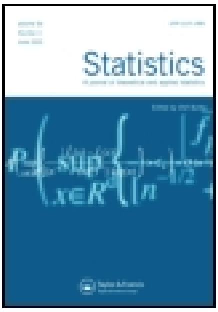

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The beta generalized gamma distribution

Gauss M. Cordeiro [a], Fredy Castellares [b], Lourdes C. Montenegro [b] & Mário de Castro [c]

[a] Departamento de Estatística, Universidade Federal de Pernambuco, 50740-540, Recife-PE, Brazil

[b] Departamento de Estatística, Universidade Federal de Minas Gerais, Av. Antônio Carlos, 6627, 31270-901, Belo Horizonte-MG, Brazil

[c] Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Avenida Trabalhador São-carlense, 400, 13566-590, São Carlos-SP, Brazil

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The beta generalized gamma distribution

Gauss M. Cordeiro[a], Fredy Castellares[b], Lourdes C. Montenegro[b] and Mário de Castro[c]*

[a] *Departamento de Estatística, Universidade Federal de Pernambuco, 50740-540 Recife-PE, Brazil;* [b] *Departamento de Estatística, Universidade Federal de Minas Gerais, Av. Antônio Carlos, 6627, 31270-901 Belo Horizonte-MG, Brazil;* [c] *Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Avenida Trabalhador São-carlense, 400, 13566-590, São Carlos-SP, Brazil*

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For the first time, a new five-parameter distribution, called the beta generalized gamma distribution, is introduced and studied. It contains at least 25 special sub-models such as the beta gamma, beta Weibull, beta exponential, generalized gamma (GG), Weibull and gamma distributions and thus could be a better model for analysing positive skewed data. The new density function can be expressed as a linear combination of GG densities. We derive explicit expressions for moments, generating function and other statistical measures. The elements of the expected information matrix are provided. The usefulness of the new model is illustrated by means of a real data set.

Keywords: beta generalized distribution; expected information matrix; generalized gamma distribution; mean deviation; moment

1. Introduction

The most general form of the gamma distribution is the three parameter generalized gamma (GG) distribution studied by Stacy [1]. It includes as special sub-models the exponential, Weibull, gamma, Rayleigh, among other models. This distribution is suitable for modelling data with different types of hazard rate functions: increasing, decreasing, bathtub shaped and unimodal, which makes it particularly useful for estimating individual hazard functions. The GG distribution has been used in several research areas such as engineering, hydrology and survival analysis. Its probability density function (pdf) is given by

$$g_{\beta,\lambda,c}(x) = \frac{c\lambda^{c\beta}}{\Gamma(\beta)} x^{c\beta-1} \exp\{-(\lambda x)^c\}, \quad x > 0, \tag{1}$$

where $\lambda > 0$ is a scale parameter, $\beta > 0$ and $c > 0$ are shape parameters and $\Gamma(\beta) = \int_0^\infty w^{\beta-1}\,\mathrm{e}^{-w}\,\mathrm{d}w$ is the gamma function. The Weibull, gamma and half-normal distributions correspond to $\beta = 1, c = 1$ and $\beta = 1/2, c = 2$, respectively. In addition, the log-normal distribution is a limiting special case when $\beta \to \infty$.

*Corresponding author. Email: mcastro@icmc.usp.br

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The cumulative distribution function (cdf) corresponding to Equation (1) is

$$G_{\beta,\lambda,c}(x) = \frac{\gamma(\beta, (\lambda x)^c)}{\Gamma(\beta)}, \tag{2}$$

where $\gamma(\beta, x) = \int_0^x w^{\beta-1}\,\mathrm{e}^{-w}\,\mathrm{d}w$ is the incomplete gamma function. Stacy and Mihram [2], Harter [3] and Hager and Bain [4] encountered some difficulties in developing maximum likelihood procedures and large sample inference for its parameters. On the other hand, Prentice [5] re-parameterized it in such a way that the inference can be fairly easily handled. Lawless [6], using Prentice's re-parametrization, developed exact inference procedures concerning the quantiles and scale parameters from uncensored samples and DiCiccio [7] proposed approximate conditional inference methods for location and scale parameters. Recently, Huang and Hwang [8] presented a simple method for estimating the model parameters, using its characterization and moment estimation. Cox *et al.* [9] presented a parametric survival analysis and taxonomy of hazard rate functions. Further, Almpanidis and Kotropoulos [10] studied a text-independent automatic phone segmentation algorithm based on the GG distribution and Nadarajah [11] analysed some incorrect references with respect to the use of this distribution in electrical and electronic engineering. An iterative estimation method for its parameters was implemented in S-PLUS by Gomes *et al.* [12]. Tadikamalla [13] proposed a simple rejection method for sampling directly from the GG distribution without generating gamma variates. This method is only applicable for $\beta > 1$.

