The exponentiated Weibull (EW) distribution.
The EW distribution (Nassar and Eissa, 2004) is an extension of the well-known Weibull distribution. The EW family contains distributions with non-monotone failure rates in addition to a broad class of monotone failure rates. In practice, many lifetime data display bathtub shape or upside-down bathtub shape failure rates and so the EW distribution provides a more realistic model than those with monotone failure rates. Taking the Weibull survival function $S(x) = \exp(-x^{\beta})$ with a scalar parameter equal to one and a shape parameter β, we obtain from (14), for $0 < \alpha \leq 1$ and under the first-activation mechanism in Table 1, the EW density function given by

$$f_T(t; \boldsymbol{\theta}) = \alpha\beta t^{\beta-1}\exp(-t^{\beta})\{1-\exp(-t^{\beta})\}^{\alpha-1}, \tag{22}$$

where $\boldsymbol{\theta} = (\alpha, \beta)$. For this restricted parameter space, the selection mechanism gives a new biological interpretation for the EW distribution.

- The Kumaraswamy G family of distributions.
Consider starting from a parent continuous distribution function $G(t)$. A natural way of generating families of distributions on some other support is to apply the quantile function to a family of distributions on the interval (0, 1). Based on the Kumaraswamy distribution on this interval, Cordeiro and de Castro (2010) defined the Kumaraswamy G (Kw-G) family of distributions by

$$F_T(t) = 1 - \{1 - [G(t)]^a\}^b, \tag{23}$$

where $a > 0$ and $b > 0$ are two additional parameters to control skewness through the relative tail weights. They presented some examples of (23) such as the Kw-normal, Kw-gamma, Kw-Weibull, Kw-Gumbel and Kw-inverse Gaussian distributions. Because of its tractable distribution function (23), the Kw-G family of distributions can be used quite effectively for inferential purposes even if the data are censored. Eq. (23) is easily obtained (for $b < 1$) from the Sibuya(b) pgf (14), under the last-activation mechanism in Table 1, by considering the Exp-$G(a)$ distribution as the parent distribution and $p_0 = 0$. In a different way, the Kw-G distribution can be derived by two mechanisms applied in sequence, which hold only for $a < 1$ and $b < 1$: the Sibuya(a) pgf under the first-activation mechanism applied to $G(t)$ gives $H(t) = [G(t)]^a$ and then the Sibuya(b) pgf under the last-activation mechanism applied to $H(t)$ yields (23), both cases with $p_0 = 0$. We have

$$G \rightarrow \text{Sibuya} + \text{first mechanism} \rightarrow \text{Exp-}G(a) \rightarrow \text{Sibuya} + \text{last mechanism} \rightarrow \text{Kw-G}.$$

- The Kumaraswamy–Weibull (KwW) distribution.
  Cordeiro et al. (2010b) introduced the KwW distribution that contains as special sub-models the exponentiated Weibull, exponentiated Rayleigh, exponentiated exponential, Kumaraswamy exponential (KwE) and Weibull distributions. Taking $G(x) = [1 - \exp\{-(\lambda x)^c\}]^a$, from (15) under the last-activation mechanism in Table 1, we obtain the KwW density function (for $t > 0$) given by

$$f_T(t;\boldsymbol{\theta}) = abc\lambda^c t^{c-1}\exp\{-(\lambda t)^c\}[1-\exp\{-(\lambda t)^c\}]^{a-1}\{1-[1-\exp\{-(\lambda t)^c\}]^a\}^{b-1}, \tag{24}$$

  where $\boldsymbol{\theta} = (a, b, c, \lambda)$ and $0 < b < 1$.
  In view of the selection mechanism considered here, we have a new biological interpretation for the KwW distribution, which is quite different from the physical interpretation given by Cordeiro et al. (2010b).
- A generalized modified Weibull (GMW) distribution.
  Carrasco et al. (2008) proposed a four-parameter generalization of the Weibull distribution, which is capable of modeling a bathtub shaped hazard rate function. This distribution has a number of well-known lifetime distributions as special cases including Weibull, extreme value, exponentiated Weibull, generalized Rayleigh and modified Weibull distributions. Now, by taking $S(x) = \exp\{-\alpha y^\gamma \exp(\lambda y)\}$, from (14) and under the first-activation mechanism in Table 1, we obtain the generalized modified Weibull density function given by

