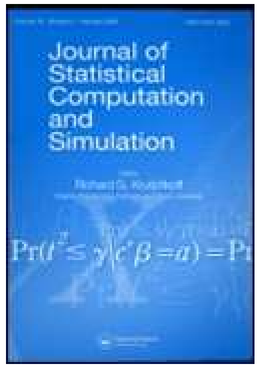

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The Log-Exponentiated Generalized Gamma Regression Model for Censored Data

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Abstract

For the first time, we introduce a generalized form of the exponentiated generalized gamma distribution (Cordeiro et al., 2010) that is the baseline for the log-exponentiated generalized gamma regression model. The new distribution can accommodate increasing, decreasing, bathtub and unimodal shaped hazard functions. A second advantage is that it includes classical distributions reported in lifetime literature as special cases. We obtain explicit expressions for the moments of the baseline distribution of the new regression model. The proposed model can be applied to censored data since it includes as sub-models several widely-known regression models. It therefore can be used more effectively in the analysis of survival data. We obtain maximum likelihood estimates for the model parameters by considering censored data. We show that our extended regression model is very useful by means of two applications to real data.

Keywords: Censored data; Exponentiated generalized gamma distribution; Log-gamma generalized regression; Survival function.

1 Introduction

Standard lifetime distributions usually present very strong restrictions to produce bathtub curves, and thus appear to be unappropriate for interpreting data with this characteristic. The gamma distribution is the most popular model for analyzing skewed data. The generalized gamma

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distribution (GG) was introduced by Stacy (1962) and includes as special sub-models: the exponential, Weibull, gamma and Rayleigh distributions, among others. The GG distribution is suitable for modeling data with hazard rate function of different forms: increasing, decreasing, bathtub and unimodal, which makes it useful for estimating individual hazard functions as well as both relative hazards and relative times (Cox, 2008). Recently, the GG distribution has been used in several research areas such as engineering, hydrology and survival anlaysis. Ortega et al. (2003) discussed influence diagnostics in GG regression models, Nadarajah and Gupta (2007) used this distribution with application to drought data, Cox et al. (2007) presented a parametric survival analysis and taxonomy of GG hazard functions and Ali et al. (2008) derived the exact distributions of the product X_1X_2 and the quotient X_1/X_2, when X_1 and X_2 are independent GG random variables providing applications of their results to drought data from Nebraska. Further, Gomes et al. (2008) focused on the parameter estimation, Ortega et al. (2008) compared three types of residuals based on the deviance component in GG regression models under censored observations, Cox (2008) discussed and compared the F-generalized family with the GG model, Almpanidis and Kotropoulos (2008) presented a text-independent automatic phone segmentation algorithm based on the GG distribution and Nadarajah (2008a) analyzed some incorrect references with respect to the use of this distribution in electrical and electronic engineering. More recently, Barkauskas et al. (2009) modeled the noise part of a spectrum as an autoregressive moving average (ARMA) model with innovations following the GG distribution, Malhotra et al. (2009) provided a unified analysis for wireless system over generalized fading channels that is modeled by a two parameter GG model and Xie and Liu (2009) analyzed three-moment auto conversion parametrization based on the GG distribution. Further, Ortega et al. (2009) proposed a modified GG regression model to allow the possibility that long-term survivors may be presented in the data and Cordeiro et al. (2010) proposed the exponentiated generalized gamma (EGG) distribution. This distribution due to its flexibility in accommodating many forms of the risk function seems to be an important model that can be used in a variety of problems in survival analysis.

In the last decade, new classes of distributions for modeling survival data based on extensions of the Weibull distribution were developed. Mudholkar et al. (1995) introduced the exponentiated Weibull (EW) distribution, Xie and Lai (1995) presented the additive Weibull distribution, Lai et al. (2003) proposed the modified Weibull (MW) distribution and Carrasco et al. (2008) defined the generalized modified Weibull (GMW) distribution. Furthermore, the main motivation for the use of the EGG distribution is that it contains as special sub-models several distributions such as the generalized gamma (GG), EW, exponentiated exponential (EE) (Gupta and Kundu, 1999) and generalized Rayleigh (GR) (Kundu and Raqab, 2005) distributions. The EGG distribution can model four types of the failure rate function (i.e. increasing, decreasing, unimodal and bathtub) depending on the values of its parameters. It is also suitable for testing goodness-of-fit of some special sub-models, such as the GG, EW, Weibull and GR distributions.

Different forms of regression models have been proposed in survival analysis. Among them, the location-scale regression model (Lawless, 2003) is distinguished since it is frequently used in clinical trials. In this article, we propose an extension of the EGG distribution and obtain some of its structural properties. Further, we propose a location-scale regression model based on this distribution, referred to as the log-exponentiated generalized gamma (LEGG) regression model, which is a feasible alternative for modeling the four existing types of failure rate functions.

The article is organized as follows. In Section 2, we define an extended version of the EGG distribution. In Section 3, we consider its moments, generating function, mean deviations, reliability, order statistics and their moments. In Section 4, we provide a simulation study. In Section 5, we define the LEGG distribution and derive an expansion for its moments. In Section 6, we propose a LEGG regression model for censored data. We consider the method of maximum likelihood to estimate the model parameters and derive the observed information matrix. In Section 7, we give two applications using well-known data sets to demonstrate the applicability of the proposed regression model. Section 8 ends with some concluding remarks.

2 The Exponentiated Generalized Gamma Distribution

Cordeiro et al. (2010) proposed the EGG distribution with four parameters $\alpha > 0$, $\tau > 0$, $k > 0$ and $\lambda > 0$ to extend the GG distribution (Stacy, 1962) that should be able to fit various types of data. The probability density function (pdf) of the EGG distribution has the form

$$f(t) = \frac{\lambda\tau}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right]\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1},\ t>0, \quad (1)$$

where $\Gamma(\cdot)$ is the gamma function, $\gamma(k,x) = \int_0^x w^{k-1}\,e^{-w}dw$ is the incomplete gamma function and $\gamma_1(k,x) = \gamma(k,x)/\Gamma(k)$ is the incomplete gamma function ratio. The function $\gamma_1(k,x)$ is simply the cumulative distribution function (cdf) of a standard gamma distribution with shape parameter k. In (1), α is a scale parameter and τ, k and λ are shape parameters. The Weibull and GG distributions arise as special sub-models of (1) when $\lambda = k = 1$ and $\lambda = 1$, respectively. The EGG distribution approaches the log-normal distribution when $\lambda = 1$ and $k \to \infty$.

If T is a random variable with density (1), we write $T \sim \text{EGG}(\alpha,\tau,k,\lambda)$. The survival and hazard rate functions corresponding to (1) are

$$S(t) = 1 - F(t) = 1 - \left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda} \quad (2)$$

and

$$h(t) = \frac{\lambda\tau}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right]\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1}\left\{1-\left[\gamma_1\left(k,\left(\frac{t}{\alpha}\right)^{\tau}\right)\right]^{\lambda}\right\}^{-1}, \quad (3)$$

respectively. The EGG distribution has a hazard rate function that involves a complicated function, although it can be easily computed numerically. Moreover, it is quite flexible for modeling survival data.

We consider a generalized form of the EGG density function (1) given by

$$f(t) = \frac{\lambda\,|\tau|}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right]\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1},\ t>0, \tag{4}$$

where τ is not zero and the other parameters are positive. If $\tau > 0$, (4) is the EGG distribution and if $\lambda = 1$ and $\tau > 0$ it becomes the GG distribution. The special case $\lambda = k = 1$ and $\tau > 0$ gives the Weibull distribution. For $\lambda = k = 1$ and $\tau < 0$, it yields to the reciprocal Weibull (or inverse Weibull) distribution. Other special sub-models of the EGG distribution are discussed by Cordeiro et al. (2010). Plots of the EGG density function for selected values of $\tau > 0$ and $\tau < 0$ are given in Figure 1.

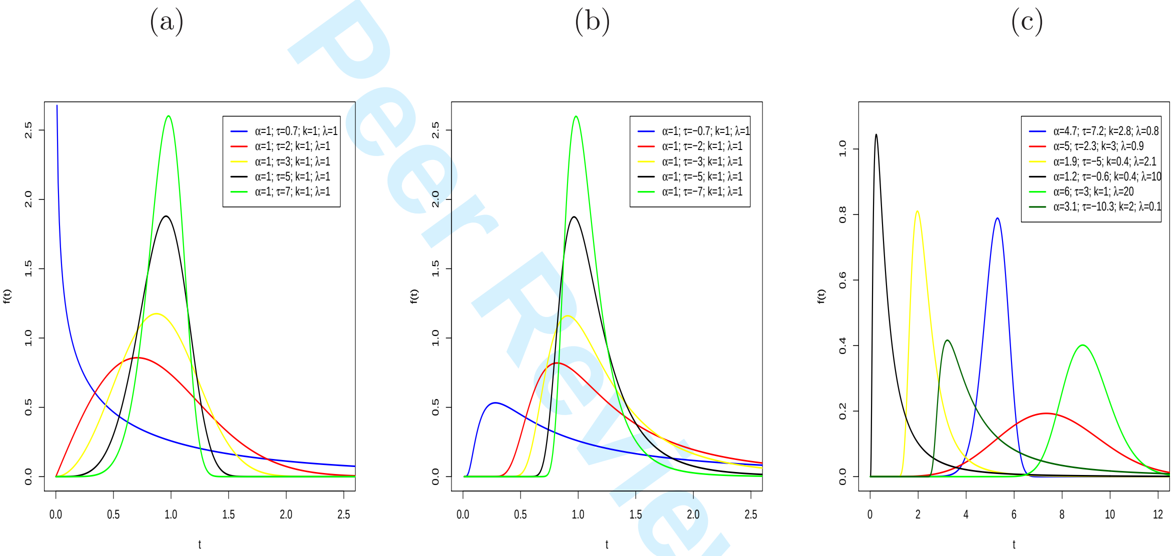

Figure 1: The EGG density function: (a) For some values of $\tau > 0$. (b) For some values of $\tau < 0$. (c) For some values of $\tau > 0$ and $\tau < 0$.

The cdf of the EGG distribution can be defined by

$$F(t) = \begin{cases} \left\{\gamma_1\left[k,(\frac{t}{\alpha})^{\tau}\right]\right\}^{\lambda} & \text{if} \quad \tau > 0, \\ 1-\left\{\gamma_1\left[k,(\frac{t}{\alpha})^{\tau}\right]\right\}^{\lambda} & \text{if} \quad \tau < 0. \end{cases} \tag{5}$$

The survival function corresponding to (5) is

$$S(t) = \begin{cases} 1 - \left\{\gamma_1\left[k, (\frac{t}{\alpha})^\tau\right]\right\}^\lambda & \text{if} \quad \tau > 0, \\ \left\{\gamma_1\left[k, (\frac{t}{\alpha})^\tau\right]\right\}^\lambda & \text{if} \quad \tau < 0. \end{cases} \tag{6}$$

Let $g_{\alpha,\tau,k}(t)$ be the density function of the GG(α, τ, k) distribution (Stacy and Mihram, 1965) given by

$$g_{\alpha,\tau,k}(t) = \frac{|\tau|}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k-1}\exp\left[-\left(\frac{t}{\alpha}\right)^\tau\right], \; t > 0. \tag{7}$$

Theorem 1. If $T \sim$ EGG$(\alpha, \tau, k, \lambda)$, we have the representation

$$f(t) = \sum_{m,i=0}^{\infty} w(m, i, k, \lambda)\, g_{\alpha,\tau,k(m+1)+i}(t), \quad t > 0, \tag{8}$$

where $g_{\alpha,\tau,k(m+1)+i}(t)$ is the $GG(\alpha, \tau, k(m + 1)i)$ density function defined by (7) and the weighted coefficients $w(m, i, k, \lambda)$ are given by

$$w(m, i, k, \lambda) = \sum_{m,i=0}^{\infty} \frac{\lambda\, s_m(\lambda)\, c_{m,i}\, \Gamma[k(m + 1) + i]}{\Gamma(k)^{m+1}}, \tag{9}$$

where the quantity $s_m(\lambda)$ is given by (37) and the constants $c_{m,i}$ can be determined from the recurrence equation (40). Clearly, the coefficients satisfy $\sum\limits_{m,i=0}^{\infty} w(m, i, k, \lambda) = 1$,

Some mathematical properties of the EGG distribution can follow directly from those properties of the GG distribution, since equation (8) is expressed in terms of a linear combination of GG densities. For example, the ordinary, inverse and factorial moments and moment generating function (mgf) of the EGG distribution can be derived directly from those quantities of the GG distribution. The proof of the theorem is in Appendix B.

3 General Properties

3.1 Moments

Here, we give two different expansions for the moments of the EGG distribution. Let $\mu'_r = E(T^r)$ be the rth ordinary moment of the EGG distribution. First, we obtain an infinite sum representation for μ'_r from equation (8). The rth moment of the GG(α, τ, k) distribution is

$$\mu'_{r,GG} = \frac{\alpha^r\, \Gamma(k + r/\tau)}{\Gamma(k)}.$$

Equation (8) then immediately gives

$$\mu'_r = \lambda\, \alpha^r \sum_{m,i=0}^{\infty} \frac{s_m(\lambda)\, c_{m,i}\, \Gamma([k(m+1)+i]+r/\tau)}{\Gamma(k)^{m+1}}. \qquad (10)$$

Equation (10) has the inconvenient that depends on the constants $c_{m,i}$ that can be calculated recursively from (40).

