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The Weibull-Geometric Distribution

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In this paper we introduce, for the first time, the Weibull-Geometric distribution which generalizes the exponential-geometric distribution proposed by Adamidis and Loukas [2]. The hazard function of the last distribution is monotone decreasing but the hazard function of the new distribution can take more general forms. Unlike the Weibull distribution, the proposed distribution is useful for modeling unimodal failure rates. We derive the cumulative distribution and hazard functions, the density of the order statistics and obtain expressions for its moments and for the moments of the order statistics. We give expressions for the Rényi and Shannon entropies. The maximum likelihood estimation procedure is discussed and an algorithm EM (Dempster et al. [5]; McLachlan and Krishnan [11]) is provided for estimating the parameters. We obtain the information matrix and discuss inference methods. Applications to real data sets are given to show the flexibility and potentiality of the proposed distribution.

Keywords: EM algorithm, Exponential distribution, Geometric distribution, Hazard function, Information matrix, Maximum likelihood estimation, Weibull distribution.

1. Introduction

Several distributions have been proposed in the literature to model lifetime data. Adamidis and Loukas [2] introduced the two-parameter exponential-geometric (EG) distribution with decreasing failure rate. Kus [8] introduced the exponential-Poisson distribution (following the same idea of the EG distribution) with decreasing failure rate and discussed various of its properties. Marshall and Olkin [10] presented a method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Adamidis et al. [1] proposed the extended exponential-geometric (EEG) distribution which generalizes the EG distribution and discussed various of its statistical properties along with its reliability features. The hazard function of the EEG distribution can be monotone decreasing, increasing or constant.

The Weibull distribution is one of the most commonly used lifetime distribution in modeling lifetime data. In practice, it has been shown to be very flexible in modeling various types of lifetime distributions with monotone failure rates but it is

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not useful for modeling the bathtub shaped and the unimodal failure rates which are common in reliability and biological studies. In this paper we introduce the Weibull-geometric (WG) distribution that generalizes the EG and Weibull distributions and study some of its properties. The paper is organized as follows. In Section 2, we define the WG distribution and plot its probability density function (pdf). In Section 3, some properties of the new distribution are given. We obtain the cumulative distribution function (cdf), survivor and hazard functions and the pdf of the order statistics. We also give expressions for its moments and for the moments of the order statistics. Maximum likelihood estimation using the algorithm EM is studied in Section 4 and asymptotic methods of inference are discussed in Section 5. Illustrative examples based on two real data sets are given in Section 6. Finally, Section 7 concludes the paper.

2. The WG distribution

The EG distribution (Adamidis and Loukas [2]) can be obtained by compounding an exponential with a geometric distribution. In fact, if X follows an exponential distribution with parameter βZ, where Z is a geometric random variable with parameter p, then the marginal distribution of X has the EG distribution with parameters (β, p). The Weibull distribution extends the exponential distribution and then it is natural to generalize the EG distribution by replacing in the above compounding mechanism the exponential by the Weibull distribution.

Suppose that $\{Y_i\}_{i=1}^{Z}$ are independent and identically distributed (iid) random variables following the Weibull distribution $W(\beta, \alpha)$ with scale parameter $\beta > 0$, shape parameter $\alpha > 0$ and pdf

$$g(x; \beta, \alpha) = \alpha\beta^{\alpha} x^{\alpha-1} e^{-(\beta x)^{\alpha}}, \quad x > 0,$$

and N a discrete random variable having a geometric distribution with probability function $P(n; p) = (1-p)p^{n-1}$ for $n \in \mathbb{N}$ and $p \in (0,1)$. The marginal pdf of $X = \min\{Y_i\}_{i=1}^{N}$ is

$$f(x; p, \beta, \alpha) = \alpha\beta^{\alpha}(1-p)x^{\alpha-1}e^{-(\beta x)^{\alpha}}\{1 - p\, e^{-(\beta x)^{\alpha}}\}^{-2}, \quad x > 0, \tag{1}$$

which defines the WG distribution. It is evident that (1) is much more flexible than the Weibull distribution. The EG distribution is a special case of the WG distribution for $\alpha = 1$. When p approaches zero, the WG distribution leads to the Weibull $W(\beta, \alpha)$ distribution. Figure 1 plots the WG density for some values of the vector $\phi = (\beta, \alpha)$ for $p = 0.01, 0.2, 0.5$ and 0.9. For all values of parameters, the WG density tends to zero as $x \to \infty$.