$$f_T(t;\boldsymbol{\theta}) = \alpha\beta t^{\gamma-1}(\gamma+\lambda t)\exp\{\lambda t - \alpha y t^\gamma \exp(\lambda t)\}[1-\exp\{-\alpha t^\gamma \exp(\lambda t)\}]^{\beta-1}, \tag{25}$$

  where $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \lambda)$ and $0 < \beta < 1$. The selection mechanism then provides a new biological interpretation for the GMW distribution from the first-activation viewpoint.
- The exponential power series (EPS) distribution
  Chahkandi and Ganjali (2009) introduced a new lifetime family of distributions (with decreasing failure rate) by combining a truncated at zero power series with some exponential distributions. Consequently, we consider $S(t) = \exp(-\beta t)$ and the power series mass function

$$p_n(\alpha) = P(N = n;\alpha) = \frac{a_n\alpha^n}{A(\alpha)}, \quad n = 0, 1, \ldots, \tag{26}$$

  where $a_n > 0$, $A(\alpha) = \sum_n a_n\alpha^n$ and $\alpha > 0$. The family (26) of distributions includes as special cases the binomial, Poisson, negative binomial and logarithmic distributions, among others. The corresponding pgf is $A_N(s;\alpha) = \frac{A(\alpha s)}{A(\alpha)}$ and $p_0 = \frac{a_0}{A(\alpha)}$. Under the first-activation mechanism given in Table 1, we obtain the density function

$$f_T(t;\boldsymbol{\theta}) = \frac{\alpha\beta\exp(-\beta t)\left.\frac{dA_N(s;\alpha)}{ds}\right|_{s=\exp(-\beta t)}}{A(\alpha)-a_0}, \tag{27}$$

  where $\boldsymbol{\theta} = (\alpha, \beta)$. Estimation of these parameters by maximum likelihood procedure and its related EM algorithm can be found in Chahkandi and Ganjali (2009).
- Beta generalized (BG) distribution.
  Given a parent distribution $G(t;\theta)$ with the parameter vector θ and the density function $g(t;\theta)$, the BG distribution may be characterized by the density function

$$f_{\mathcal{BG}}(t;\theta,a,b) = B(a,b)^{-1}g(t;\theta)G(t;\theta)^{a-1}[1-G(t;\theta)]^{b-1}, \tag{28}$$

  where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ denotes the beta function, $\Gamma(\cdot)$ the gamma function and $a > 0$ and $b > 0$ are additional shape parameters to those in θ. If T is a random variable with pdf (28), we write $T \sim \mathcal{BG}(G;\theta,a,b)$. The density function $f_{\mathcal{BG}}(t;\theta,a,b)$ will be most tractable when both functions $G(t;\theta)$ and $g(t;\theta)$ have simple analytic expressions. Except for some special choices of these functions, $f_{\mathcal{BG}}(t;\theta,a,b)$ could be complicated to deal with in full generality. Some BG distributions have been discussed in recent literature. For example, Eugene et al. (2002); Nadarajah and Kotz (2004) and Nadarajah and Gupta (2004) defined the beta normal, beta Gumbel and beta Fréchet distributions by taking $G(t)$ to be the cdf of the normal, Gumbel and Fréchet distributions, respectively, and studied some of their properties. More than twenty *BG* distributions have been developed by several authors during the past ten years. It should be emphasized that for a and b positive integers, (28) reduces to the density function of the ath order statistic from the G distribution in a sample of size $a + b - 1$. Here, we provide a simple interpretation when b is real less than one and a is any positive real.
  The random variable T admits the simple stochastic representation $T = G^{-1}(V)$, where V follows a beta distribution with parameters a and b. Using this transformation, the cdf corresponding to (28) can be expressed as

$$F_{\mathcal{BG}}(t;\theta,a,b) = I_{G(t;\theta)}(a,b) = [B(a,b)]^{-1}\int_0^{G(t;\theta)}\omega^{a-1}(1-\omega)^{b-1}d\omega, \tag{29}$$

  where $I_x(a,b)$ denotes the incomplete beta ratio function.