Theorem 2. If $T \sim$EGG(α,τ,k,λ), the rth moment of T reduces to

$$\mu'_r = \frac{\lambda\, \alpha^r\, \mathrm{sgn}(\tau)}{\Gamma(k)} \sum_{j=0}^{\infty} \sum_{m=0}^{j} (-1)^{j+m} \binom{\lambda-1}{j} \binom{j}{m} I\left(k, \frac{r}{\tau}, m\right), \qquad (11)$$

where $A(k, r/\tau, m) = k^{-m}\, \Gamma\big(r/\tau + k(m+1)\big)$ and

$$I\left(k, \frac{r}{\tau}, m\right) = A(k, r/\tau, m)\, F_A^{(m)}\big(r/\tau + k(m+1); k, \cdots, k; k+1, \cdots, k+1; -1, \cdots, -1\big). \qquad (12)$$

Hence, as an alternative representation to (10), the rth moment of the EGG distribution follows from both equations (11) and (12) as an infinite weighted sum of the Lauricella functions of type A. The proof of the theorem is in Appendix B.

3.2 Moment Generating Function

Theorem 3. If $T \sim$EGG(α,τ,k,λ), the mgf of T reduces to

$$M(t) = \sum_{m,i=0}^{\infty} w(m,i,k,\lambda)\, M_{\alpha,\tau,k(m+1)+i}(s), \qquad (13)$$

where $w(m,i,k,\lambda)$ is defined by (9) and

$$M_{\alpha,\tau,k(m+1)+i}(s) = \frac{\mathrm{sgn}(\tau)}{\Gamma(k(m+1)+i)} \sum_{m=0}^{\infty} \Gamma\left(\frac{m}{\tau} + k(m+1) + i\right) \frac{(s\alpha)^m}{m!}.$$

The proof of the theorem is in Appendix B.

3.3 Deviations

If T has the EGG density function $f(t)$, the deviations about the mean $\delta_1 = E(\mid T - \mu'_1 \mid)$ and about the median $\delta_2 = E(\mid T - m \mid)$ can be calculated from the relations

$$\delta_1 = 2\mu'_1 F(\mu'_1) - 2I(\mu'_1) \quad \text{and} \quad \delta_2 = \mu'_1 - 2I(m), \qquad (14)$$

where $F(a)$ is easily obtained from (5), m is the solution of $F(m) = 1/2$ and $I(s) = \int_0^s t\, f(t)dt$. This integral can be determined from (8) as

$$I(s) = \sum_{m,i=0}^{\infty} w(m,i,k,\lambda)\, J(\alpha,\tau,[k(m+1)+i],s), \tag{15}$$

where

$$J(\alpha,\tau,k,s) = \int_0^s t\, g_{\alpha,\tau,k}(t)dt.$$

We can obtain from the density of the GG(α,τ,k) distribution by setting $x = t/\alpha$

$$J(\alpha,\tau,k,s) = \frac{\alpha|\tau|}{\Gamma(k)} \int_0^{s/\alpha} x^{\tau k} \exp(-x^{\tau})dx.$$

The substitution $w = x^{\tau}$ yields $J(\alpha,\tau,k,s)$ in terms of the incomplete gamma function

$$J(\alpha,\tau,k,s) = \frac{\alpha\,\mathrm{sgn}(\tau)}{\Gamma(k)} \int_0^{(s/\alpha)^{\tau}} w^{k+\tau^{-1}-1} \exp(-w)dw = \gamma(k+\tau^{-1},(s/\alpha)^{\tau}).$$

Hence, inserting the last result into (15) gives

$$I(s) = \sum_{m,i=0}^{\infty} w(m,i,k,\lambda)\gamma(k(m+1)+i+\tau^{-1},(s/\alpha)^{\tau}). \tag{16}$$

The mean deviations of the EGG distribution can be obtained from equations (14) and (16).

3.4 Reliability

In the context of reliability, the stress-strength model describes the life of a component which has a random strength X_1 that is subjected to a random stress X_2. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X_1 > X_2$. For X_1 and X_2 independent random variables having a common EGG distribution, the reliability

$$R = R = Pr(X_2 < X_1) = \int_0^{\infty} f(t)\, F(t)dt,$$

where $f(t)$ and $F(t)$ are calculated from (4) and (5), respectively, can be written explicitly as follows:

- For $\tau > 0$,

$$R = \sum_{m,i=0}^{\infty} \frac{|\tau|\, w(m,i,k,\lambda)}{\alpha\Gamma[k(m+1)+i]} \int_0^\infty \exp\left[-\left(\frac{t}{\alpha}\right)^\tau\right] \left(\frac{t}{\alpha}\right)^{\tau[k(m+1)+i]-1} \times \left(1-\left\{\gamma_1\left[k, \left(\frac{t}{\alpha}\right)^\tau\right]\right\}^\lambda\right) dt.$$

Setting $x = \left(\frac{t}{\alpha}\right)^\tau$ in the last equation gives

$$R = \sum_{m,i=0}^{\infty} \frac{\mathrm{sgn}(\tau)\, w(m,i,k,\lambda)}{\Gamma[k(m+1)+i]} \int_0^\infty x^{k(m+1)+i-1} \exp(-x)\, \gamma_1(k,x)^\lambda dx.$$

By equation (36) in Appendix A, we obtain

$$R = \sum_{m,i,j=0}^{\infty} \sum_{m_1=0}^{j} v_1(m,m_1,i,j,k,\lambda)\, I[k(m+1),i,m_1], \qquad (17)$$

where

$$v_1(m,m_1,i,j,k,\lambda) = \frac{(-1)^{j+m_1}\,\mathrm{sgn}(\tau)\,\lambda\, s_m(\lambda)\, c_{m,i}}{\Gamma(k)^{m+1}} \binom{\lambda}{j} \binom{j}{m_1},$$

and

$$I[k(m+1),i,m_1] = \int_0^\infty x^{k(m+1)+i-1} \exp(-x)\, \gamma_1(k,x)^{m_1} dx.$$

Defining $C = [k(m+1)]^{-m_1}\Gamma\big(i+k(m+m_1+1)\big)$ and using the Lauricella function of type A (see Appendix B), this integral can be expressed as (see Nadarajah, 2008b, equation (23))

$$I\,[k(m+1),i,m_1] = C\, F_A^{(m_1)}\big(i+k(m+m_1+1); k(m+1),\ldots,k(m+1); k(m+1)+1,\ldots,k(m+1)+1; -1,\ldots,-1\big). \qquad (18)$$

- For $\tau < 0$,

$$R = \sum_{m,i=0}^{\infty} \frac{|\tau|\, w(m,i,k,\lambda)}{\alpha\,\Gamma[k(m+1)+i]} \int_0^\infty \exp\left[-\left(\frac{t}{\alpha}\right)^\tau\right] \left(\frac{t}{\alpha}\right)^{\tau[k(m+1)+i]-1} \times \left(1-\left\{\gamma_1\left[k, \left(\frac{t}{\alpha}\right)^\tau\right]\right\}^\lambda\right) dt.$$

Setting $x = \left(\frac{t}{\alpha}\right)^{\tau}$ in the last equation yields

$$R = 1 - \sum_{m,i=0}^{\infty} \frac{\mathrm{sgn}(\tau)\, w(m,i,k,\lambda)}{\Gamma[k(m+1)]+i} \int_0^{\infty} x^{k(m+1)+i-1} \exp(-x)\, \gamma_1(k,x)^{\lambda} dx.$$

Again, by equation (36), we obtain

$$R = 1 - \sum_{m,i,j=0}^{\infty} \sum_{m_1=0}^{j} v_2(m,m_1,i,j,k,\tau,\lambda)\, I\left[k(m+1),i,m_1\right], \quad (19)$$

where

$$v_2(m,m_1,i,j,k,\tau,\lambda) = \frac{(-1)^{j+m_1}\mathrm{sgn}(\tau)}{\Gamma[k(m+1)+i]} \binom{\lambda}{j} \binom{j}{m_1},$$

$w(m,i,k,\lambda)$ is just defined after (8) and $I\left[k(m+1),i,m_1\right]$ is given by (18).

3.5 Order Statistics

The density function $f_{i:n}(t)$ of the ith order statistic, for $i = 1,\ldots,n$, from random variables $T_1,\ldots,T_n$ having density (4), can be written as

$$f_{i:n}(t) = \frac{1}{B(i,n-i+1)} f(t) \sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^l\, F(t)^{i+l-1}, \quad (20)$$

where $f(t)$ and $F(t)$ are the pdf and cdf of the EGG distribution, respectively, and $B(\cdot,\cdot)$ denotes the beta function. Let $f_{\alpha,\tau,k,\lambda}(t)$ be the density function of the EGG(α,τ,k,λ) distribution. Plugging (4) and (5) in the last equation and after some algebra, we can write:

- For $\tau > 0$,

$$f_{i:n}(t) = \frac{1}{B(i,n-i+1)} \sum_{l=0}^{n-i} \frac{(-1)^l}{(i+l)} \binom{n-i}{l} f_{\alpha,\tau,k,\lambda(i+l)}(t). \quad (21)$$

Equation (21) gives the density function of the ith order statistic as a finite linear combination of EGG densities. Hence, the moments of the order statistics can be calculated directly from (21) using equations (11) and (12). The rth moment of the ith order statistic, say $\mu'_{r(i:n)}$, reduces to

$$\mu'_{r(i:n)} = \frac{\lambda\, \alpha^r \mathrm{sgn}(\tau)}{B(i,n-i+1)} \sum_{l=0}^{n-i} \sum_{j=0}^{\infty} \sum_{m=0}^{j} \frac{(-1)^{l+j+m}}{\Gamma(k)} \binom{n-i}{l} \binom{\lambda(i+l)-1}{j} \binom{j}{m} I\left(k,\frac{r}{\tau},m\right), \quad (22)$$

where $I(k,r/\tau,m)$ comes from (12).

- For $\tau < 0$,

$$f_{i:n}(t) = \frac{\lambda|\tau|}{\alpha\Gamma(k)B(i,n-i+1)}\left(\frac{t}{\alpha}\right)^{\tau k-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right]\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1}$$
$$\times \sum_{l=0}^{n-i}\binom{n-i}{l}(-1)^l\left(1-\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{i+l-1}.$$

Using the binomial expansion in the last expression, we have

$$f_{i:n}(t) = \frac{1}{B(i,n-i+1)}\sum_{l=0}^{n-i}\sum_{j_1=0}^{i+l-1}\frac{(-1)^{l+j_1}}{(1+j_1)}\binom{n-i}{l}\binom{i+l-1}{j_1}f_{\alpha,\tau,k,\lambda(1+j_1)}(t). \quad (23)$$

Equation (23) shows that the density function of the ith order statistic is a finite linear combination of EGG densities and then its moments can be determined directly from (23) using equations (11) and (12). We obtain

$$\mu'_{r(i:n)} = \frac{\lambda\,\alpha^r\text{sgn}(\tau)}{B(i,n-i+1)}\sum_{l=0}^{n-i}\sum_{j_1=0}^{i+l-1}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\frac{(-1)^{l+j+j_1+m}}{\Gamma(k)}\binom{n-i}{l}\binom{i+l-1}{j_1}$$
$$\times \binom{\lambda(1+j_1)-1}{j}\binom{j}{m}I\left(k,\frac{r}{\tau},m\right). \quad (24)$$

4 Simulation study

We perform some Monte Carlo simulation studies to assess on the finite sample behavior of the maximum likelihood estimators (MLEs) of α, τ, k and λ. The results were obtained from 3000 Monte Carlo replications and the simulations were carried out using the software R. In each replication, a random sample of size n is drawn from the EGG(α,τ,k,λ) distribution and the parameters were estimated by maximum likelihood. The EGG random variable was generated using the inversion method. The true parameter values used in the data generating processes are $\alpha = 0.1$, $\tau = 0.4$, $k = 1.2$ and $\lambda = 2.0$. The mean estimates of the four model parameters and the corresponding root mean squared errors (RMSEs) for sample sizes $n = 50$, $n = 100$, $n = 200$ and $n = 300$ are listed in Table 1.

The figures in Table 1 show that the biases and RMSEs of MLEs of α, τ, k and λ decay toward zero when the sample size increases, as expected. There is a small sample bias in the estimation of the model parameters. Future research should be conducted to obtain bias corrections for these estimators.