For $\alpha > 1$, the WG density is unimodal (see appendix A) and the mode $x_0 = \beta^{-1}u^{1/\alpha}$ is obtained by solving the nonlinear equation

$$u + pe^{-u}\left(u + \frac{\alpha - 1}{\alpha}\right) = \frac{\alpha - 1}{\alpha}. \tag{2}$$

The WG pdf can be expressed as an infinite mixture of Weibull distributions with the same shape parameter α. If $|z| < 1$ and $k > 0$, we have the series representation

$$(1 - z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k + j)}{\Gamma(k)j!} z^j. \tag{3}$$

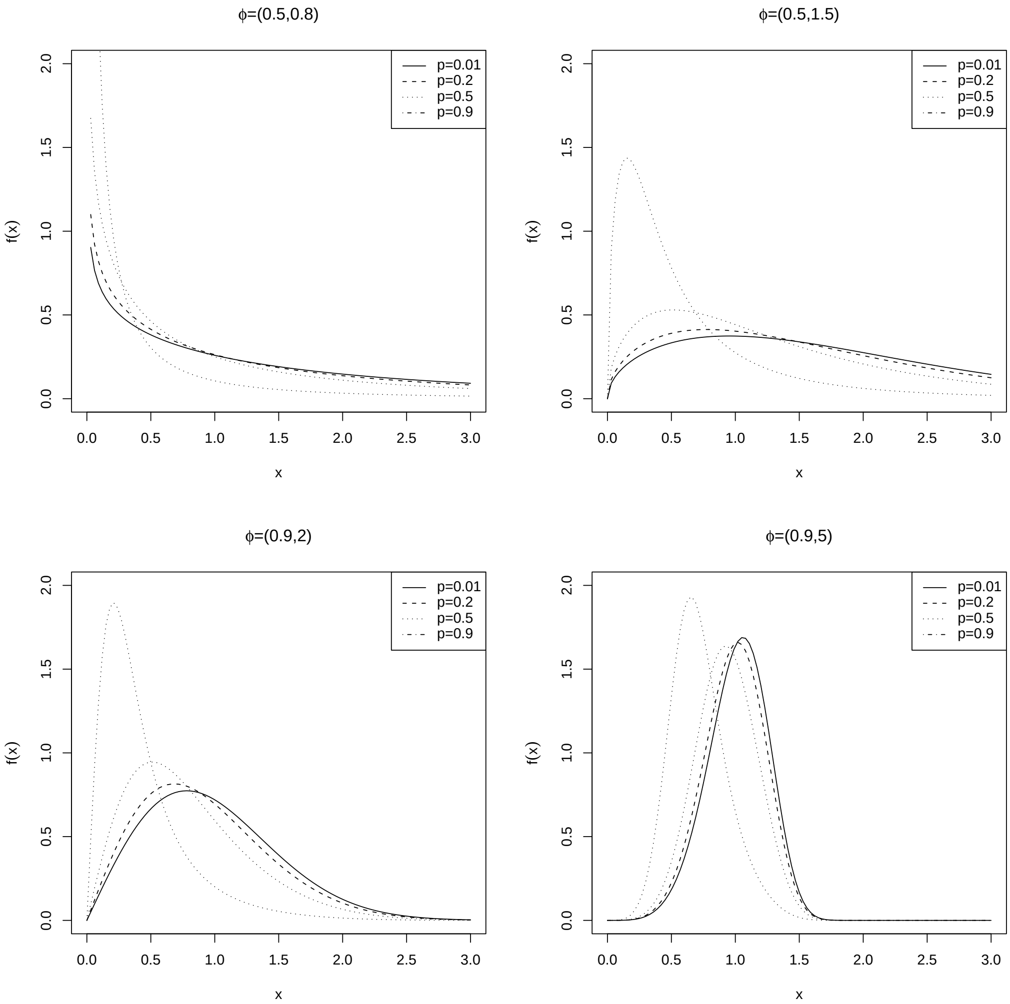


Figure 1. Pdf of the WG distribution for selected values of the parameters.

Expanding $\{1 - p\,e^{-(\beta x)^\alpha}\}^{-2}$ as in (3), we rewrite (1) as

$$f(x; p, \beta, \alpha) = \alpha\beta^\alpha(1-p)x^{\alpha-1}e^{-(\beta x)^\alpha}\sum_{j=0}^{\infty}(j+1)p^j e^{-j(\beta x)^\alpha}.$$

From the Weibull pdf given before, we obtain

$$f(x; p, \beta, \alpha) = (1-p)\sum_{j=0}^{\infty} p^j g(x; \beta(j+1)^{1/\alpha}, \alpha). \tag{4}$$

Hence, some mathematical properties (cdf, moments, percentiles, moment generating function, factorial moments, etc.) of the WG distribution can be obtained using (4) from the corresponding properties of the Weibull distribution.

3. Properties of the WG distribution

3.1. *The distribution and hazard functions and order statistics*

Let X be a random variable such that X follows the WG distribution with parameters p, β and α. In the sequel, the distribution of X will be referred to $WG(p, \beta, \alpha)$. Its cdf is given by

$$F(x) = \frac{1 - e^{-(\beta x)^{\alpha}}}{1 - p\, e^{-(\beta x)^{\alpha}}}, \quad x > 0. \tag{5}$$

The survivor and hazard functions are