Consider starting from an arbitrary baseline cdf $G(x)$, Eugene *et al.* [14] defined a class of beta generalized (BG) distributions by

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)} \omega^{a-1}(1-\omega)^{b-1}\,\mathrm{d}\omega, \tag{3}$$

where $a > 0$ and $b > 0$ are two additional shape parameters whose role is to introduce skewness and to vary tail weight, $B(a,b) = \int_0^1 \omega^{a-1}(1-\omega)^{b-1}\,\mathrm{d}\omega$ is the beta function and $I_y(a,b) = B(a,b)^{-1}\int_0^y w^{a-1}(1-w)^{b-1}\,\mathrm{d}w$ is the incomplete beta function ratio. One major benefit of this class of distributions is its ability of fitting skewed data that cannot be properly fitted by existing distributions. If $b = 1$, $F(x) = G(x)^a$ and then F is usually called the exponentiated G distribution (or the Lehmann type-I distribution). See, for example, the exponentiated Weibull [15] and exponentiated exponential [16] distributions. Eugene *et al.* [14], Nadarajah and Kotz [17], Nadarajah and Gupta [18] and Nadarajah and Kotz [19] defined the beta normal, beta Gumbel, beta Fréchet and beta exponential distributions by taking $G(x)$ to be the cdf of the normal, Gumbel, Fréchet and exponential distributions, respectively. Furthermore, Pescim *et al.* [20] proposed the beta-G half-normal distribution, which contains some important distributions as special cases, such as the half-normal and generalized half-normal distributions, the last one defined by Cooray and Ananda [21]. Paranaíba *et al.* [22] defined the beta Burr XII distribution, which contains as special sub-models some well-known distributions discussed in the literature, such as the logistic, Weibull and Burr XII distributions, among several others. Zografos and Balakrishnan [23] characterized the beta generated family using the maximum entropy principle and defined adequate conditions for which the Kullback–Leibler entropy is maximized. Further, they defined a new family of generated distributions from a parent distribution G and a two-parameter gamma random variable Z using the transformation $T = G^{-1}(1-\mathrm{e}^{-Z})$. The distribution of T is completely different from the beta-G distribution studied in this article.

The pdf corresponding to Equation (3) can be written as

$$f(x) = \frac{g(x)}{B(a,b)} G(x)^{a-1}\{1-G(x)\}^{b-1}, \tag{4}$$

where $g(x) = \mathrm{d}G(x)/\mathrm{d}x$ is the baseline density function. The density $f(x)$ will be most tractable when both $G(x)$ and $g(x) = \mathrm{d}G(x)/\mathrm{d}x$ have simple analytic expressions.

Inserting the cdf (2) in Equation (3), we obtain (for $x > 0$) the beta generalized gamma (BGG) cumulative function with five positive parameters

$$F(x) = I_{\gamma(\beta,(\lambda x)^c)/\Gamma(\beta)}(a,b) = \frac{1}{B(a,b)} \int_0^{\gamma(\beta,(\lambda x)^c)/\Gamma(\beta)} \omega^{a-1}(1-\omega)^{b-1}\,\mathrm{d}\omega, \tag{5}$$

which involves simultaneously the incomplete beta and gamma functions. The density and hazard rate functions associated with Equation (5) are (for $x > 0$)

$$f(x) = \frac{c\lambda^{c\beta}x^{c\beta-1}\exp\{-(\lambda x)^c\}\gamma(\beta,(\lambda x)^c)^{a-1}\{\Gamma(\beta)-\gamma(\beta,(\lambda x)^c)\}^{b-1}}{B(a,b)\Gamma(\beta)^{a+b-1}} \tag{6}$$

and

$$h(x) = \frac{c\lambda^{c\beta}x^{c\beta-1}\exp\{-(\lambda x)^c\}\gamma(\beta,(\lambda x)^c)^{a-1}\{\Gamma(\beta)-\gamma(\beta,(\lambda x)^c)\}^{b-1}}{B(a,b)\Gamma(\beta)^{a+b-1}[1-I_{\gamma(\beta,(\lambda x)^c)/\Gamma(\beta)}(a,b)]}, \tag{7}$$

respectively. Equations (5)–(7) are straightforward to compute using any statistical software with numerical facilities. The density function (6) allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology. We study some structural properties of this distribution since it extends several well-known distributions.

The GG model with parameters β, λ and c is clearly a special sub-model for $a = b = 1$, with a continuous crossover towards cases with different shapes (e.g. a particular combination of skewness and kurtosis). The BGG distribution also contains the exponentiated generalized gamma distribution as a special sub-model when $b = 1$. For $a = 1$, Equation (6) reduces to a new distribution which is called the Lehmann type-II GG distribution. The beta exponential [19] and beta Weibull [24] distributions are special sub-models for $\beta = c = 1$ and $\beta = 1$, respectively. As pointed out by a referee, if Y follows a beta gamma distribution with parameters $(\beta, \lambda = 1, a, b)$, then $X = \lambda^{-1}Y^{1/c} \sim \mathrm{BGG}(\beta, \lambda, c, a, b)$. In Figure 1, we plot the density function for selected values of a, b and c when $\lambda = \beta = 1$ including some special cases listed in Table 1. The parameters of the GG distribution are identifiable, and then it seems that lack of identifiability is not a concern.

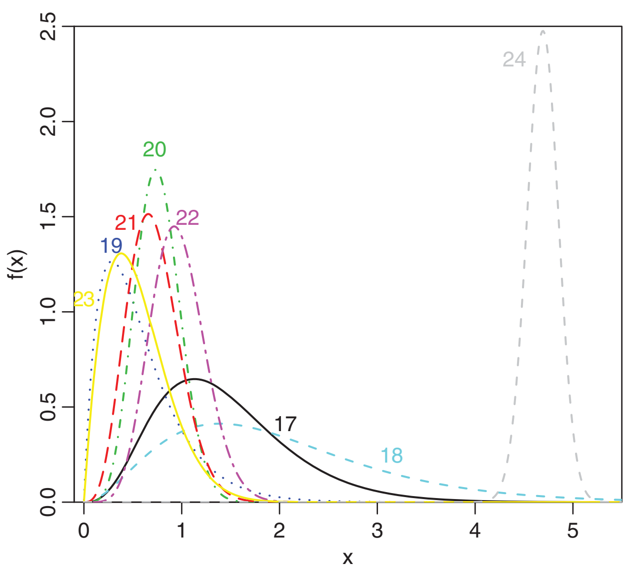

Figure 1. pdfs for some distributions in Table 1.