Table 1: Mean estimates and RMSEs of α, τ, k and λ

n	Parameter	Mean	RMSE
50	α	0.1616	0.2885
	τ	0.5159	1.0505
	k	2.0871	2.3444
	λ	2.5206	2.2346
100	α	0.1325	0.2011
	τ	0.4106	0.1152
	k	1.7954	1.8229
	λ	2.4522	2.0495
200	α	0.1180	0.1433
	τ	0.3991	0.0762
	k	1.5024	1.2270
	λ	2.3891	1.9163
300	α	0.1221	0.1318
	τ	0.3997	0.0657
	k	1.3492	1.0322
	λ	2.3936	1.7632

5 The Log-Exponentiated Generalized Gamma Distribution

Henceforth, let T be a random variable having the EGG density function (4) and $Y = \log(T)$. It is easy to verify that the density function of Y obtained by replacing $k = q^{-2}$, $\tau = (\sigma\sqrt{k})^{-1}$ and $\alpha = \exp[\mu - \tau^{-1}\log(k)]$ reduces to

$$\begin{aligned} f(y) &= \frac{\lambda|q|(q^{-2})^{q^{-2}}}{\sigma\Gamma(q^{-2})}\exp\left\{q^{-1}\left(\frac{y-\mu}{\sigma}\right) - q^{-2}\exp\left[q\left(\frac{y-\mu}{\sigma}\right)\right]\right\} \\ &\times \left\{\gamma_1\left[q^{-2}, q^{-2}\exp\left\{q\left(\frac{y-\mu}{\sigma}\right)\right\}\right]\right\}^{\lambda-1}, \end{aligned} \quad (25)$$

where $-\infty < y < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$, $\lambda > 0$ and q is different from zero. We consider an extended form including the case $q = 0$ (Lawless, 2003). Thus, the density of Y can be written as

$$f(y) = \begin{cases} \frac{\lambda|q|(q^{-2})^{q^{-2}}}{\sigma\Gamma(q^{-2})}\exp\left\{q^{-1}\left(\frac{y-\mu}{\sigma}\right) - q^{-2}\exp\left[q\left(\frac{y-\mu}{\sigma}\right)\right]\right\}\times \\ \left\{\gamma_1\left[q^{-2}, q^{-2}\exp\left\{q\left(\frac{y-\mu}{\sigma}\right)\right\}\right]\right\}^{\lambda-1} & \text{if} \quad q \neq 0, \\ \frac{\lambda}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right]\Phi^{(\lambda-1)}\left(\frac{y-\mu}{\sigma}\right) & \text{if} \quad q = 0, \end{cases} \quad (26)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution. The sub-model defined by $q = 0$ is exactly the skew normal distribution, whereas if $\lambda = 1$ and $q = 1$, we obtain the extreme value distribution. Further, the sub-model $\lambda = 1$ corresponds to the log-gamma generalized distribution and, if in addition $q = -1$, it reduces to the log-inverse Weibull distribution. We refer to equation (26) as the LEGG distribution, say $Y \sim \text{LEGG}(\mu, \sigma, q, \lambda)$, where $\mu \in \mathbb{R}$ is the location parameter, $\sigma > 0$ is the scale parameter and q and λ are shape parameters. So,

$$\text{if} \quad T \sim \text{EGG}(\alpha, \tau, k, \lambda) \quad \text{then} \quad Y = \log(T) \sim \text{LEGG}(\mu, \sigma, q, \lambda).$$

The plots of the density function (26) for selected values of λ, $\mu = 0$ and $\sigma = 1$ for $q < 0$, $q > 0$ and $q = 0$ are given in Figure 2. These plots show that the LEGG distribution could be very flexible for modeling its kurtosis. The corresponding survival function is

(a) (b) (c)

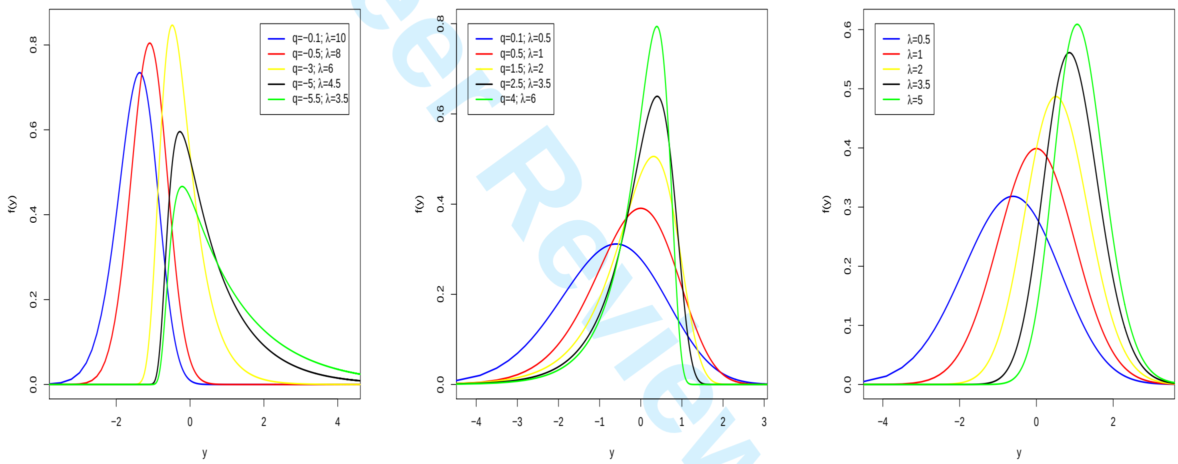

Figure 2: The LEGG density curves: (a) For some values of $q < 0$. (b) For some values of $q > 0$. (c) For some values of $q = 0$.

- $q > 0$

$$\begin{aligned} S(y) &= 1 - F(y) = P(Y > y) = P(\mu + \sigma Z > y) = P\left(Z > \frac{y - \mu}{\sigma}\right) = P(Z > z) \\ &= \int_z^{\infty} \frac{\lambda q (q^{-2})^{q^{-2}}}{\Gamma(q^{-2})} \exp\left\{q^{-1} u - q^{-2} \exp(qu)\right\} \left\{\gamma_1 \left[q^{-2}, q^{-2} \exp(qu)\right]\right\}^{\lambda - 1} du; \end{aligned}$$

- $q < 0$

$$\begin{aligned} S(y) &= 1 - F(y) = P(Y > y) = P(\mu + \sigma Z > y) = P\left(Z > \frac{y-\mu}{\sigma}\right) = P(Z > z) \\ &= \int_z^\infty \frac{-\lambda q (q^{-2})^{q^{-2}}}{\Gamma(q^{-2})} \exp\left\{q^{-1}u - q^{-2}\exp(qu)\right\}\left\{\gamma_1\left[q^{-2}, q^{-2}\exp(qu)\right]\right\}^{\lambda-1} du. \end{aligned}$$

These integrals have explicit forms

$$S(y) = \begin{cases} 1 - \left\{\gamma_1\left(q^{-2}, q^{-2}\exp\left[q(\frac{y-\mu}{\sigma})\right]\right)\right\}^\lambda & \text{if} \quad q > 0, \\ \left\{\gamma_1\left(q^{-2}, q^{-2}\exp\left[q(\frac{y-\mu}{\sigma})\right]\right)\right\}^\lambda & \text{if} \quad q < 0, \\ 1 - \Phi^\lambda\left(\frac{y-\mu}{\sigma}\right) & \text{if} \quad q = 0. \end{cases} \tag{27}$$

Theorem 4. If $Y \sim \text{LEGG}(\mu, \sigma, q, \lambda)$, then the rth moment is given by

i. For $q \neq 0$,

$$\mu_r' = \sum_{m,i=0}^{\infty} \sum_{l=0}^{r} \frac{\lambda \,\text{sgn}(q) s_m(\lambda)\, c_{m,i}}{\Gamma\left(q^{-2}\right)^{m+1}} \binom{r}{l} \left[\frac{2\sigma}{q}\log(|q|) + \mu\right]^{r-l} \left[\dot{\Gamma}(q^{-2}(m+1)+i)\right]^{\frac{\sigma l}{q}}, \tag{28}$$

where $\dot{\Gamma}(p) = \frac{\partial \Gamma(p)}{\partial p}$.

ii. For $q = 0$,

$$\mu_r' = \lambda \sum_{p=0}^{\infty} \sum_{j=0}^{r} s_p(\lambda) \binom{r}{j} \sigma^j \mu^{r-j} \tau_{j,p}, \tag{29}$$

where $s_p(\lambda)$ is given by (37), and when $j + p - l$ is even, we obtain

$$\begin{aligned} \tau_{j,p} &= 2^{j/2} \pi^{-(p+1/2)} \sum_{\substack{l=0 \\ (j+p-l)\, even}}^{p} \binom{p}{l} 2^{-l} \pi^l \,\Gamma\left(\frac{j+p-l+1}{2}\right) \times \\ & F_A^{(p-l)}\left(\frac{j+p-l+1}{2}; \frac{1}{2}, \cdots, \frac{1}{2}; \frac{3}{2}, \cdots, \frac{3}{2}; -1, \cdots, -1\right). \end{aligned} \tag{30}$$

Inserting (30) in (29) yields the rth moment of the LEGG distribution when $q = 0$. The proof of Theorem 4 is given in Appendix B.

The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. Plots of the skewness and kurtosis for selected values of μ and σ versus λ are shown in Figures 3 and 4, respectively.

(a) (b)

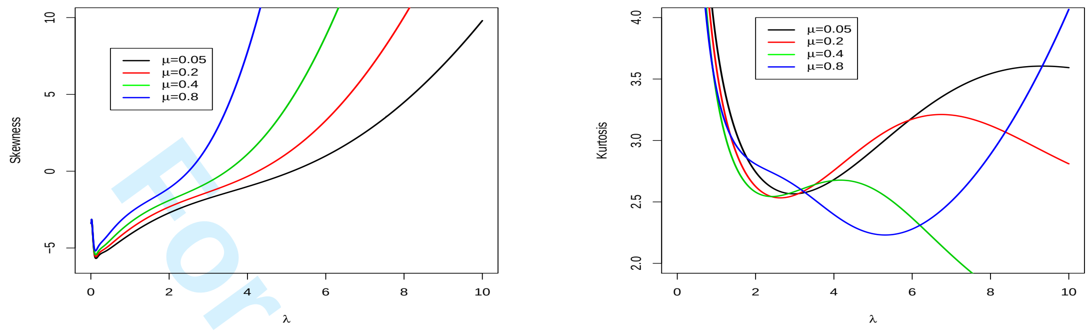

Figure 3: Skewness and kurtosis of the LEGG distribution for some values of μ with $\sigma = 0.8$, $\lambda = 2$ and $q = 2.5$.

(a) (b)

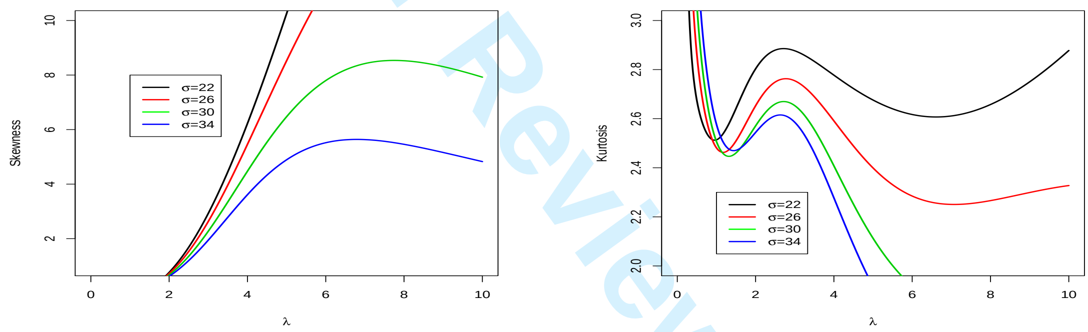

Figure 4: Skewness and kurtosis of the LEGG distribution for some values of σ with $\mu = 0.001$, $\lambda = 2$ and $q = 0.5$.

6 The Log-Exponentiated Generalized Gamma Regression Model

In many practical applications, the lifetimes are affected by explanatory variables such as the cholesterol level, blood pressure, weight and many others. Parametric models to estimate univariate survival functions for censored data regression problems are widely used. A parametric model that provides a good fit to lifetime data tends to yield more precise estimates of the quantities of interest. If $Y \sim \text{LEGG}(\mu, \sigma, q, \lambda)$, we define the standardized random variable $Z = (Y - \mu)/\sigma$

having density function

$$f(z) = \begin{cases} \frac{\lambda|q|}{\Gamma(q^{-2})}(q^{-2})^{q^{-2}}\exp\{q^{-1}z - q^{-2}\exp(qz)\}\{\gamma_1[q^{-2}, q^{-2}\exp(qz)]\}^{\lambda-1} & \text{if} \quad q \neq 0, \\ \frac{\lambda}{\sqrt{2\pi}}\exp\big(-\frac{z^2}{2}\big)\Phi^{(\lambda-1)}(z) & \text{if} \quad q = 0. \end{cases} \tag{31}$$

We write $Z \sim \text{LEGG}(0, 1, q, \lambda)$. Now, we propose a linear location-scale regression model linking the response variable y_i and the explanatory variable vector $\mathbf{x}_i^T = (x_{i1}, \ldots, x_{ip})$ by

$$y_i = \mathbf{x}_i^T\boldsymbol{\beta} + \sigma z_i, \; i = 1, \ldots, n, \tag{32}$$

where the random error z_i has density function (31), $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^T$, $\sigma > 0$, $\lambda > 0$ and $-\infty < q < \infty$ are unknown parameters. The parameter $\mu_i = \mathbf{x}_i^T\boldsymbol{\beta}$ is the location of y_i. The location parameter vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T$ is is given by a linear model $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$, where $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)^T$ is a known model matrix. The LEGG model (32) opens new possibilities for fitted many different types of data. It contains as special sub-models the following well-known regression models. For $\lambda = q = 1$, we obtain the classical Weibull regression model (see, Lawless, 2003). If $\sigma = 1$ and $\sigma = 0.5$, in addition to $\lambda = q = 1$, the LEGG regression model reduces to the exponential and Rayleigh regression models, respectively. For $q = 1$, we obtain the log-exponentiated Weibull regression model introduced (Mudholkar et al., 1995). See, also, Cancho et al. (1999), Ortega et al. (2006), Cancho et al. (2008) and Hashimoto et al. (2010). If $\sigma = 1$, in addition to $q = 1$, the LEGG regression model reduces to the log-exponentiated exponential regression model. If $\sigma = 0.5$, in addition to $q = 1$, the LEGG model reduces to the log-generalized Rayleigh regression model. For $\lambda = 1$, we obtain the log-gamma generalized (LGG) regression model (Lawless, 2003; Ortega et al., 2003). More recently, the LGG distribution has been used in several research areas. See, for example, Ortega et al. (2008, 2009). For $q = -1$, it gives the log-generalized inverse Weibull regression model. If $\lambda = 1$, in addition to $q = -1$, we obtain the log-inverse Weibull regression model (Gusmão et al., 2009). Finally, for $q = 0$, it yields the log-exponentiated normal regression model.