$$S(x) = \frac{(1-p)e^{-(\beta x)^{\alpha}}}{1 - p\, e^{-(\beta x)^{\alpha}}}, \quad x > 0 \tag{6}$$

and

$$h(x) = \alpha\beta^{\alpha}x^{\alpha-1}\{1 - p\, e^{-(\beta x)^{\alpha}}\}^{-1}, \quad x > 0, \tag{7}$$

respectively.

The hazard function (7) is decreasing for $0 < \alpha \le 1$. However, for $\alpha > 1$ it can take different forms. As the WG distribution converges to the Weibull distribution when $p \to 0^{+}$, the hazard function for very small values of p can be decreasing, increasing and almost constant. When $p \to 1^{-}$, the WG distribution converges to a distribution degenerate in zero. Hence, the parameter p can be interpreted as a concentration parameter. Figure 2 illustrates some of the possible shapes of the hazard function for selected values of the vector $\phi = (\beta, \alpha)$ for $p = 0.01, 0.2, 0.5$ and 0.9. These plots show that the hazard function of the new distribution is quite flexible.

We now calculate the pdf of the order statistics. Let $X_1, \ldots, X_n$ be iid random variables such that $X_i \sim WG(p, \beta, \alpha)$ for $i = 1, \ldots, n$. The pdf of the ith order statistic, say $X_{i:n}$, is given by (for $x > 0$)

$$f_{i:n}(x) = \frac{\alpha\beta^{\alpha}(1-p)^{n-i+1}}{B(i, n-i+1)}x^{\alpha-1}e^{-(n-i+1)(\beta x)^{\alpha}}\frac{\{1 - e^{-(\beta x)^{\alpha}}\}^{i-1}}{\{1 - pe^{-(\beta x)^{\alpha}}\}^{n+1}}, \tag{8}$$

where $B(a, b) = \int_0^{\infty} \omega^{a-1}(1-\omega)^{b-1}d\omega$ is the beta function. Let $g_{i:n}(x)$ be the pdf of the ith order statistic in a sample of size n from the Weibull distribution with parameters β and α. We have

$$g_{i:n}(x) = \frac{\alpha\beta^{\alpha}}{B(i, n-i+1)}x^{\alpha-1}e^{-(n-i+1)(\beta x)^{\alpha}}\{1 - e^{-(\beta x)^{\alpha}}\}^{i-1}.$$