Table 1. Some particular cases of the BGG distribution.

				$a = b = 1$			
Case			c	λ	β	Distribution	References
(1)			c	λ	β	GG	[1]
(2)			1	λ	β	Gamma	
(3)			1	$\frac{1}{2}$	$n/2$	Chi-square	
(4)			1	λ	1	Exponential	
(5)			c	λ	1	Weibull	
(6)			2	λ	1	Rayleigh	
(7)			2	λ	$\frac{3}{2}$	Maxwell	
(8)			2	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	Folded normal	
(9)			c	λ	∞	Log-normal	
						$b = 1$	
	a		c	λ	β		
(10)	a		1	λ	β	Exponentiated gamma	
(11)	a		1	$\frac{1}{2}$	$n/2$	Exponentiated chi-square	
(12)	a		1	λ	1	Exponentiated exponential	[16]
(13)	a		c	λ	1	Exponentiated Weibull	
(14)	a		2	λ	1	Exponentiated Rayleigh	
(15)	a		2	λ	$\frac{3}{2}$	Exponentiated Maxwell	
(16)	a		2	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	Exponentiated folded normal	
(17)	a		c	λ	∞	Exponentiated log-normal	
	a	b	c	λ	β		
(18)	a	b	1	λ	β	Beta gamma	[25]
(19)	a	b	1	$\frac{1}{2}$	$n/2$	Beta chi-square	
(20)	a	b	1	λ	1	Beta exponential	[19]
(21)	a	b	c	λ	1	Beta Weibull	[24,26]
(22)	a	b	2	λ	1	Beta Rayleigh	
(23)	a	b	2	λ	$\frac{3}{2}$	Beta Maxwell	
(24)	a	b	2	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	Beta folded normal	
(25)	a	b	c	λ	∞	Beta log-normal	

For a and b are positive integers, the BGG density function becomes the density function of the ath order statistic from the GG distribution in a sample of size $a + b - 1$.

Let $\{X_n\}$ be a sequence of iid random variables with an arbitrary continuous distribution G. Consider the random variable Z, independent from $\{X_n\}$, having the exponentialized G distribution, that is, its cdf is $F_Z(x) = G(x)^a$. We assume $b < 1$ and define a discrete random variable M, independent from $\{X_n\}$ and Z, with probabilities (for $n = 1, 2, \ldots$)

$$p_n = \frac{(1-b)_n}{B(a,b)(a+n)(1-\alpha)n!},$$

where $\alpha = [B(a,b)a]^{-1}$, $(1-b)_n = (1-b)(2-b)\ldots(n-b)$ and $\sum_{n=1}^{\infty} p_n = 1$. Next, we define $R = \max\{X_1, \ldots, X_M, Z\}$. Let X be a random variable having the beta-G distribution with additional shape parameters a and b. Given R, Z and α, we can demonstrate (if $0 < b < 1$) that the density function of X is a simply mixture of the densities of Z and R with proportions α and $1 - \alpha$, respectively.

An algorithm to generate X can be obtained as follows: if V is sampled from a beta distribution with parameters (a, b), then $X = \lambda^{-1}[H_{1,\beta}^{-1}(V)]^{1/c}$ follows the BGG$(\beta, \lambda, c, a, b)$ distribution, where $H^{-1}_{1,\beta}(u)$ denotes the gamma quantile function with parameters $(1, \beta)$. Hence, random samples from the BGG distribution can be generated using freely available software [27]. Further, while the transformation (3) is not analytically tractable in the general case, the formulae related to the BGG distribution turn out manageable as it is shown in this article.

The rest of the paper is organized as follows. In Section 2, we demonstrate that the BGG density function can be expressed as a linear combination of GG density functions. This is an important result to provide some mathematical properties of the BGG distribution directly from those properties of the GG model. Some explicit expansions for the moments, moment-generating function (mgf), mean deviations, Bonferroni and Lorenz curves, order statistics and their moments and Rényi entropy are provided in Section 3. Estimation by the method of maximum likelihood is presented in Section 4. Section 5 illustrates an application of the new model to a real data set. Section 6 provides some concluding remarks.

2. The BGG density expansion

The cdf $G_{\beta,\lambda,c}(x)$ of the GG distribution is usually straightforward to compute numerically using statistical software with numerical facilities. However, we demonstrate that the BGG density function can be written as a linear combination of GG densities.

For b real non-integer, we expand the binomial term in Equation (4) to yield

$$f(x) = \frac{g(x)}{B(a,b)} \sum_{r=0}^{\infty} w_r G(x)^{a+r-1}, \tag{8}$$

where $w_j = w_j(b) = (-1)^j \binom{b-1}{j}$. If b is an integer, the above sum stops at $b-1$. If a is an integer, Equation (8) gives the beta-G density function in terms of an infinite power series of $G(x)$.