6.1 Maximum Likelihood Estimation

Consider a sample $(y_1, \mathbf{x}_1), \ldots, (y_n, \mathbf{x}_n)$ of n independent observations, where each random response is defined by $y_i = \min\{\log(t_i), \log(c_i)\}$. We assume non-informative censoring such that the observed lifetimes and censoring times are independent. Let F and C be the sets of individuals for which y_i is the log-lifetime or log-censoring, respectively. Conventional likelihood estimation techniques can be applied here. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (\lambda, \sigma, q, \boldsymbol{\beta}^T)^T$ from model (32) has the form $l(\boldsymbol{\theta}) = \sum\limits_{i\in F} l_i(\boldsymbol{\theta}) + \sum\limits_{i\in C} l_i^{(c)}(\boldsymbol{\theta})$, where $l_i(\boldsymbol{\theta}) = \log[f(y_i)]$, $l_i^{(c)}(\boldsymbol{\theta}) = \log[S(y_i)]$, $f(y_i)$ is the density (26) and $S(y_i)$ is the survival function (27) of Y_i. The total log-likelihood function for $\boldsymbol{\theta}$ reduces to

$$l(\boldsymbol{\theta}) = \begin{cases} \sum_{i \in F} \log\Big\{ \frac{\lambda q (q^{-2})^{q^{-2}}}{\sigma \Gamma(q^{-2})} \exp\big[q^{-1} z_i - q^{-2} \exp(q z_i)\big] \big\{ \gamma_1 \big[q^{-2}, q^{-2}\exp(q z_i)\big] \big\}^{\lambda - 1} \Big\} & \\ + \sum_{i \in C} \log\Big\{ 1 - \{\gamma_1 [q^{-2}, q^{-2} \exp(q z_i)]\}^{\lambda} \Big\}, & \text{if} \quad q > 0, \\ \sum_{i \in F} \log\Big\{ \frac{-\lambda q (q^{-2})^{q^{-2}}}{\sigma \Gamma(q^{-2})} \exp\big[q^{-1} z_i - q^{-2} \exp(q z_i)\big] \big\{ \gamma_1 \big[q^{-2}, q^{-2}\exp(q z_i)\big] \big\}^{\lambda - 1} \Big\} & \\ + \lambda \sum_{i \in C} \log\Big\{ \gamma_1 [q^{-2}, q^{-2} \exp(q z_i)] \Big\}, & \text{if} \quad q < 0, \\ \sum_{i \in F} \log\Big\{ \frac{\lambda}{\sqrt{2\pi}\sigma} \exp(-\tfrac{1}{2} z_i^2) \Phi^{(\lambda-1)}(z_i) \Big\} + \sum_{i \in C} \log\big[1 - \Phi^{\lambda}(z_i)\big], & \text{if} \quad q = 0, \end{cases} \tag{33}$$

where $z_i = (y_i - \mathbf{x}_i^T \boldsymbol{\beta})/\sigma$. The MLE $\widehat{\boldsymbol{\theta}}$ of the vector of unknown parameters can be calculated by maximizing the log-likelihood (33). The maximization of (33) follows the same two steps for obtaining the MLE of $\boldsymbol{\theta}$ for the uncensored case. In general, it is reasonable to expect that the shape parameter q belongs to the interval $[-3, 3]$. We fixed, in the first step of the iterative process, different values of q in this interval. Then, we obtain the MLEs $\widetilde{\lambda}(q)$, $\widetilde{\sigma}(q)$ and $\widetilde{\boldsymbol{\beta}}(q)$ and the maximized log-likelihood function $\mathrm{L}_{\max}(q)$ is determined. We use, in this step, a subroutine NLMixed of SAS. In the second step, the log-likelihood $\mathrm{L}_{\max}(q)$ is maximized, and then $\widehat{q}$ is obtained. The MLEs of λ, σ and $\boldsymbol{\beta}$ are, respectively, given by $\widehat{\lambda} = \widetilde{\lambda}(\widehat{q})$, $\widehat{\sigma} = \widetilde{\sigma}(\widehat{q})$ and $\widehat{\boldsymbol{\beta}} = \widetilde{\boldsymbol{\beta}}(\widehat{q})$. Initial values for $\boldsymbol{\beta}$ and σ are obtained by fitting the Weibull regression model with $\lambda = 1$ and $q = 1$. The fit of the LEGG model yields the estimated survival function for y_i ($\hat{z}_i = (y_i - \mathbf{x}_i^T \hat{\boldsymbol{\beta}})/\hat{\sigma}$) given by

$$\hat{S}(y_i; \hat{\lambda}, \hat{\sigma}, \hat{q}, \widehat{\boldsymbol{\beta}}^T) = \begin{cases} 1 - \Big\{ \gamma_1 \big(\hat{q}^{-2}, \hat{q}^{-2} \exp[\hat{q}(\hat{z}_i)] \big) \Big\}^{\hat{\lambda}}, & \text{if} \quad q > 0, \\ \Big\{ \gamma_1 \big(\hat{q}^{-2}, \hat{q}^{-2} \exp[\hat{q}(\hat{z}_i)] \big) \Big\}^{\hat{\lambda}}, & \text{if} \quad q < 0, \\ 1 - \Phi^{\hat{\lambda}}(\hat{z}_i), & \text{if} \quad q = 0. \end{cases} \tag{34}$$

Under conditions that are fulfilled for the parameter vector $\boldsymbol{\theta}$ in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is multivariate normal $N_{p+2}(0, K(\boldsymbol{\theta})^{-1})$, where $K(\boldsymbol{\theta})$ is the information matrix. The asymptotic covariance matrix $K(\boldsymbol{\theta})^{-1}$ of $\widehat{\boldsymbol{\theta}}$ can be approximated by the inverse of the $(p+2) \times (p+2)$ observed information matrix $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$. The elements of the observed information matrix $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$, namely $-\mathbf{L}_{\lambda\lambda}$, $-\mathbf{L}_{\lambda\sigma}$, $-\mathbf{L}_{\lambda\beta_j}$, $-\mathbf{L}_{\sigma\sigma}$, $-\mathbf{L}_{\sigma\beta_j}$ and $-\mathbf{L}_{\beta_j\beta_s}$ for $j, s = 1, \ldots, p$, are given in Appendix C. The approximate multivariate normal distribution $N_{p+2}(0, -\ddot{\mathbf{L}}(\boldsymbol{\theta})^{-1})$ for $\widehat{\boldsymbol{\theta}}$ can be used in the classical way to construct approximate confidence regions for some parameters in $\boldsymbol{\theta}$.

We can construct likelihood ratio (LR) statistics for comparing some special sub-models with the LEGG model. We consider the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T)^T$, where $\boldsymbol{\theta}_1$ is a subset of parameters of interest and $\boldsymbol{\theta}_2$ is a subset of remaining parameters. The LR statistic for testing the null hypothesis $H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ versus the alternative hypothesis $H_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$ is $w = 2\{\ell(\widehat{\boldsymbol{\theta}}) - \ell(\widetilde{\boldsymbol{\theta}})\}$, where $\widetilde{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\theta}}$ are the estimates under the null and alternative hypotheses, respectively. The statistic w is asymptotically (as $n \to \infty$) distributed as χ^2_k, where k is the dimension of the subset of parameters $\boldsymbol{\theta}_1$ of interest.

7 Applications

In this section, we provide two applications to show the usefulness of the proposed regression model.

Example 1: The diabetic retinopathy study

We consider a data set analyzed by Huster et al. (1989), Liang et al. (1993) and Wada and Hotta (2000). Patients with diabetic retinopathy in both eyes and 20/100 or better visual acuity for both eyes were eligible for the study. One eye was randomly selected for the treatment and the other was observed without treatment. The patients were followed for two consecutively completed 4 month follow-ups and the endpoint was the occurrence of visual acuity less than 5/200. We choose only the treatment time. A 50% sample of the high-risk patients defined by diabetic retinopathy criteria was taken for the data set ($n = 197$) and the percentage of censored observations was 72.4%. The variables involved in the study are: t_i - failure time for the treatment (in min); $cens_i$ - censoring indicator (0=censoring, 1=lifetime observed); x_{i1} - age (0 = patient is an adult diabetic, 1 = patient is a juvenile diabetic). We adopt the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \sigma z_i,$$

where the random variable y_i has the LEGG distribution (26) for $i = 1, 2, \ldots, 197$. The MLEs of the model parameters are calculated using the NLMixed procedure in SAS. In order to estimate λ, σ and $\boldsymbol{\beta}$ of the LEGG regression model, we take some values for q. We choose the value $\hat{q} = -1.743$ that maximizes the likelihood function over several values of $q \in (-3, 3)$. Hence, this value is assumed for the MLE of the parameter q. Iterative maximization of the logarithm of the likelihood function (33) starts with initial values for $\boldsymbol{\beta}$ and σ taken from the fitted log-Weibull regression model (with $\lambda = 1$). The MLEs (approximate standard errors and p-values in parentheses) are: $\hat{\lambda} = 1.263(0.537)$, $\hat{\sigma} = 2.929(0.659)$, $\hat{\beta}_0 = 4.009(2.494)(0.109)$ and $\hat{\beta}_1 = 0.710(0.397)(0.075)$. The explanatory variable x_1 is marginally significant for the model at the significance level of 10%.

In order to assess if the model is appropriate, the empirical survival function and the estimated survival function (34) from the fitted LEGG regression model are plotted in Figure 5a. In fact, the LEGG regression model provides a good fit for these data.

Example 2: Voltage data

Lawless (2003) reports an experiment in which specimens of solid epoxy electrical-insulation were studied in an accelerated voltage life test. The sample size was $n = 60$ and the percentage of censored observations was 10%. We considered three levels of voltage $52.5, 55.0$ and 57.5. The variables involved in the study are: t_i - failure time for epoxy insulation specimens (in min); $cens_i$ - censoring indicator (0=censoring, 1=lifetime observed); x_{i1} - voltage (kV). Now, we consider the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \sigma z_i,$$

where the random variable y_i follows the LEGG distribution (26) for $i = 1, 2, \ldots, 60$. We choose the value of q that maximizes the likelihood function over selected values of $q \in (-3, 3)$ yielding $\hat{q} = -0.5393$. Hence, this value is assumed for the MLE of the parameter q. Iterative maximization of the logarithm of the likelihood function (33) starts with initial values for $\boldsymbol{\beta}$ and σ taken from the fitted log-Weibull regression model (with $\lambda = 1$). The MLEs (approximate standard errors and p-values in parentheses) are: $\hat{\lambda} = 1.0153(0.144)$, $\hat{\sigma} = 0.902(0.443)$, $\hat{\beta}_0 = 16.048(4.112)(0.0002)$ and $\hat{\beta}_1 = -0.178(0.066)(0.009)$. The fitted LEGG regression model shows that x_1 is significant at (5%) and that there is a significant difference between the voltages 52.5, 55.0 and 57.5 for the survival times.

In order to assess if the model is appropriate, the empirical survival function and estimated survival functions of the LEGG regression model are plotted in Figure 5b for different voltage levels. We conclude that the LEGG regression model provides a good fit to these data.

8 Concluding Remarks

We introduce a generalized form of the exponentiated generalized gamma (EGG) distribution whose hazard rate function accommodates the four types of shape forms, i.e. increasing, decreasing, bathtub and unimodal. We derive expansions for its moments, moment generating function, mean deviations, reliability, order statistics and their moments. Further, we define the log-exponentiated generalized gamma (LEGG) distribution and propose a LEGG regression model very suitable for modeling censored and uncensored lifetime data. The new regression model serves as a good alternative for lifetime data analysis, since we can adopt goodness of fit tests for several widely known regression models as special sub-models. We demonstrate in two applications to real data that the LEGG model can produce better fits than the usual models.