Equation (8) can be rewritten in terms of $g_{i:n}(x)$ as

$$f_{i:n}(x) = (1-p)^{n-i+1}\{1 - pe^{-(\beta x)^{\alpha}}\}^{-(n+1)}g_{i:n}(x).$$

Further, we can express the pdf of $X_{i:n}$ as an infinite mixture of Weibull order

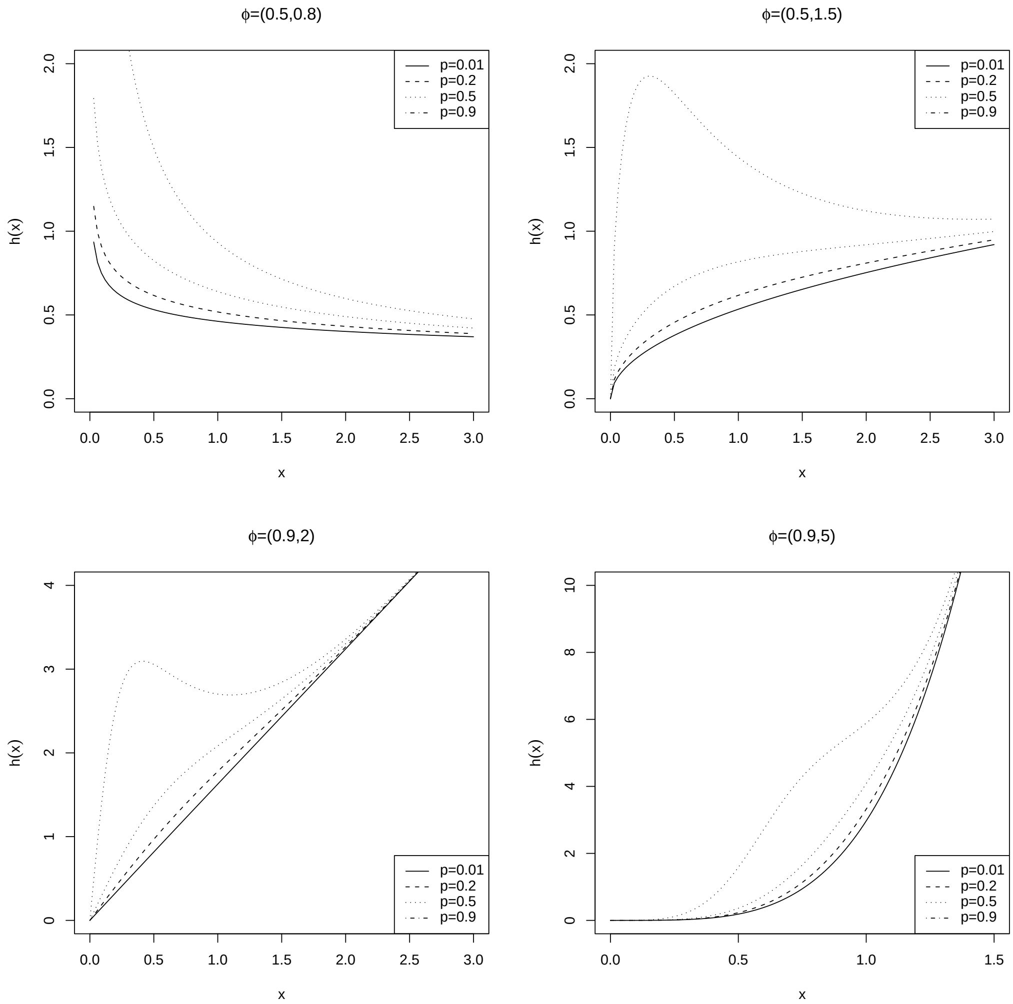


Figure 2. Hazard rate function of the WG distribution for selected values of the parameters.