An integer power of the GG cumulative distribution can be expanded as

$$G_{\beta,\lambda,c}(x)^r = \frac{(\lambda x)^{rc\beta}}{\Gamma(\beta)^r} \sum_{m=0}^{\infty} c_{r,m} (\lambda x^c)^m, \tag{9}$$

where the quantities $c_{r,m}$ (for $m = 1, 2, \ldots$) are easily determined recursively from $c_{j,0} = a_0^j$ and

$$c_{j,i} = (ia_0)^{-1} \sum_{m=1}^{i} [(j+1)m - i] a_m c_{j,i-m}, \tag{10}$$

where $a_m = (-1)^m / [(a+m)m!]$. The coefficients $c_{j,i}$ for any i can be calculated directly from $c_{j,0}, \ldots, c_{j,i-1}$ and, therefore, from $a_0, \ldots, a_i$.

From Equations (1), (8) and (9), we obtain an expansion for the BGG density function (for $a > 0$ integer)

$$f_{\text{int}}(x) = \sum_{r,m=0}^{\infty} q_{\text{int}}(r, m, \beta, \lambda, a, b) g_{(r+a)\beta+m,\lambda,c}(x), \tag{11}$$

where

$$q_{\text{int}}(r, m, \beta, \lambda, a, b) = \frac{\lambda^{m(1-c)} \Gamma((r+a)\beta + m) w_r c_{r+a-1,m}}{B(a,b)\Gamma(\beta)^{r+a}}.$$

However, if a is a real non-integer, we have to derive another power-series expansion for $f(x)$. First, we use the expansion for any $\alpha > 0$ real non-integer as

$$G(x)^{\alpha} = \sum_{r=0}^{\infty} s_r(\alpha) G(x)^r, \tag{12}$$

where $s_r(\alpha) = \sum_{j=r}^{\infty} (-1)^{r+j} \binom{\alpha}{j}\binom{j}{r}$. The power-series expansion (12) is required for the BGG density expansion (a real) and for the density of the BGG order statistics. Inserting Equation (12) in Equation (8), the beta-G density function can be expressed as a power series of $G(x)$ (for a real non-integer)

$$f(x) = \frac{g(x)}{B(a,b)} \sum_{r=0}^{\infty} t_r G(x)^r, \tag{13}$$

where $t_r = t_r(a,b) = \sum_{j=0}^{\infty} w_j s_r(a+j-1)$.

Analogously, from Equations (1), (9) and (13), we obtain (for $a > 0$ a real non-integer)

$$f_{\text{real}}(x) = \sum_{r,m=0}^{\infty} q_{\text{real}}(r,m,\beta,\lambda,a,b) g_{(r+1)\beta+m,\lambda,c}(x), \tag{14}$$

where

$$q_{\text{real}}(r,m,\beta,\lambda,a,b) = \frac{\lambda^{m(1-c)}\Gamma((r+1)\beta+m) t_r c_{r,m}}{B(a,b)\Gamma(\beta)^{r+1}}.$$

Equations (11) and (14) are the main results of this section. They allow us to derive some BGG mathematical properties from those properties of the GG distribution.

3. Properties of the BGG distribution

3.1. *Moments and generating function*

Here and henceforth, X stands for a random variable having the BGG density function (6). For a integer, Equation (11) gives the sth moment of X as

$$E(X^s) = \lambda^{-s} \sum_{r,m=0}^{\infty} q_{\text{int}}(r,m,\beta,\lambda,a,b) \frac{\Gamma((r+a)\beta+m+s/c)}{\Gamma((r+a)\beta+m)}.$$

In a similar manner, for a real non-integer, Equation (14) gives $E(X^s)$.

The moments of the BGG distribution can be expressed as linear functions of the corresponding moments of GG distributions. These expansions are readily computed numerically using standard statistical software. They (and other expansions in this article) can also be evaluated in symbolic computation software such as Mathematica and Maple. In numerical applications, a large natural number N can be used in the sums instead of infinity.

We provide explicit expressions for the mgf $M(t) = E[\exp(tX)]$ of X. First, the mgf of the GG($\beta^{\star}, \lambda, c$) distribution is given by [2]

$$M_{\beta^{\star},\lambda,c}(s) = \frac{1}{\Gamma(\beta^{\star})} \sum_{m=0}^{\infty} \Gamma\left(\frac{m}{c} + \beta^{\star}\right) \frac{(s/\lambda)^m}{m!}.$$

For $a > 0$ real non-integer, it follows from Equation (14) and the GG generating function with $\beta^\star = (r+1)\beta + m$

$$M_{\text{real}}(t) = \sum_{r,m=0}^{\infty} q_{\text{real}}(r, m, \beta, \lambda, a, b,)M_{(r+1)\beta+m,\lambda,c}(t). \tag{15}$$

Analogously, for $a > 0$ integer, we can express the mgf of X as

$$M_{\text{int}}(t) = \sum_{r,m=0}^{\infty} q_{\text{int}}(r, m, \beta, \lambda, a, b)M_{(r+a)\beta+m,\lambda,c}(t). \tag{16}$$

Clearly, special formulas for the mgf of all sub-models of the BGG distribution can be easily derived from Equations (15) and (16) by the substitution of known parameters.