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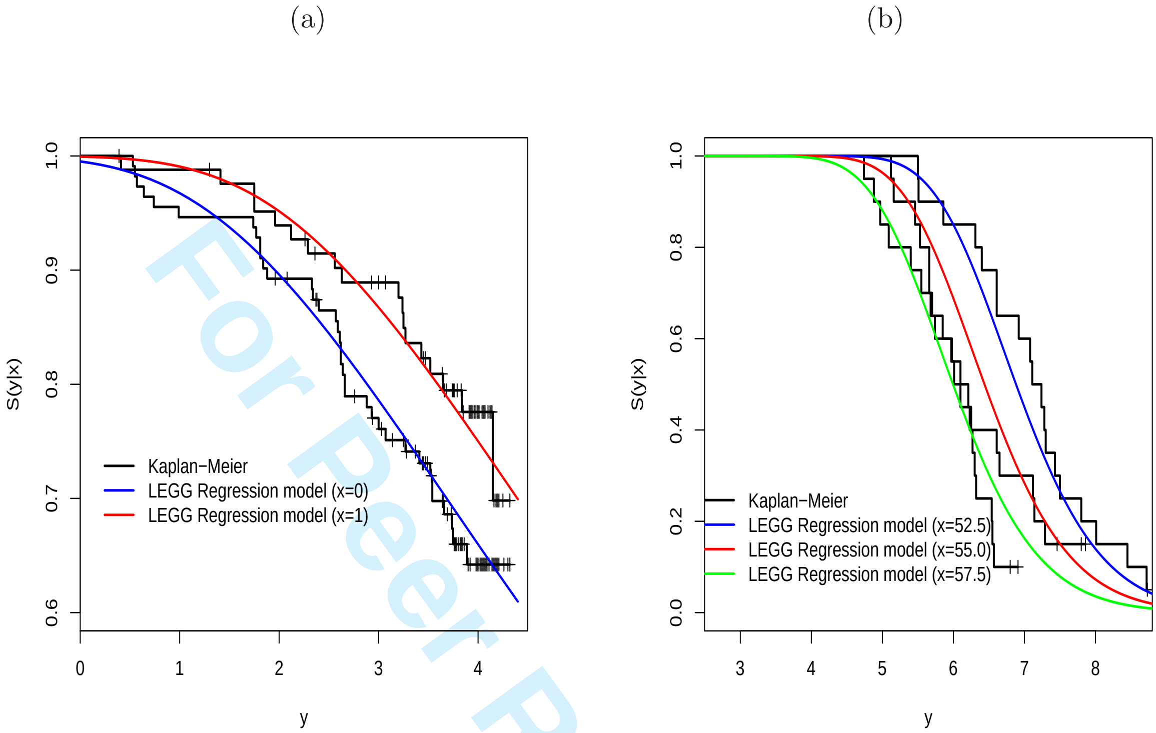

Figure 5: Estimated survival function by fitting the LEGG regression model and the empirical survival (a) For each level of the diabetic retinopathy study. (b) For each level of the voltage data.

Appendix A

We derive an expansion for $\gamma_1(k,x)^{\lambda-1}$ for any $\lambda > 0$ real non-integer. By simple binomial expansion since $0 < \gamma_1(k,x) < 1$, we can write

$$\gamma_1(k,x)^{\lambda-1} = \sum_{j=0}^{\infty}\sum_{m=0}^{j}(-1)^{j+m}\binom{\lambda-1}{j}\binom{j}{m}\gamma_1(k,x)^{m}, \qquad (35)$$

which always converges. We can substitute $\sum_{j=0}^{\infty}\sum_{m=0}^{j}$ for $\sum_{m=0}^{\infty}\sum_{j=m}^{\infty}$ to obtain

$$\gamma_1(k,x)^{\lambda-1} = \sum_{m=0}^{\infty} s_m(\lambda)\,\gamma_1(k,x)^{m}, \qquad (36)$$

where

$$s_m(\lambda) = \sum_{j=m}^{\infty} (-1)^{j+m} \binom{\lambda-1}{j} \binom{j}{m}. \tag{37}$$

A power series expansion for the incomplete gamma function ratio is given by

$$\gamma_1(k,x) = \frac{x^k}{\Gamma(k)} \sum_{i=0}^{\infty} \frac{(-x)^i}{(k+i)i!}. \tag{38}$$

We use an equation in Section 0.314 of Gradshteyn and Ryzhik (2000) for a power series raised to a positive integer m

$$\left(\sum_{i=0}^{\infty} a_i\, x^i\right)^m = \sum_{i=0}^{\infty} c_{m,i}\, x^i, \tag{39}$$

where the coefficients $c_{m,i}$ (for $i = 1, 2, \ldots$) satisfy the recurrence relation

$$c_{m,i} = (i a_0)^{-1} \sum_{p=1}^{i} (mp - i + p)\, a_p\, c_{m,i-p} \tag{40}$$

and $c_{m,0} = a_0^m$. Here, $c_{m,i}$ can be calculated from $c_{m,0}, \ldots, c_{m,i-1}$ and also be expressed explicitly as a function of $a_0, \ldots, a_i$, although it is not necessary for programming numerically our expansions using any software with numerical facilities. By equation (39), we obtain

$$\gamma_1(k,x)^m = \frac{x^{km}}{\Gamma(k)^m} \sum_{i=0}^{\infty} c_{m,i}\, x^i, \tag{41}$$

whose coefficients $c_{m,i}$ are determined from (40) with $a_p = (-1)^p/(k+p)p!$. Combining (36) and (41), we can rewrite (36) as

$$\gamma_1(k,x)^{\lambda-1} = \sum_{m,i=0}^{\infty} \frac{s_m(\lambda)\, c_{m,i}}{\Gamma(k)^m}\, x^{km+i}. \tag{42}$$

Appendix B

Proof of Theorem 1: From (4) and (42), the EGG$(\alpha, \tau, k, \lambda)$ density function can be written as

$$f(t) = \frac{\lambda\, |\tau|}{\alpha\, \Gamma(k)} \exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right] \sum_{m,i=0}^{\infty} \frac{s_m(\lambda)\, c_{m,i}}{\Gamma(k)^m} \left(\frac{t}{\alpha}\right)^{\tau[k(m+1)+i]-1}, \; t > 0.$$

We can express $f(t)$ as a linear combination given by

$$f(t) = \sum_{m,i=0}^{\infty} w(m,i,k,\lambda)\, g_{\alpha,\tau,k(m+1)+i}(t), \quad t>0,$$

where $g_{\alpha,\tau,k(m+1)+i}(t)$ denotes the $GG(\alpha,\tau,k(m+1)i)$ density function defined by (7) and the weighted coefficients $w(m,i,k,\lambda)$ are

$$w(m,i,k,\lambda) = \sum_{m,i=0}^{\infty} \frac{\lambda\, s_m(\lambda)\, c_{m,i}\, \Gamma[k(m+1)+i]}{\Gamma(k)^{m+1}}. \qquad \square$$

Proof of Theorem 2: The rth moment of the EGG distribution comes from (4) as

$$\mu_r' \;=\; \frac{\lambda\,|\tau|\,\alpha^{r-1}}{\Gamma(k)} \int_0^{\infty} \left(\frac{t}{\alpha}\right)^{\tau k+r-1} \exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right] \left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1} dt.$$

Setting $x = \left(\frac{t}{\alpha}\right)^{\tau}$, we have

$$\mu_r' = \frac{\lambda\,\alpha^r\,\mathrm{sgn}(\tau)}{\Gamma(k)} \int_0^{\infty} x^{k+\frac{r}{\tau}-1} \exp(-x)\,\gamma_1\,(k,x)^{\lambda-1}\,dx. \tag{43}$$

Equation (35) for $\gamma_1(k,x)^{\lambda-1}$ leads to

$$\gamma_1(k,x)^{\lambda-1} = \sum_{j=0}^{\infty}\sum_{m=0}^{j} (-1)^{j+m} \binom{\lambda-1}{j}\binom{j}{m}\gamma_1(k,x)^m.$$

Inserting the last equation into (43) and interchanging terms gives

$$\mu_r' = \frac{\lambda\,\alpha^r\,\mathrm{sgn}(\tau)}{\Gamma(k)} \sum_{j=0}^{\infty}\sum_{m=0}^{j} (-1)^{j+m} \binom{\lambda-1}{j}\binom{j}{m} I\left(k,\frac{r}{\tau},m\right),$$

where

$$I\left(k,\frac{r}{\tau},m\right) = \int_0^{\infty} x^{k+\frac{r}{\tau}-1} \exp(-x)\,\gamma_1\,(k,x)^m\,dx.$$

We calculate the last integral using the series expansion (38) for the incomplete gamma function

$$I\left(k,\frac{r}{\tau},m\right) = \int_0^{\infty} x^{k+\frac{r}{\tau}-1} \exp(-x) \left[\frac{x^k}{\Gamma(k)} \sum_{i=0}^{\infty} \frac{(-x)^i}{(k+i)i!}\right]^m dx.$$

This integral can be determined from equations (24) and (25) of Nadarajah (2008b) in terms of the Lauricella function of type A (Exton, 1978; Aarts, 2000) defined by

$$F_A^{(n)}\left(a; b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n\right) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!},$$

where $(a)_i = a(a+1)\cdots(a+i-1)$ is the ascending factorial defined by (with the convention that $(a)_0 = 1$). Numerical routines for the direct computation of the Lauricella function of type A are available, see Exton (1978) and Mathematica (Trott, 2006). We obtain

$$I\left(k, \frac{r}{\tau}, m\right) = k^{-m}\,\Gamma\big(r/\tau + k(m+1)\big) \times F_A^{(m)}\big(r/\tau + k(m+1); k, \ldots, k; k+1, \ldots, k+1; -1, \ldots, -1\big). \qquad \square$$

Proof of Theorem 3: Suppose that T is a random variable having a $\mathrm{GG}(\alpha, \tau, k)$ density function (7). First, we provide a closed form expression for the mgf of T using the Wright function (Wright, 1935). Setting $u = t/\alpha$, we have

$$M_{\alpha,\tau,k}(s) = \frac{|\tau|}{\Gamma(k)} \int_0^{\infty} \exp(s\alpha u)\, u^{\tau k-1} \exp(-u^{\tau}) du.$$

Expanding the exponential in Taylor series and using $\int_0^{\infty} u^{\tau k+m-1} \exp(-u^{\tau}) du = \tau^{-1}\,\Gamma(k + m/\tau)$, we obtain

$$M_{\alpha,\tau,k}(s) = \frac{\mathrm{sgn}(\tau)}{\Gamma(k)} \sum_{m=0}^{\infty} \Gamma\left(\frac{m}{\tau} + k\right) \frac{(s\alpha)^m}{m!}. \tag{44}$$

Equation (44) holds for any τ different from zero. However, if $\tau > 1$, we can simplify it by using the Wright generalized hypergeometric function (Wright, 1935) defined by

$${}_p\Psi_q\left[\begin{array}{c} (\alpha_1, A_1), \cdots, (\alpha_p, A_p) \\ (\beta_1, B_1), \cdots, (\beta_q, B_q) \end{array} ; x\right] = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j\, m)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j\, m)} \frac{x^m}{m!}. \tag{45}$$

This function exists if $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$. By combining (44) and (45), we have

$$M_{\alpha,\tau,k}(s) = \frac{1}{\Gamma(k)}\, {}_1\Psi_0\left[\begin{array}{c} (1, 1/\tau) \\ - \end{array} ; s\alpha\right]. \tag{46}$$

Finally, the mgf of the EGG distribution follows from (8) as

$$M(t) = \sum_{m,i=0}^{\infty} w(m, i, k, \lambda)\, M_{\alpha,\tau,k(m+1)+i}(s),$$

where $M_{\alpha,\tau,k(m+1)+i}(s)$ is easily obtained from equations (44) or (46). $\square$

Proof of Theorem 4: The rth moment of the LEGG distribution can be obtained from (26)

- For $q \neq 0$, we have

$$\mu'_r = E(Y^r) = \int_{-\infty}^{\infty} y^r \frac{\lambda|q|\,(q^{-2})^{q^{-2}}}{\sigma\Gamma(q^{-2})} \exp\left\{q^{-1}\left(\frac{y-\mu}{\sigma}\right) - q^{-2}\exp\left[q\left(\frac{y-\mu}{\sigma}\right)\right]\right\} \times \left(\gamma_1\left\{q^{-2}, q^{-2}\exp\left[q\left(\frac{y-\mu}{\sigma}\right)\right]\right\}\right)^{\lambda-1} dy.$$

Setting $x = q^{-2}\exp\left[q\left(\frac{y-\mu}{\sigma}\right)\right]$ in the last equation yields

$$\mu'_r = \frac{\lambda\,\mathrm{sgn}(q)}{\Gamma(q^{-2})} \int_0^{\infty} \left\{\frac{\sigma}{q}[\log(x) + 2\log(|q|)] + \mu\right\}^r x^{q^{-2}-1} \exp(-x)\, \gamma_1\left(q^{-2}, x\right)^{\lambda-1} dx.$$

Using expansion (42) for $\gamma_1(q^{-2}, x)^{\lambda-1}$ leads to

$$\gamma_1(q^{-2}, x)^{\lambda-1} = \sum_{m,i=0}^{\infty} \frac{s_m(\lambda)\, c_{m,i}}{\Gamma(q^{-2})^m}\, x^{q^{-2}m+i}.$$

Inserting the last equation into μ'_r and interchanging terms, we obtain

$$\mu'_r = \frac{\lambda\,\mathrm{sgn}(q)}{\Gamma(q^{-2})} \int_0^{\infty} \left\{\frac{\sigma}{q}[\log(x) + 2\log(|q|)] + \mu\right\}^r x^{q^{-2}-1} \exp(-x) \sum_{m,i=0}^{\infty} \frac{s_m(\lambda)\, c_{m,i}}{\Gamma(q^{-2})^m}\, x^{q^{-2}m+i}.$$

The binomial expansion in $\left\{\frac{\sigma}{q}[\log(x) + 2\log(|q|)] + \mu\right\}^r$ gives

$$\mu'_r = \sum_{m,i=0}^{\infty} \sum_{l=0}^{r} \frac{\lambda\,\mathrm{sgn}(q) s_m(\lambda)\, c_{m,i}}{\Gamma(q^{-2})^{m+1}} \binom{r}{l} \left[\frac{2\sigma}{q}\log(|q|) + \mu\right]^{r-l} \times \frac{\sigma l}{q} \int_0^{\infty} \log(x)\, \exp(-x)\, x^{q^{-2}(m+1)+i-1} dx.$$

The last integral is given in Prudnikov et al. (1986, vol 1, Section 2.6.21, integral 1) and then

$$\mu'_r = \sum_{m,i=0}^{\infty} \sum_{l=0}^{r} \frac{\lambda\,\mathrm{sgn}(q) s_m(\lambda) c_{m,i}}{\Gamma(q^{-2})^{m+1}} \binom{r}{l} \left[\frac{2\sigma}{q}\log(|q|) + \mu\right]^{r-l} \left[\dot{\Gamma}(q^{-2}(m+1) + i)\right]^{\frac{\sigma l}{q}},$$

where $\dot{\Gamma}(p) = \frac{\partial\Gamma(p)}{\partial p}$.