statistic densities. Using (3) in (8), we obtain

$$f_{i:n}(x) = (1-p)^{n-i+1}\,\frac{n!(n+j-i)!}{(n+j)!(n-i)!}\sum_{j=0}^{\infty}\binom{n+j}{n}p^j g_{i:n+j}(x). \tag{9}$$

Hence, equation (9) shows that some mathematical properties of the WG order statistics can be obtained immediately from the corresponding properties of the Weibull order statistics.

3.2. *Quantiles and moments*

The quantile γ (x_γ) of the WG distribution follows from (5) as

$$x_\gamma = \beta^{-1}\left\{\log\left(\frac{1-p\,\gamma}{1-\gamma}\right)\right\}^{1/\alpha}.$$

In particular, the median is simply $x_{0.5} = \beta^{-1}\{\log(1-p)\}^{1/\alpha}$.

The rth moment of X is given by

$$E(X^r) = \alpha\beta^\alpha(1-p)\int_0^\infty x^{r+\alpha-1}e^{-(\beta x)^\alpha}\left\{1-p\,e^{-(\beta x)^\alpha}\right\}^{-2}dx.$$

Expanding the term $\{1-p\,e^{-(\beta x)^\alpha}\}^{-2}$ as in (3) yields

$$E(X^r) = \frac{(1-p)\Gamma(r/\alpha+1)}{p\,\beta^r}L(p;r/\alpha),$$

where $L(p;a)=\sum_{j=1}^\infty p^j j^{-a}$ is Euler's polylogarithm function (see, Erdelyi et al. [6], p. 31) which is readily available in standard software such as Mathematica.

Figure 3 plots the skewness and kurtosis of the WG distribution as functions of p for $\beta=1$ and some values of α. When $p\to1^-$, the coefficients of skewness and kurtosis tend to zero as expected, since the WG distribution converges to a degenerate distribution (in zero) when $p\to1^-$.

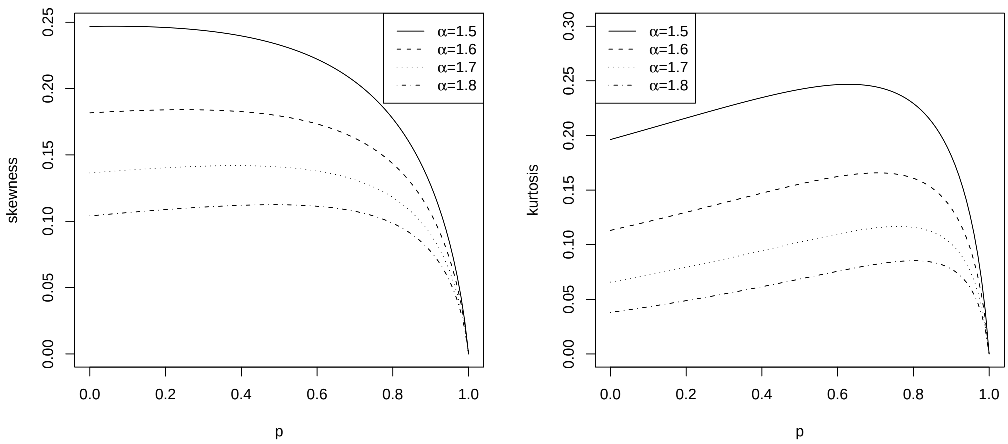


Figure 3. Skewness and kurtosis of the WG distribution as functions of p for $\beta=1$ and some values of α.