3.2. *Mean deviations*

The mean deviations of X about the mean $\mu = E(X)$ and about the median m are $\delta_1 = \int_0^\infty | x - \mu | f(x)\,\mathrm{d}x$ and $\delta_2 = \int_0^\infty | x - m | f(x)\,\mathrm{d}x$, respectively. The median is the solution of the nonlinear equation $I_{[\gamma(\beta,(\lambda m)^c)/\Gamma(\beta)]}(a, b) = \frac{1}{2}$. Defining the integral $L(s) = \int_0^s xf(x)\,\mathrm{d}x$, these measures can be expressed as

$$\delta_1 = 2\mu F(\mu) - 2L(\mu) \quad \text{and} \quad \delta_2 = E(X) - 2L(\mu), \tag{17}$$

where $F(\mu)$ is easily calculated from Equation (3). Now, we derive formulas for the integral $L(s)$. Setting $\rho_r(s) = \int_0^s xg_{\beta,\lambda,c}(x)G_{\beta,\lambda,c}(x)^r\,\mathrm{d}x$, $r \in \mathbb{N}$, it follows from Equations (8) and (13) for $a > 0$ integer and $a > 0$ real non-integer

$$L(s) = \frac{1}{B(a,b)} \sum_{r=0}^{\infty} w_r \rho_{a+r-1}(s) \quad \text{and} \quad L(s) = \frac{1}{B(a,b)} \sum_{r=0}^{\infty} t_r \rho_r(s), \tag{18}$$

respectively. By calculating the integral $\rho_r(s)$, we obtain

$$\rho_r(s) = \frac{1}{\lambda\Gamma(\beta)^{r+1}} \sum_{m=0}^{\infty} \frac{\Gamma((1+r)\beta + m)c_{r,m}}{\Gamma((1+r)\beta + m + c^{-1})} \gamma((1+r)\beta + m + c^{-1}, \lambda^c s^c). \tag{19}$$

Hence, we can calculate the mean deviations in Equation (17) from Equations (18) and (19).

Bonferroni and Lorenz curves are useful in fields like reliability, economics, demography, insurance and medicine. For the BGG distribution, these curves can be calculated (for a given $0 < \pi < 1$) from $B(\pi) = (\pi\mu)^{-1}L(q)$ and $K(\pi) = \mu^{-1}L(q)$, respectively, where $\mu = E(X)$, $q = F^{-1}(\pi)$ is the BGG quantile function obtained by inverting Equation (5) and $L(q)$ can be determined from Equation (18).

3.3. *Order statistics*

Order statistics make their appearance in many areas of statistical theory and practice. The density $f_{i:n}(x)$ of the ith order statistic for $i = 1, \ldots, n$ from data values $X_1, \ldots, X_n$ having the BGG$(\beta, \lambda, c, a, b)$ distribution can be written as

$$f_{i:n}(x) = \frac{g(x)G(x)^{a-1}\{1 - G(x)\}^{b-1}}{B(a,b)B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{i+j-1}. \tag{20}$$

We now demonstrate that $f_{i:n}(x)$ can be expressed as a linear combination of GG density functions. First, we provide expansions for a power (integer or real non-integer) of $F(x)$ as infinite power series of the baseline $G(x)$. For a integer and a real non-integer, the beta-G cumulative distribution follows by integrating Equations (8) and (13) as

$$F(x)=\frac{1}{B(a,b)}\sum_{r=0}^{\infty}w_r^{*}G(x)^{a+r}\quad\text{and}\quad F(x)=\frac{1}{B(a,b)}\sum_{r=0}^{\infty}t_r^{*}G(x)^{r},\tag{21}$$

respectively, where $w_r^{*}=(-1)^r\binom{b-1}{r}/(a+r)$, $t_r^{*}=\sum_{j=0}^{\infty}w_j^{*}s_r(a+j)$ and $s_r(a+j)$ is defined in Section 3. For $a>0$ integer and $b>0$ real non-integer, using Equations (9), (20) and (21), we can write

$$f_{i:n}(x)=\sum_{j=0}^{n-i}\sum_{r,l=0}^{\infty}(-1)^{j+l}\binom{b-1}{l}\binom{n-i}{j}\frac{c_{i+j-1,r}^{*}g(x)G(x)^{r+l+a(i+j)-1}}{B(a,b)^{i+j}B(i,n-i+1)},\tag{22}$$

where $c_{j,i}^{*}=(iw_0^{*})^{-1}\sum_{m=1}^{i}[(j+1)m-i]w_m^{*}c_{j,i-m}^{*}$ and $c_{j,0}^{*}=(w_0^{*})^{j}$ are calculated from Equation (10). If b is an integer, the index l in the sum (22) stops at $b-1$. In the same way, for $a>0$ real non-integer, we have

$$f_{i:n}(x)=\sum_{j=0}^{n-i}\sum_{r,l=0}^{\infty}(-1)^{j+l}\binom{b-1}{l}\binom{n-i}{j}\frac{d_{i+j-1,r}g(x)G(x)^{r+l+a-1}}{B(a,b)^{i+j}B(i,n-i+1)},\tag{23}$$

where $d_{i+j-1,r}=(rt_0^{*})^{-1}\sum_{m=1}^{r}[(i+j)m-r]t_m^{*}d_{i+j-1,r-m}$, and $d_{i+j-1,0}=(t_0^{*})^{i+j-1}$. If b is an integer, the index l in the sum (23) stops at $b-1$.