- For $q = 0$, we obtain

$$\mu'_r = E(Y^r) = \frac{\lambda}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} y^r \exp\Big[-\frac{1}{2}\Big(\frac{y-\mu}{\sigma}\Big)^2\Big]\Phi^{(\lambda-1)}\Big(\frac{y-\mu}{\sigma}\Big)dy.$$

Setting $y = \mu + \sigma z$ and using (36) for $\Phi^{(\lambda-1)}(z)$, we have

$$\mu'_r = \lambda \sum_{p=0}^{\infty}\sum_{j=0}^{r} s_p(\lambda)\binom{r}{j}\sigma^j \mu^{r-j}\tau_{j,p},$$

where $s_p(\lambda)$ is given by (37). We define the integral for j and p non-negative integers

$$\tau_{j,p} = \int_{-\infty}^{\infty} z^j \phi(z)\Phi^p(z)dz,$$

where $\phi(z)$ is the standard normal density. The standard normal cdf can be written as

$$\Phi(x) = \frac{1}{2}\left\{1 + \mathrm{erf}\left(\frac{\mathrm{x}}{\sqrt{2}}\right)\right\}, \quad x \in \mathbb{R}.$$

By the binomial expansion and interchanging terms, we obtain

$$\tau_{j,p} = \frac{1}{2^j\sqrt{2\pi}}\sum_{l=0}^{p}\binom{p}{l}\int_{-\infty}^{\infty} x^j \exp(-x^2/2)\,\mathrm{erf}\left(\frac{\mathrm{x}}{\sqrt{2}}\right)^{\mathrm{p-l}}\mathrm{dx}.$$

Using the series expansion for the error function erf(.)

$$\mathrm{erf(x)} = \frac{2}{\sqrt{\pi}}\sum_{\mathrm{m}=0}^{\infty}\frac{(-1)^{\mathrm{m}}\mathrm{x}^{2\mathrm{m}+1}}{(2\mathrm{m}+1)\mathrm{m}!},$$

we can solve the last integral by equations (9)-(11) of Nadarajah (2008b). When $j + p - l$ is even, we obtain

$$\tau_{j,p} = 2^{j/2}\pi^{-(p+1/2)} \sum_{\substack{l=0 \\ (j+p-l)\ even}}^{p}\binom{p}{l}2^{-l}\pi^l\Gamma\left(\frac{j+p-l+1}{2}\right)\times F_A^{(p-l)}\left(\frac{j+p-l+1}{2};\frac{1}{2},\cdots,\frac{1}{2};\frac{3}{2},\cdots,\frac{3}{2};-1,\cdots,-1\right). \qquad \square$$

Appendix C: Matrix of second derivatives $-\ddot{\mathbf{L}}(\boldsymbol{\theta})$

Here, we provide the formulas to obtain the second-order partial derivatives of the log-likelihood function. After some algebraic manipulations, we obtain

$$\mathbf{L}_{\alpha,\alpha} = \frac{n\tau k}{\alpha^2} - \frac{\tau}{\alpha^2}\sum_{i=1}^{n}(v_i + \tau v_i) - \frac{(\lambda-1)\tau}{\alpha^2\Gamma(k)}\sum_{i=1}^{n}\left\{\frac{-v_i^k \exp(-v_i)}{\gamma_1(k,v_i)} - \frac{\tau k v_i^k \exp(-v_i)}{\gamma_1(k,v_i)} + \frac{\tau v_i^{(k+1)}\exp(-v_i)}{\gamma_1(k,v_i)} + \frac{\tau}{\Gamma(k)}\frac{[v_i^k]^2[\exp(-v_i)]^2}{[\gamma_1(k,v_i)]^2}\right\},$$

$$\mathbf{L}_{\alpha,\tau} = -\frac{nk}{\alpha} + \frac{1}{\alpha}\sum_{i=1}^{n}[v_i + v_i\log(v_i)] - \frac{(\lambda-1)}{\alpha\Gamma(k)}\sum_{i=1}^{n}\left\{\frac{v_i^k\exp(-v_i)}{\gamma_1(k,v_i)} + \frac{v_i^k\exp(-v_i)\log(v_i^k)}{\gamma_1(k,v_i)} - \frac{v_i^{(k+1)}\exp(-v_i)\log(v_i)}{\gamma_1(k,v_i)} - \frac{1}{\Gamma(k)}\frac{[v_i^k]^2[\exp(-v_i)]^2\log(v_i)}{[\gamma_1(k,v_i)]^2}\right\},$$

$$\mathbf{L}_{\alpha,k} = -\frac{n\tau}{\alpha} - \frac{\tau(\lambda-1)}{\alpha\Gamma(k)}\sum_{i=1}^{n}v_i^k\exp(-v_i)\left\{\frac{\log(v_i)}{\gamma_1(k,v_i)} - \frac{[\dot{\gamma}_1(k,v_i)]_k}{\Gamma(k)[\gamma_1(k,v_i)]^2}\right\},$$

$$\mathbf{L}_{\alpha,\lambda} = -\frac{\tau}{\alpha\Gamma(k)}\sum_{i=1}^{n}\frac{v_i^k\exp(-v_i)}{\gamma_1(k,v_i)}, \qquad \mathbf{L}_{\tau,\lambda} = \frac{1}{\tau\Gamma(k)}\sum_{i=1}^{n}\frac{v_i^k\exp(-v_i)\log(v_i)}{\gamma_1(k,v_i)},$$

$$\mathbf{L}_{\tau,\tau} = -\frac{n}{\tau^2} - \frac{1}{\tau^2}\sum_{i=1}^{n}v_i[\log(v_i)]^2 + \frac{(\lambda-1)}{\tau^2\Gamma(k)}\sum_{i=1}^{n}\left\{\frac{v_i^k\exp(-v_i)\log(v_i)^{k+1}}{\gamma_1(k,v_i)} - \frac{(v_i)^{k+1}\exp(-v_i)[\log(v_i)]^2}{\gamma_1(k,v_i)} - \frac{[v_i^k]^2[\exp(-v_i)]^2[\log(v_i)]^2}{\Gamma(k)[\gamma_1(k,v_i)]^2}\right\},$$

$$\mathbf{L}_{\tau,k} = \frac{1}{\tau}\sum_{i=1}^{n}\log(v_i) + \frac{(\lambda-1)}{\tau\Gamma(k)}\sum_{i=1}^{n}v_i^k\exp(-v_i)\log(v_i)\left\{\frac{\log(v_i)}{\gamma_1(k,v_i)} - \frac{[\dot{\gamma}_1(k,v_i)]_k}{\Gamma(k)[\gamma_1(k,v_i)]^2}\right\},$$

$$\mathbf{L}_{k,k} = -n\lambda\psi'(k) + \frac{(\lambda-1)}{\Gamma(k)}\sum_{i=1}^{n}\left\{\frac{[\ddot{\gamma}_1(k,v_i)]_{kk}}{\gamma_1(k,v_i)} - \frac{\{[\dot{\gamma}_1(k,v_i)]_k\}^2}{\Gamma(k)[\gamma_1(k,v_i)]^2}\right\},$$

$$\mathbf{L}_{k,\lambda} = -n\psi(k) + \frac{1}{\Gamma(k)}\sum_{i=1}^{n}\left\{\frac{[\dot{\gamma}_1(k,v_i)]_k}{\gamma_1(k,v_i)}\right\}, \qquad \mathbf{L}_{\lambda,\lambda} = \frac{-n}{\lambda^2},$$

where

$$v_i = (\frac{t_i}{\alpha})^{\tau}, \quad [\dot{\gamma}_1(k, v_i)]_k = \int_0^{v_i} w^{k-1} \exp(-w) \log(w) dw,$$

$$[\ddot{\gamma}_1(k, v_i)]_{kk} = \int_0^{v_i} w^{k-1} \exp(-w) [\log(w)]^2 dw,$$

$\psi(.)$ is the digamma function and $\psi'(.)$ is the polygamma function.

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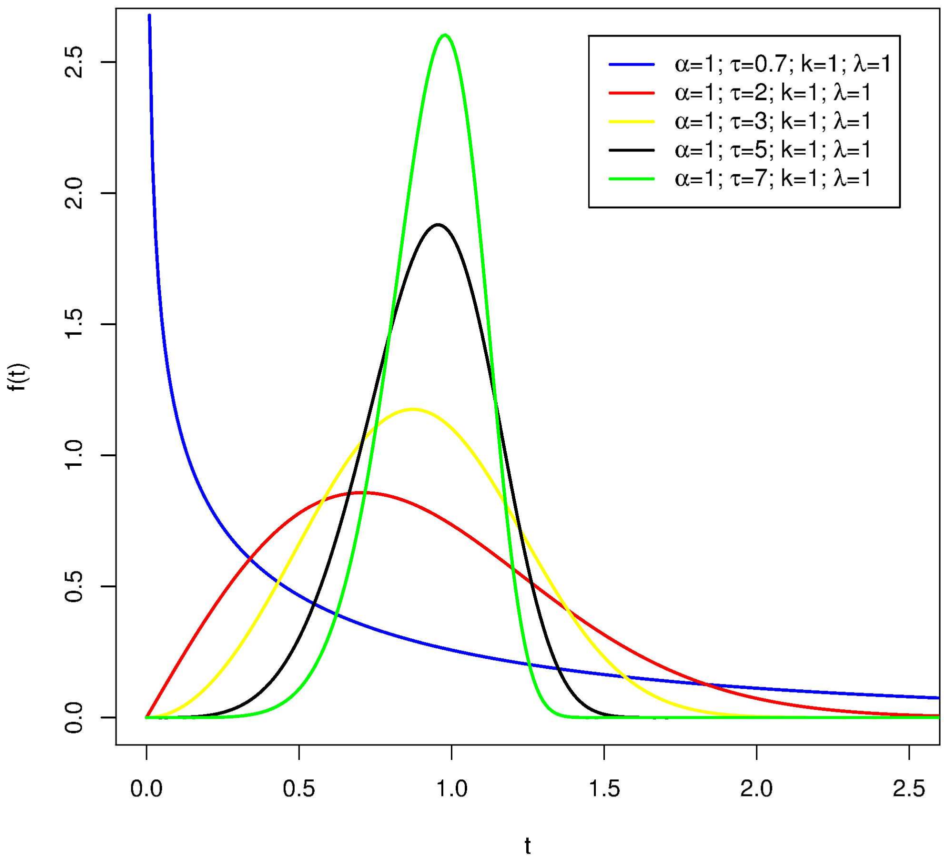

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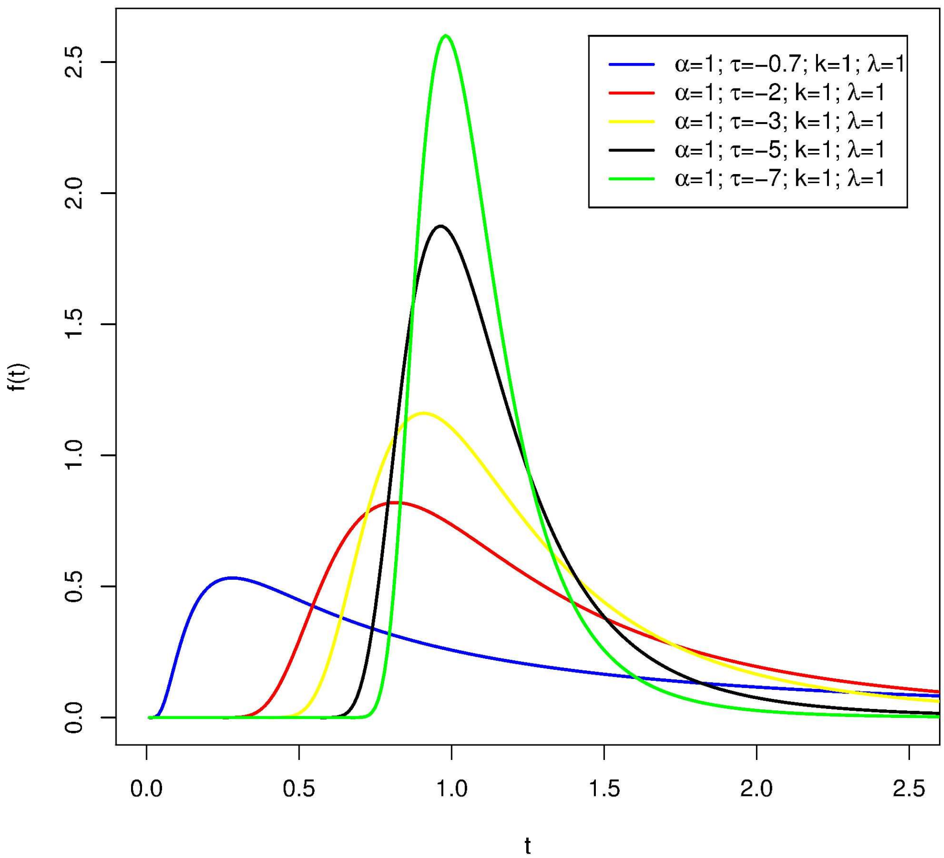