The rth moment of the ith order statistic $X_{i:n}$ is given by

$$E(X_{i:n}^r) = \frac{\alpha\beta^\alpha(1-p)^{n-i+1}}{B(i,n-i+1)}\int_0^\infty x^{\alpha+r-1}e^{-(n-i+1)(\beta x)^\alpha}\frac{\{1-e^{-(\beta x)^\alpha}\}^{i-1}}{\{1-pe^{-(\beta x)^\alpha}\}^{n+1}}dx.$$

Expanding the term $\{1-pe^{-(\beta x)^\alpha}\}^{-(n+1)}$ as in (3) and using the binomial expansion for $\{1-e^{-(\beta x)^\alpha}\}^{i-1}$, the rth moment of $X_{i:n}$ becomes

$$E(X_{i:n}^r) = \frac{(1-p)^{n-i+1}\Gamma(r/\alpha+1)}{B(i,n-i+1)\beta^r}\sum_{j=0}^\infty\sum_{k=0}^{i-1}\frac{(-1)^k\binom{n+j}{n}\binom{i-1}{k}p^j}{(n+j+k-i+1)^{r/\alpha+1}}. \qquad (10)$$

We now give an alternative expression to (10) by using a result due to Barakat and Abdelkader [3]. We have

$$E(X_{i:n}^r) = r\sum_{k=n-i+1}^{n}(-1)^{k-n+i-1}\binom{k-1}{n-i}\binom{n}{k}\int_0^\infty x^{r-1}S(x)^k dx,$$

where $S(x)$ is the survivor function (6).

Using the expansion (3) and changing variables $u = (k+j)(\beta x)^\alpha$, we have

$$\begin{aligned}\int_0^\infty x^{r-1}S(x)^k dx &= (1-p)^k \sum_{j=0}^{\infty} \binom{k+j-1}{k-1} p^j \int_0^\infty x^{r-1} e^{-(k+j)(\beta x)^\alpha} dx \\ &= \frac{(1-p)^k}{\alpha\beta^r} \int_0^\infty u^{r/\alpha-1} e^{-u} du \sum_{j=0}^{\infty} \binom{k+j-1}{k-1} \frac{p^j}{(k+j)^{r/\alpha}} \\ &= \frac{(1-p)^k \Gamma(r/\alpha)}{\alpha\beta^r} \sum_{j=0}^{\infty} \binom{k+j-1}{k-1} \frac{p^j}{(k+j)^{r/\alpha}}.\end{aligned}$$

Hence,

$$E(X_{i:n}^r) = \frac{\Gamma(r/\alpha+1)}{(-1)^{n-i+1}\beta^r} \sum_{j=0}^{\infty} \sum_{k=n-i+1}^{n} (-1)^k \binom{n}{k} \binom{k-1}{n-i} \binom{k+j-1}{k-1} \frac{p^j(1-p)^k}{(k+j)^{r/\alpha}}. \tag{11}$$

Expressions (10) and (11) give the moments of the order statistics and can be compared numerically. Table 1 gives numerical values for the first four moments of the order statistics $X_{1:15}$, $X_{7:15}$ and $X_{15:15}$ from (10) and (11) with the index j stopping at 100 and by numerical integration. We take the parameter values as $p = 0.8$, $\beta = 0.4$ and $\alpha = 2$. The figures in this table show good agreement among the three methods.

$X_{i:15} \downarrow$	rth moment$\rightarrow$	$r=1$	$r=2$	$r=3$	$r=4$
	Expression (10)	0.25717	0.08697	0.035116	0.016265
$i=1$	Expression (11)	0.25717	0.08697	0.035116	0.016265
	Numerical	0.26102	0.08795	0.035408	0.016364
	Expression (10)	0.96660	0.98827	1.06643	1.21249
$i=7$	Expression (11)	0.98502	0.99295	1.06784	1.21298
	Numerical	0.96674	0.98836	1.06649	1.21253
	Expression (10)	3.33109	11.97872	46.35371	192.32090
$i=15$	Expression (11)	3.33126	11.97875	46.35375	192.32090
	Numerical	3.33126	11.97875	46.35375	192.32090

Table 1. First four moments of some order statistics from (10) and (11) and by numerical integration.