Let $\tau_{s,r}=\int_0^{\infty}x^{s}G_{\beta,\lambda,c}(x)^{r}$ be the probability weighted moment of the GG(β,λ,c) distribution given by

$$\tau_{s,r}=\lambda^{-s}\Gamma(\beta)^{-(r+1)}\sum_{m=0}^{\infty}c_{r,m}\Gamma\left((1+r)\beta+m+\frac{s}{c}\right).$$

The sth ordinary moment of the ith order statistic, say $X_{i:n}$, for $a>0$ integer, follows from Equation (22) as

$$E(X_{i:n}^{s})=\sum_{j=0}^{n-i}\sum_{r,l=0}^{\infty}\frac{(-1)^{j+l}c_{r,i+j-1}^{*}\binom{n-i}{j}\binom{b-1}{l}}{B(a,b)^{i+j}B(i,n-i+1)}\tau_{s,r+a(i+j)+l-1},\tag{24}$$

where the coefficient $c_{r,i+j-1}^{*}$ was defined before. If b is an integer, the index l in the above sum stops at $b-1$. For $a>0$ real non-integer, Equation (23) gives

$$E(X_{i:n}^{s})=\sum_{j=0}^{n-i}\sum_{r,l=0}^{\infty}\frac{(-1)^{j+l}d_{r,i+j-1}\binom{n-i}{j}\binom{b-1}{l}}{B(a,b)^{i+j}B(i,n-i+1)}\tau_{s,r+a+l-1},\tag{25}$$

where $d_{r,i+j-1}$ was defined before. If b is an integer, the index l in the above sum stops at $b-1$.

3.4. *Entropy*

An entropy of a random variable X is a measure of variation of the uncertainty. One of the popular entropy measures is the Rényi entropy defined by

$$\Im(\rho)=\frac{1}{1-\rho}\log\left\{\int f(x)^{\rho}\,\mathrm{d}x\right\},$$

where $\rho > 0$ and $\rho \neq 1$. For the density function (4), we have

$$f(x)^\rho = \frac{g(x)^\rho}{B(a,b)^\rho} G(x)^{(a-1)\rho}(1-G(x))^{(b-1)\rho}.$$

If $(b-1)\rho > 0$ and $(a-1)\rho > 0, f(x)^\rho$ can be expressed as a power series of the cdf $G(x)$ as $f(x)^\rho = C(x)\sum_{r=0}^{\infty} t_r(\rho)G(x)^r$, where $t_r(\rho)$ becomes

$$t_r(\rho) = \sum_{k=0}^{\infty}(-1)^k \binom{(b-1)\rho}{k} s_r((a-1)\rho + k),$$

where the quantity $s_r((a-1)\rho + k)$ is defined in Section 2 and $C(x) = B(a,b)^{-\rho}g(x)^\rho$. The entropy measure can be rewritten as

$$\Im(\rho) = \frac{1}{1-\rho}\log\left\{\sum_{r=0}^{\infty}\frac{t_r(\rho)}{B(a,b)^\rho} J_R(\rho)\right\},$$

where

$$J_R(\rho) = \sum_{m=0}^{\infty}\frac{c_{r,m}c^\rho\lambda^{c\beta(r+\rho)+m}}{\Gamma(\beta)^{\rho+r}}\frac{\Gamma((c(\rho\beta + r\beta + m) - \rho + 1)/c)}{c(\sqrt[c]{\rho})^{c(\rho\beta+r\beta+m)-\rho+1}}.$$

4. Inference

Consider that X follows the BGG distribution and let $\theta = (\lambda, \beta, c, a, b)^{\mathrm{T}}$ be the parameter vector. Setting $t = (\lambda x)^c$, the log-likelihood for a single observation x of X, say $\ell = \ell(\lambda, \beta, c, a, b)$, becomes

$$\ell = \log(c) + \log(\lambda) + \beta\log(t) - \frac{1}{c}\log(t) - t + (a-1)\log[\gamma(\beta,t)]$$
$$+ (b-1)\log[\Gamma(\beta,t)] - \log[B(a,b)] - (a+b-1)\log[\Gamma(\beta)].$$

The expected value of the score vector vanishes. Defining $T = (\lambda X)^c$, we obtain $E(T) = \beta, E\{\log(T)\} = \psi(\beta), E\{T\log(T)\} = 1 + \beta\psi(\beta)$, where $\psi(\cdot)$ is the digamma function and

$$E\left\{\frac{\mathrm{e}^{-T}T^\beta}{\gamma(\beta,T)}\right\} = E\left\{\frac{\mathrm{e}^{-T}T^{-\beta}}{\Gamma(\beta,T)}\right\} = E\left\{\frac{\mathrm{e}^{-T}T^{-\beta}\log(T)}{\gamma(\beta,T)}\right\}$$
$$= E\left\{\frac{\mathrm{e}^{-T}T^{\beta}\log(T)}{\Gamma(\beta,T)}\right\} = 0,$$

and

$$E\left\{\frac{\gamma'(\beta,T)}{\gamma(\beta,T)}\right\} = \psi(\beta), \quad E\left\{\frac{\gamma'(\beta,T)}{\Gamma(\beta,T)}\right\} = \psi(\beta)E\left\{\frac{\gamma(\beta,T)}{\Gamma(\beta,T)}\right\}.$$