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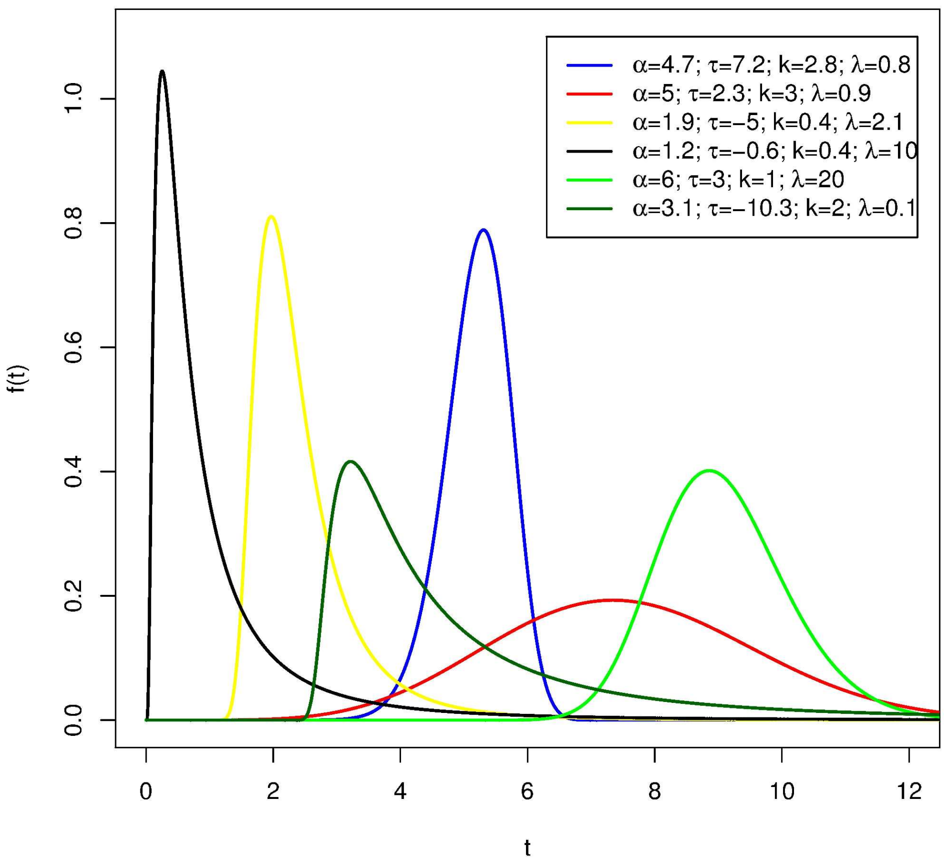

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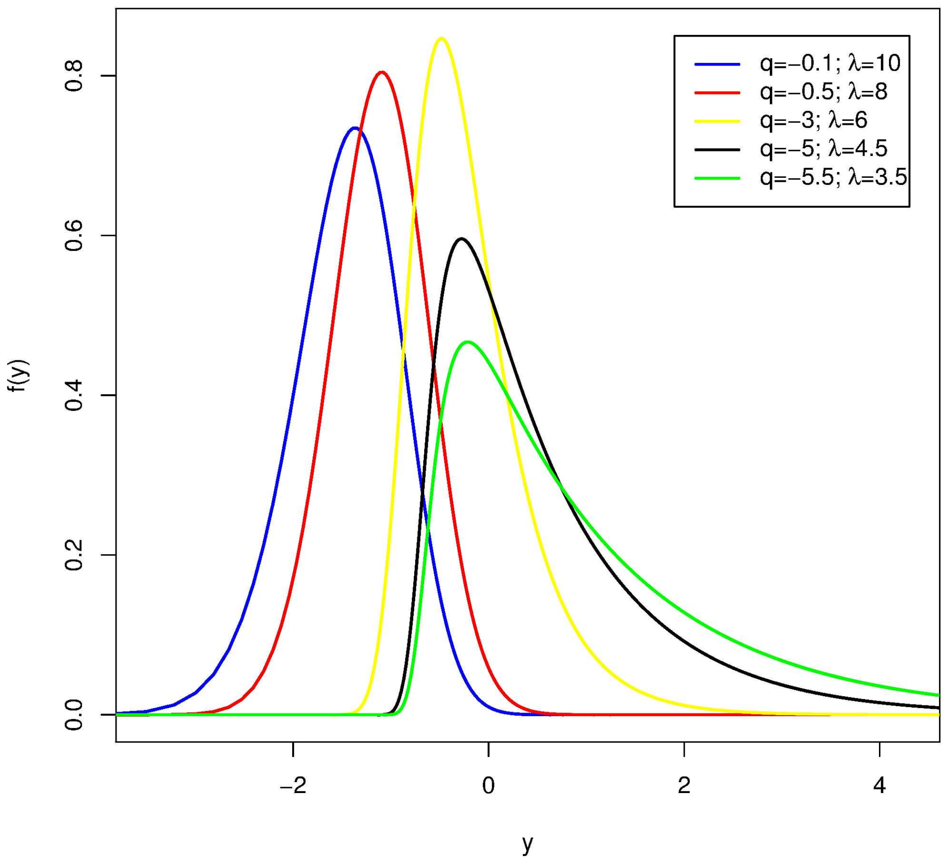

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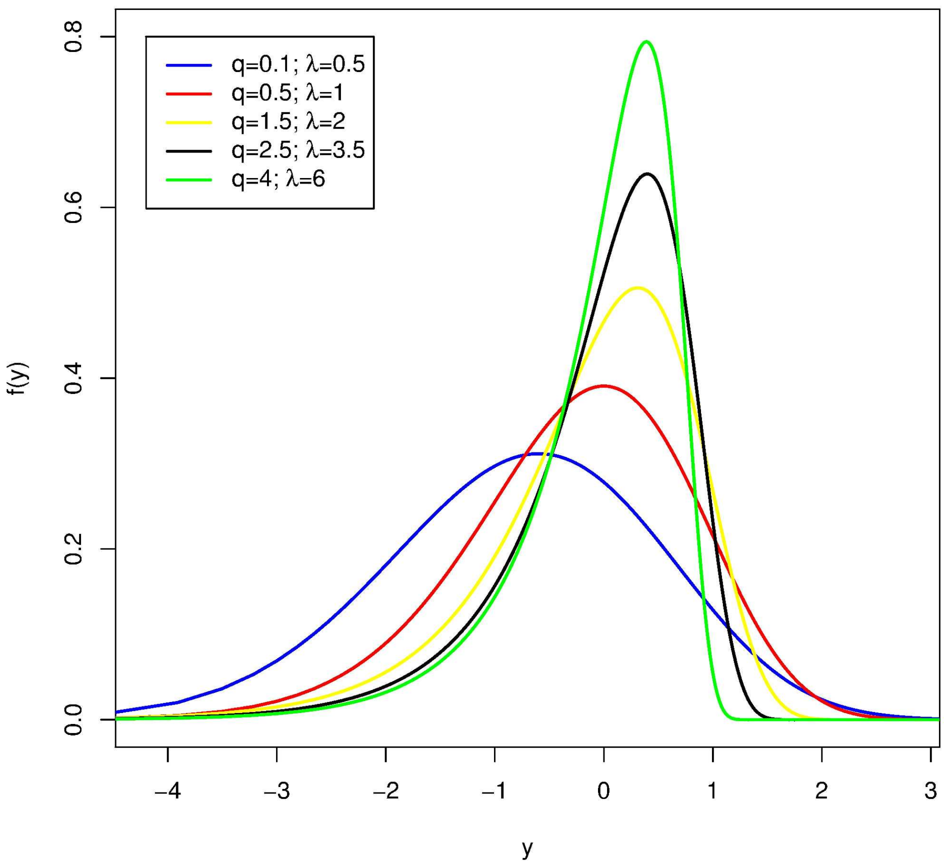

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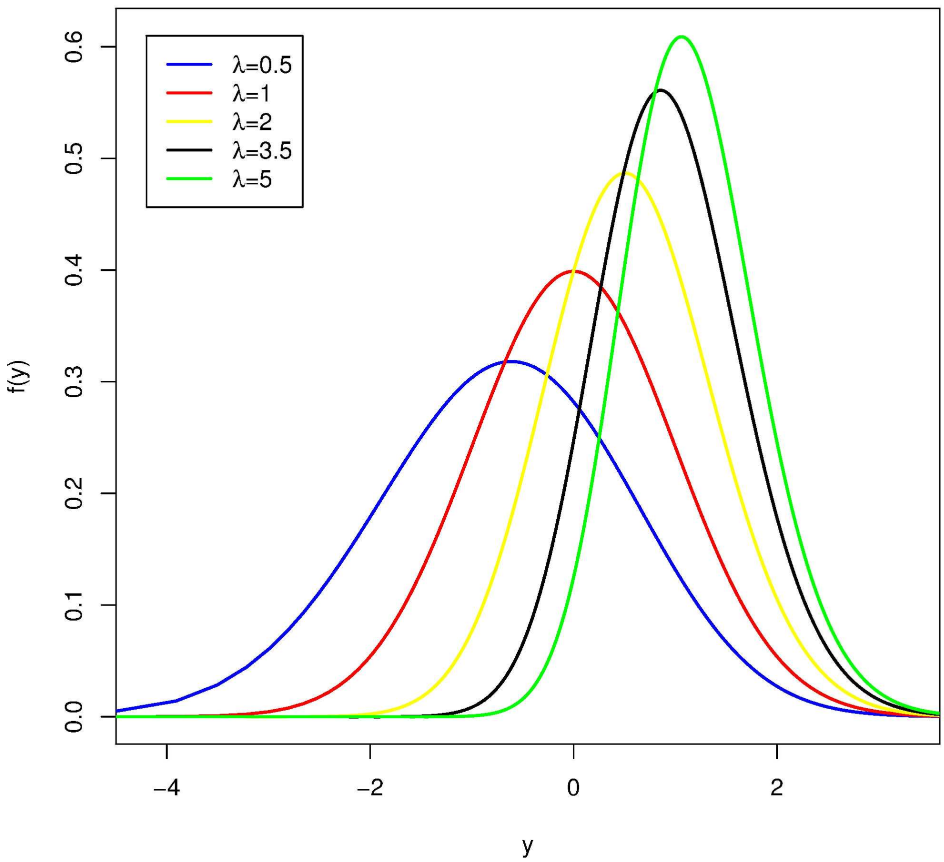

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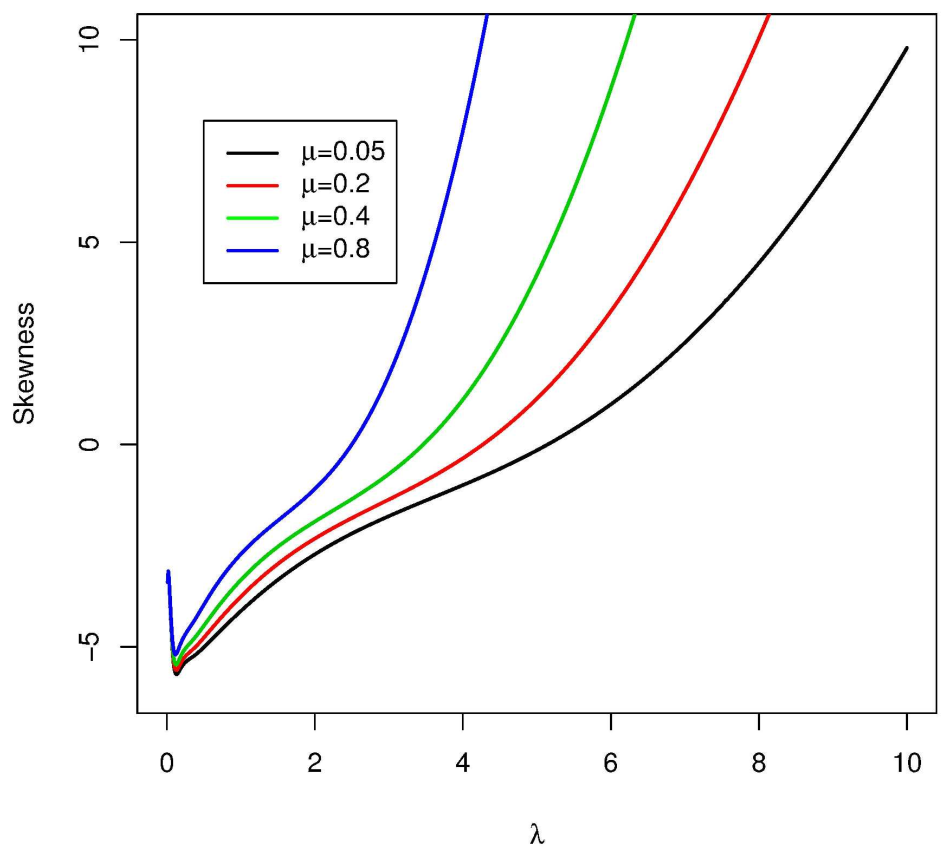