3.3. *Rényi and Shannon entropies*

Entropy has been used in various situations in science and engineering. The entropy of a random variable X is a measure of variation of the uncertainty. Rényi entropy is defined by $I_R(\gamma) = \frac{1}{1-\gamma}\log\{\int_{\mathbb{R}} f^{\gamma}(x)dx\}$, where $\gamma > 0$ and $\gamma \neq 1$. From (3), we obtain

$$\int_0^{\infty} f^{\gamma}(x;p,\beta,\alpha)dx = \frac{[\alpha\beta^{\alpha}(1-p)]^{\gamma}}{\Gamma(2\gamma)}\sum_{j=0}^{\infty} p^j \frac{\Gamma(2\gamma+j)}{j!}\int_0^{\infty} x^{(\alpha-1)\gamma}e^{-(\gamma+j)(\beta x)^{\alpha}}dx.$$

If $(\alpha-1)(\gamma-1)\geq 0$, this expression reduces to

$$\int_0^{\infty} f^{\gamma}(x;p,\beta,\alpha)dx = \frac{\Gamma(\alpha)[\alpha(1-p)]^{\gamma}}{\beta^{\alpha(1-\gamma)}\Gamma(2\gamma)}\sum_{j=0}^{\infty} p^j \frac{\Gamma(2\gamma+j)}{j!(\alpha+j)}E(Y_j^{(\alpha-1)(\gamma-1)}),$$

where Y_j follows a gamma distribution with scale parameter $(\gamma+j)^{1/\alpha}$ and shape parameter α. Then, we have

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\frac{[\alpha(1-p)]^{\gamma}\Gamma(\gamma(\alpha-1)+1)}{\beta^{1-\gamma}\Gamma(2\gamma)}\sum_{j=0}^{\infty}\frac{p^j\Gamma(2\gamma+j)}{j!(\alpha+j)^{(\alpha-1)(\gamma-1)/\alpha+1}}\right\}.$$

Shannon entropy is defined as $E\{-\log[f(X)]\}$. This is a special case obtained from $\lim_{\gamma\to 1} I_R(\gamma)$. Hence,

$$E[-\log f(X)] = -\log[\alpha\beta^{\alpha}(1-p)]-(\alpha-1)E[\log(X)]+\beta^{\alpha}E(X^{\alpha})-2E\{\log[1-pe^{-(\beta X)^{\alpha}}]\}.$$

We can show that

$$E[\log(X)] = \psi(1)/\alpha,$$
$$E(X^{\alpha}) = -\frac{(1-p)}{p\beta^{\alpha}}\log(1-p),$$
$$E\{\log[1-pe^{-(\beta X)^{\alpha}}]\} = -\frac{1-p}{p}\{1+(1-p)[1+\log(1-p)]\}.$$

Further, the Shannon entropy reduces to

$$E[-\log f(X)] = -\log[\alpha\beta^{\alpha}(1-p)] - \frac{\alpha-1}{\alpha}\psi(1) - \frac{1-p}{p}[4-2p+(3-2p)\log(1-p)].$$

4. Estimation

Let $x = (x_1,\ldots,x_n)$ be a random sample of the WG distribution with unknown parameter vector $\theta = (p,\beta,\alpha)$. The log likelihood $\ell = \ell(\theta;x)$ for θ is

$$\ell = n[\log\alpha + \alpha\log\beta + \log(1-p)] + (\alpha-1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}(\beta x_i)^{\alpha}$$
$$-2\sum_{i=1}^{n}\log[1-p\,e^{-(\beta x_i)^{\alpha}}].$$