For $0 < s \leq 1$, the well-known power series expansion for the incomplete beta ratio function holds and is given by

$$I_s(a, b) = \sum_{n=0}^{\infty} d_n s^{a+n},$$

where the coefficients d_n are positive for $b < 1$. They are given by

$$d_n = \frac{\Gamma(1 - b + n)}{(a + n)n!\Gamma(1 - b)B(a, b)}.$$

Clearly, $\sum_{n=0}^{\infty} d_n = 1$ since $I_1(a, b) = 1$. From the above expansion, let a discrete random variable M be defined by the probabilities d_n for $n = 0, 1, \ldots$ The generating function of M is then given by $A_M(s) = \sum_{n=0}^{\infty} d_n s^n$. We define the constant random variable K such that $P(K = a) = 1$ and the random variable $N = K + M$, where K and M are assumed to be independent. Exploring the combination of the pgf of N with the last-activation mechanism in Table 1 yields the BG distribution by setting $p_0 = 0$, i.e., $A_N(G(t)) = A_K(G(t))A_M(G(t)) = \sum_{n=0}^{\infty} d_n G(t)^{a+n} = F_{\mathcal{BG}}(t; \theta, a, b)$.

4. Some properties

For an arbitrary baseline cdf $G(t)$ and a discrete random variable N defined by the pgf $A_N(s)$, the unified cumulative distribution of T under the first-activation and last-activation mechanisms can be expressed from Table 1 as

$$F_T(t) = \frac{1 - A_N[1 - G(t)]}{1 - p_0} \quad \text{and} \quad F_T(t) = \frac{A_N[G(t)] - p_0}{1 - p_0},$$

respectively. From now on, a random variable Z_a is said to have the exponentiated-G distribution with parameter $a > 0$, say $Z_a \sim$ Exp-$G(a)$, if its pdf and cdf are given by

$$h_a(x) = ag(x)G^{a-1}(x) \quad \text{and} \quad H_a(x) = G^a(x),$$

respectively. Here, we demonstrate that $f_T(t)$ can be written as a mixture of exponentiated-G densities under the last-activation mechanism and a linear combination of exponentiated-G densities under the first-activation mechanism. In both cases, the weighted coefficients depend only on the probabilities of N. Under the last-activation mechanism, we have

$$F_T(t) = \frac{A_N[G(t)] - p_0}{1 - p_0} = \sum_{r=1}^{\infty} \frac{p_r}{1 - p_0}[G(t)]^r$$

and then

$$f_T(t) = \sum_{r=0}^{\infty} v_r h_r(t), \tag{30}$$

where $v_0 = 0$ and $v_r = rp_r/(1 - p_0)$ for $r = 1, 2, \ldots$.

By expanding the binomial term in the first-activation mechanism, we obtain

$$F_T(t) = \frac{1 - A_N[1 - G(t)]}{1 - p_0} = \frac{1}{1 - p_0} - \sum_{i=0}^{\infty} \sum_{r=0}^{i} \frac{(-1)^r \binom{i}{r} p_i}{1 - p_0} G(t)^r.$$

Now, by interchanging the orders of summation, we can write

$$F_T(t) = \sum_{r=1}^{\infty} q_r G(t)^r,$$

where $q_r = \sum_{i=r}^{\infty} \frac{(-1)^{r+1}\binom{i}{r} p_i}{1-p_0}$ for $r = 1, 2, \ldots$. Hence,

$$f_T(t) = \sum_{r=0}^{\infty} w_r h_r(t), \tag{31}$$

where $w_r = (r + 1)q_{r+1}$ for $r = 0, 1, \ldots$. Eq. (31) has the same form as (30), but with different weight coefficients.

So, some mathematical quantities (such as ordinary and incomplete moments, generating function and mean deviation) of the unified distribution of T in both mechanisms can be obtained by knowing those quantities for the exponentiated-G distribution. The mathematical properties of the exponentiated distributions have been studied by many authors in recent years (Nadarajah and Kotz, 2006). Now, we obtain the moments and generating function of T from (31) since they are similar to (30). The sth moment of T is given by

$$E(T^s) = \sum_{r=0}^{\infty} w_r E(Z_r^s),$$

Table 2
Parameter estimates and goodness-of-fit statistics for the fitted distributions (ordered according to A^*).

Data set	Distribution	Estimates	W^*	A^*
Boeing data	ECOMP	$(7.37 \times 10^{-3}, 1.37, 0.983)$	0.068	0.449
$(n = 213)$	EG	$(7.99 \times 10^{-3}, 0.429)$	0.074	0.484
	EP	$(7.49 \times 10^{-3}, 1.34)$	0.070	0.461
	Exponential	1.07×10^{-2}	0.165	1.018
Disasters data	ECOMP	$(1.55 \times 10^{-3}, 3.90, 1.08)$	0.067	0.432
$(n = 109)$	EP	$(1.65 \times 10^{-3}, 3.26)$	0.068	0.439
	EG	$(2.36 \times 10^{-3}, 0.619)$	0.068	0.480
	Exponential	4.15×10^{-3}	0.070	0.658