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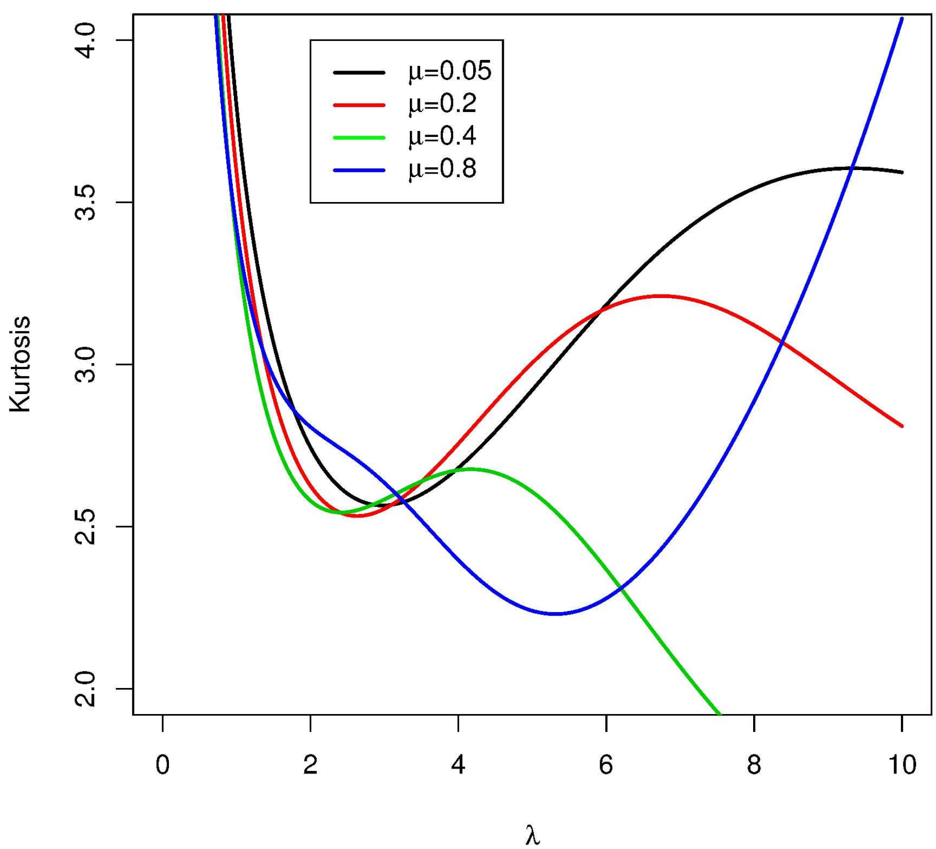

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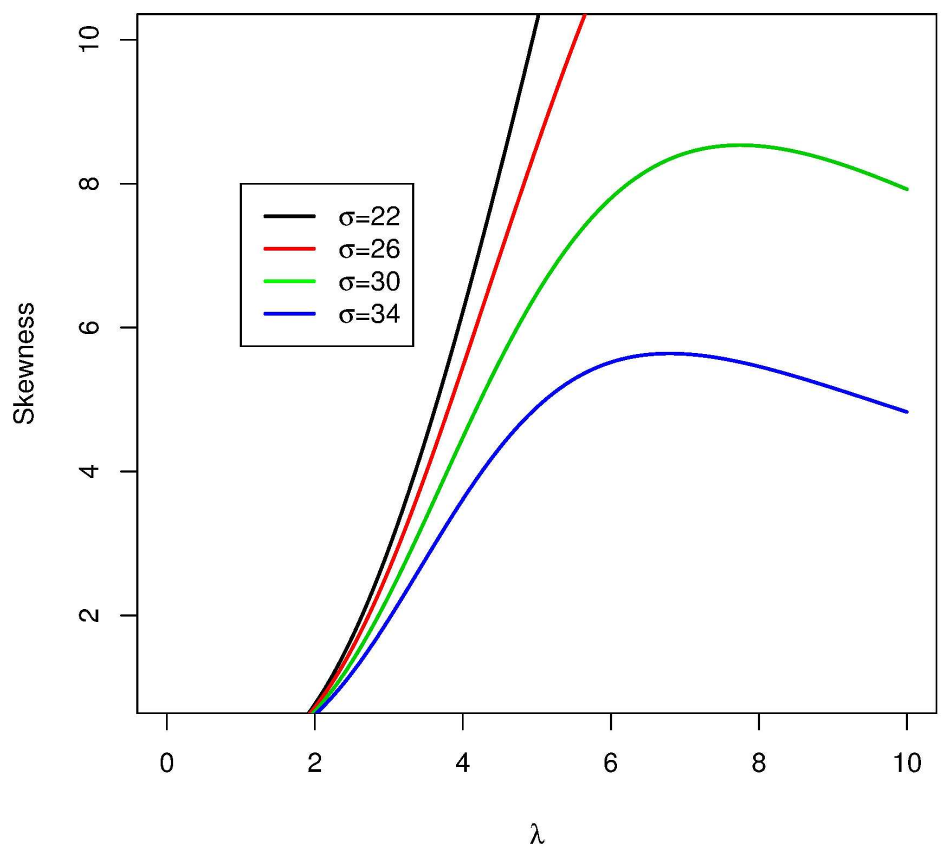

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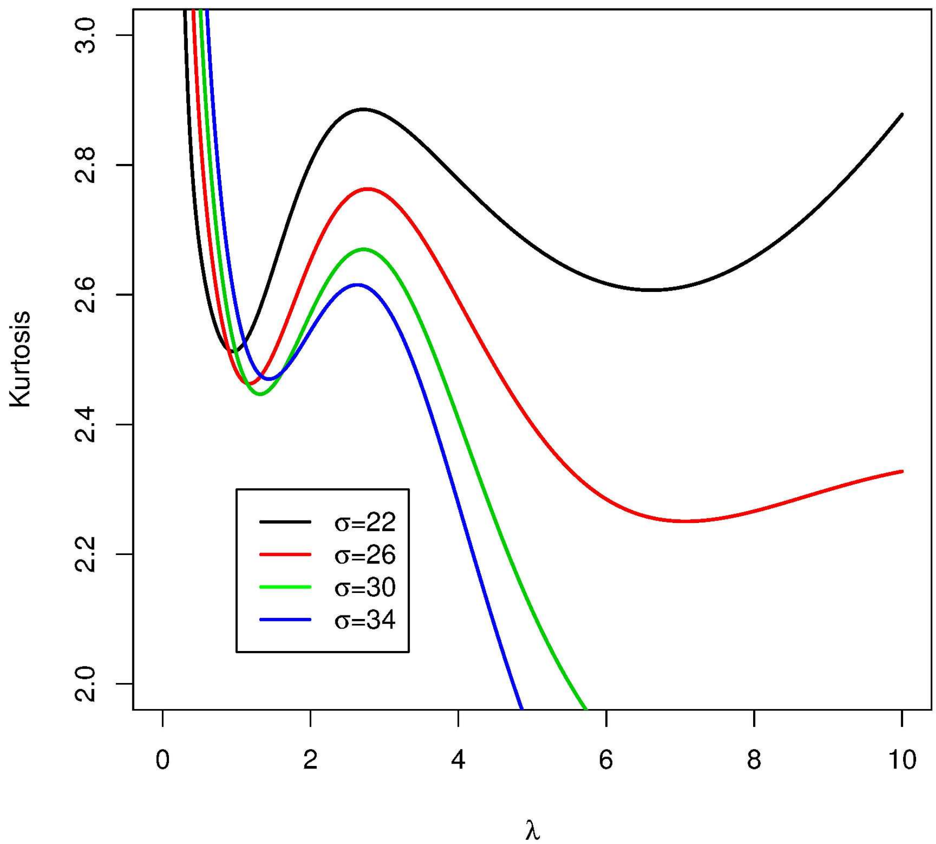

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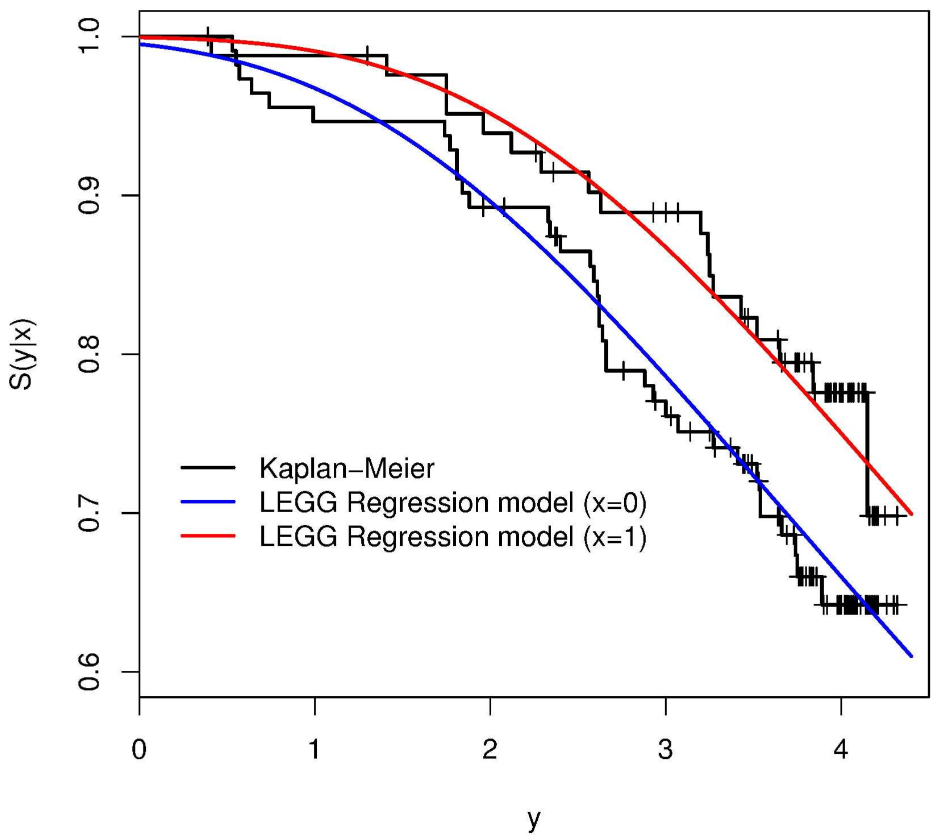

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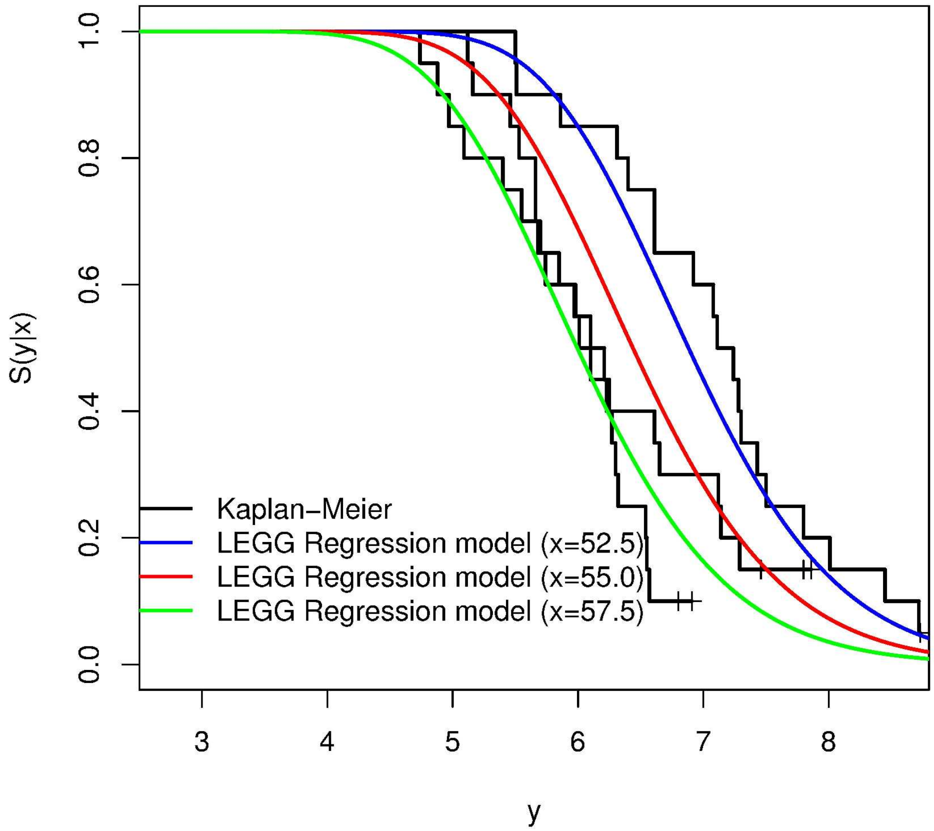

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