For a random sample $x = (x_1, \ldots, x_n)$ of size n from X, the total log-likelihood is $\ell = \sum_{i=1}^{n}\ell^{(i)}$, where $\ell^{(i)}$ is the log-likelihood for the ith observation $(i = 1, \ldots, n)$. The maximum likelihood estimate (MLE) $\hat{\theta}$ of θ can be calculated numerically. For interval estimation and hypothesis tests on the parameters in θ, we require the 5×5 unit expected information matrix

$$K(\theta) = \{\kappa_{i,j}\}, \quad i,j = \beta, \lambda, c, a, b.$$

The elements of the information matrix K are given by $\kappa_{\lambda,\lambda} = c^2\psi(\beta)/\lambda^2$, $\kappa_{\lambda,c} = 1 + \beta\psi(\beta)$, $\kappa_{\lambda,a} = \kappa_{\lambda,b} = 0$, $\kappa_{\lambda,\beta} = -c/\lambda$, $\kappa_{\beta,\beta} = \psi'(\beta)$, $\kappa_{\beta,c} = -\psi(\beta)/c$, $\kappa_{\beta,a} = \kappa_{\beta,b} = 0$,

Table 2. The AIC and BIC statistics for the fitted distributions.

	Criterion		
Distribution	$-\max \ell(\theta)$	AIC	BIC
BGG	23.4	33.4	44.2
GG	29.2	35.2	41.6
BG	30.9	38.9	47.5
G	47.9	51.9	56.2

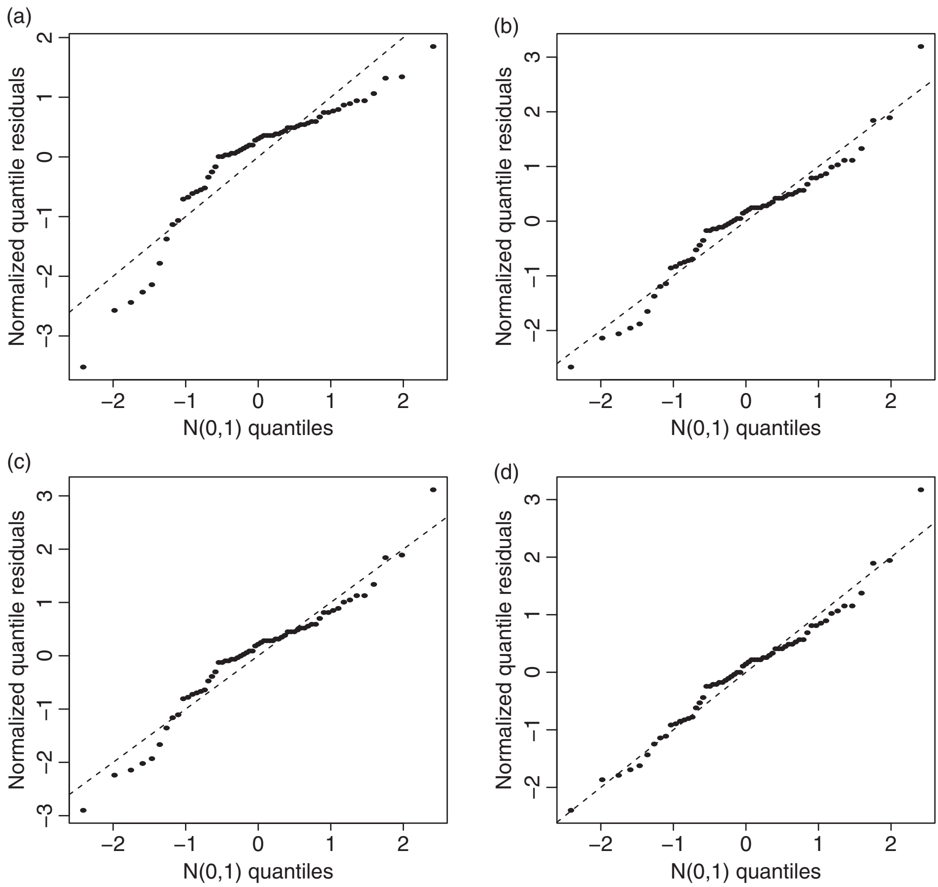

Figure 2. QQ plot of the normalized quantile residuals with an identity line for the distributions: (a) G, (b) GG, (c) BG and (d) BGG.

$\kappa_{c,c} = [1 + E(T\{\log(T)\}^2)]/c, \kappa_{c,a} = \kappa_{c,b} = 0, \kappa_{a,a} = \psi'(a) - \psi'(a+b), \kappa_{b,b} = \psi'(b) - \psi'(a+b)$ and $\kappa_{a,b} = -\psi'(a+b)$.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$ is $N_5(0, K(\theta)^{-1})$. The asymptotic multivariate normal $N_5(0, K_n(\hat{\theta})^{-1})$ distribution of $\hat{\theta}$, where $K_n(\theta) = nK(\theta)$ is the total information matrix, can be used to construct approximate confidence intervals and confidence regions for the parameters.