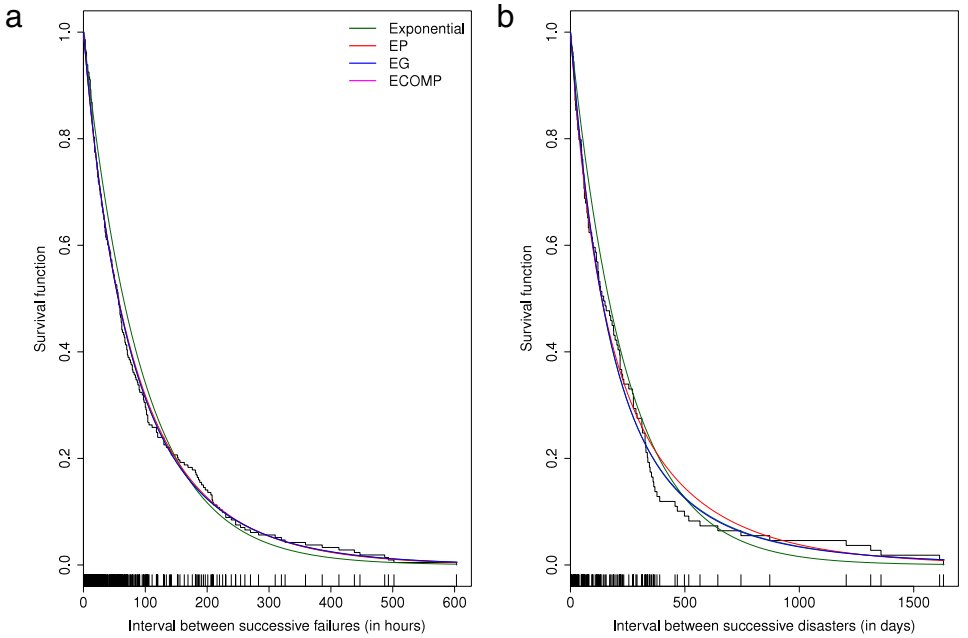

Fig. 1. Empirical survival function and some fitted distributions. (a) Boeing data. (b) Coal-mining disasters data.

where $Z_r \sim$ Exp-$G(r)$. The moments of Z_r can be derived from the quantile function of G, say $Q(u) = G^{-1}(u)$, as $E(Z_r^s) = r \int_0^1 Q(u)^s u^{r-1} du$. Similarly, the generating function of T can be expressed as

$$M_T(w) = \sum_{r=0}^{\infty} w_r M_{Z_r}(w),$$

where $M_{Z_r}(w) = r \int_0^1 \exp\{wQ(u)\} u^{r-1} du$ is the generating function of Z_r.

5. Illustrative examples

In this section, we present two illustrative examples. The first data set was presented by Proschan (1963) which consists of the interval in hours between successive failures of the air conditioning system in a fleet of Boeing 720 airplanes. The data set contains 213 observations and was also analyzed by Adamidis and Loukas (1998), Kuş (2007) and Chahkandi and Ganjali (2009), among others. The second data set, presented by Cox and Lewis (1966) and used by Adamidis and Loukas (1998), comprises 109 observations on the number of days between successive coal-mining disasters. The required computations were performed in R language (R Development Core Team, 2011). Computational code is available from the first author on request.

Table 2 lists the parameter estimates and the results of the formal goodness-of-fit tests. We apply the modified Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics proposed by Chen and Balakrishnan (1995). In general, the smaller the values of these statistics, the better the fit to the data. For both data sets, Table 2 presents the values of W^* and A^*, which indicate that the ECOMP, EG and EP distributions yield very similar fits. The fitted survival functions of these distributions superimposed on the empirical survival function in Fig. 1 and reinforce this claim. For the GEP distribution, the estimates of the parameter α in both examples are close to one, supporting the EP distribution.

6. Concluding remarks

In this work, we have used the selection mechanism proposed by Arellano-Valle et al. (2006) to formulate a very flexible family of distributions, where some structural properties are presented in detail. This unified distribution includes many of the recently proposed lifetime models as special cases, and moreover facilitates in giving a biological interpretation for them. Also, the idea of personal probability presented gives an important interpretation for the weight function, which we feel will be of interest in survival analysis. However, much more research needs to be done in order to investigate unexplored aspects of this mechanism, especially in inferential problems. We hope to motivate many important applications of this selection lifetime distribution in the future.

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