The likelihood ratio (LR) statistic is useful for comparing the BGG distribution with some of its sub-models. If we consider the partition $\theta = (\theta_1^T, \theta_2^T)^{\mathrm{T}}$, tests of hypotheses of the type

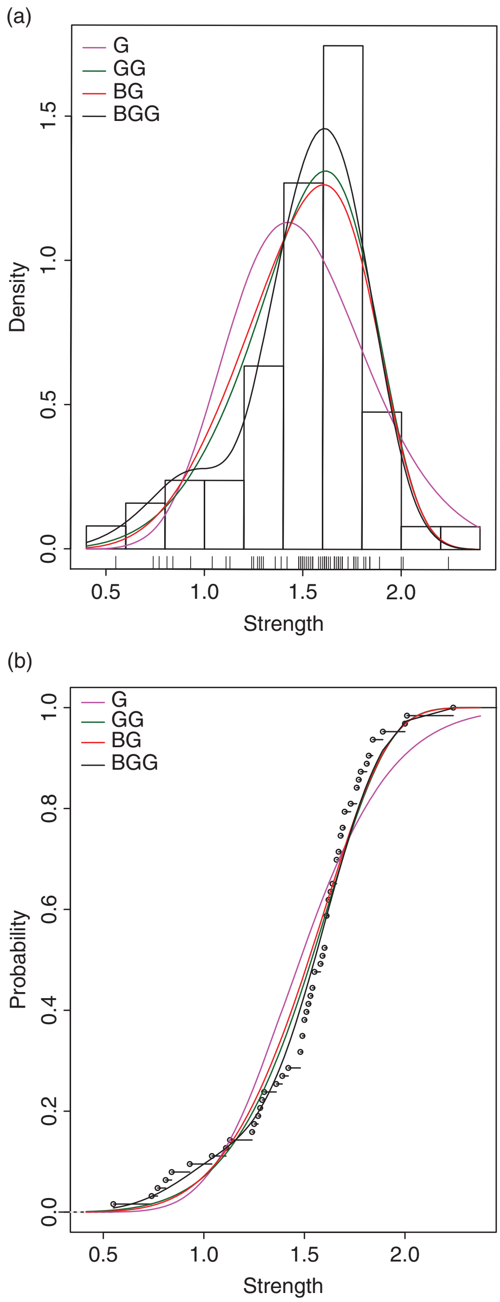

Figure 3. (a) Histogram of strength and fitted density functions and (b) empirical cumulative function of strength and fitted cumulative functions.

$H_0 : \theta_1 = \theta_1^{(0)}$ versus $H_A : \theta_1 \neq \theta_1^{(0)}$ can be performed using LR statistics. The LR statistic for testing a null model (H_0) against the BGG model (H_A) is $w = 2\{\ell(\hat{\theta}) - \ell(\tilde{\theta})\}$, where $\hat{\theta}$ and $\tilde{\theta}$ are the MLEs of θ under H_A and H_0, respectively. The LR test rejects H_0 if $w > \xi_\gamma$, where ξ_γ denotes the upper $100\gamma\%$ point of the χ_q^2 distribution and q is the difference of the dimensions of the vectors of parameters under both models. For example, we can check if the fit using the

BGG distribution is statistically 'superior' to a fit using the GG distribution for a given data set by testing $H_0 : a = b = 1$ versus $H_A : H_0$ is not true. Further, non-nested distributions can be compared based on the Akaike information criterion given by $\text{AIC} = -2\ell(\widehat{\theta}) + 2\#(\theta)$ and the Bayesian information criterion defined by $\text{BIC} = -2\ell(\widehat{\theta}) + \#(\theta)\log(n)$, where $\#(\theta)$ is the number of model parameters. The distribution with the smallest value of any of these criteria (among all distributions considered) is usually taken as the best choice for describing the given data set.

5. Application

We give an example with data from strength of 1.5 cm fibres (in unknown units). The data set contains 63 observations presented as sample one by Smith and Naylor [28]. The gamma (G), GG, BG and BGG distributions were fitted to the data. Computational code in R language [27] is available from the authors upon request. According to the classical statistics (AIC and BIC) in Table 2, the BGG and GG distributions are the best models.

The QQ plots of the normalized quantile residuals [29] in Figure 2 show the improvement in the fit achieved with the BGG distribution over the other distributions. This claim is supported by the inspection of the plots in Figure 3. We also emphasize the gain yielded by the BGG distribution in relation to the beta-G exponential distribution, recently proposed by Barreto-Souza *et al.* [30, Figure 5].

Comparing the BGG and BG distributions, the LR statistic is $w = 7.49$ (1 d.f., p-value $=$ 0.0062), whereas the BGG versus GG comparison represents a borderline situation at a 5% significance level since $w = 5.86$ (2 d.f., p-value $= 0.0533$). However, taking into account the values in Table 2 and the plots of Figures 2 and 3, we conclude that the BGG distribution provides a better fit. Parameter estimates (and estimated standard errors) for the BGG distribution are: $\hat{a} = 0.1305$ (0.1082), $\hat{b} = 0.0185$ (0.0180), $\hat{c} = 6.0008$ (0.9087), $\hat{\lambda} = 1.2151$ (0.2553) and $\hat{\beta} =$ 5.9155 (5.6735), so that the null hypothesis $H_0 : c = 1$ is strongly rejected.

6. Conclusion

We introduce the BGG distribution with two additional positive parameters because of the wide usage of the GG distribution and the fact that the current generalization provides extensions to its continuous extension to still more complex situations. The new distribution unifies more than 24 distributions and yields a general overview of these distributions for theoretical studies. In fact, the BGG distribution (6) generalizes the Weibull, exponentiated Weibull [15,16,31–34], beta exponential [19] and beta Weibull [24] distributions and other important models. The BGG density function can be expressed as a linear combination of GG density functions which allow us to derive some of its mathematical properties. The estimation of parameters is approached by the method of maximum likelihood and the expected information matrix is calculated. The usefulness of the BGG distribution is illustrated in one application to a real data set. The new model provides a rather flexible mechanism for fitting a wide spectrum of real world lifetime data in reliability, biology and other areas.

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