The score function $U(\theta) = (\partial\ell/\partial p, \partial\ell/\partial\beta, \partial\ell/\partial\alpha)^T$ has components

$$\frac{\partial\ell}{\partial p} = -n(1-p)^{-1} + 2\sum_{i=1}^{n} e^{-(\beta x_i)^\alpha}[1 - p\, e^{-(\beta x_i)^\alpha}]^{-1},$$

$$\frac{\partial\ell}{\partial\beta} = n\alpha\beta^{-1} - \alpha\beta^{\alpha-1}\sum_{i=1}^{n} x_i^\alpha\{1 + 2p\, e^{-(\beta x_i)^\alpha}[1 - p\, e^{-(\beta x_i)^\alpha}]^{-1}\},$$

$$\frac{\partial\ell}{\partial\alpha} = n\alpha^{-1} + \sum_{i=1}^{n}\log(\beta x_i) - \sum_{i=1}^{n}(\beta x_i)^\alpha\log(\beta x_i)\{1 + 2p\, e^{-(\beta x_i)^\alpha}[1 - p\, e^{-(\beta x_i)^\alpha}]^{-1}\}.$$

The maximum likelihood estimate (MLE) $\widehat{\theta}$ of θ is calculated numerically from the nonlinear equations $U(\theta) = \mathbf{0}$ using the EM algorithm (Dempster et al. [5]; McLachlan and Krishnan [11]) and defining an hypothetical complete-data distribution with density function

$$f(x, z; \theta) = \alpha\beta^\alpha(1-p)zx^{\alpha-1}p^{z-1}e^{-z(\beta x)^\alpha},$$

for $x, \beta, \alpha > 0$, $p \in (0,1)$ and $z \in \mathbb{N}$. Under this formulation, the E-step of an EM cycle requires the conditional expectation of $(Z|X;\theta^{(r)})$, where $\theta^{(r)} = (p^{(r)}, \beta^{(r)}, \alpha^{(r)})$ is the current estimate of θ. From the probability function $P(z|x;\theta) = zp^{z-1}e^{-(z-1)(\beta x)^\alpha}\{1 - pe^{-(\beta x)^\alpha}\}^2$ for $z \in N$, it follows that $E(Z|X;\theta) = \{1 + pe^{-(\beta x)^\alpha}\}\{1 - pe^{-(\beta x)^\alpha}\}^{-1}$. The EM cycle is completed with the M-step by using the maximum likelihood estimation over θ, with the missing Z's replaced by their conditional expectations given above. Hence, an EM iteration reduces to

$$p^{(r+1)} = 1 - \frac{n}{\sum_{i=1}^{n} w_i^{(r)}}, \quad \beta^{(r+1)} = n\left\{\sum_{i=1}^{n} x_i^{\alpha^{(r+1)}} w_i^{(r)}\right\}^{-1/\alpha^{(r+1)}},$$

where $\alpha^{(r+1)}$ is the solution of the nonlinear equation

$$\frac{n}{\alpha^{(r+1)}} + \sum_{i=1}^{n}\log x_i - n\frac{\sum_{i=1}^{n} w_i^{(r)} x_i^{\alpha^{(r+1)}}\log x_i}{\sum_{i=1}^{n} w_i^{(r)} x_i^{\alpha^{(r+1)}}} = 0,$$

where

$$w_i^{(r)} = \frac{1 + p^{(r)}e^{-(\beta^{(r)}x_i)^{\alpha^{(r)}}}}{1 - p^{(r)}e^{-(\beta^{(r)}x_i)^{\alpha^{(r)}}}}.$$

An implementation of this algorithm using the software R is given in Appendix B.

5. Inference

For asymptotic interval estimation and hypothesis tests on the model parameters, we require the information matrix. Invert the joint observed information matrix for the parameters p, β and α to obtain the large-sample covariance matrix of the estimates $\widehat{p}$, $\widehat{\beta}$ and $\widehat{\alpha}$ will provide confidence regions for any parameters using the asymptotic normality of these estimates. Then, asymptotic standard errors for the MLEs can be given by the square root of the diagonal elements of the estimated inverse observed information matrix, and from these quantities asymptotic confidence intervals can be formed and hypothesis tests made.The 3×3 observed information matrix $J_n = J_n(\theta)$ is given by

$$J_n = \begin{pmatrix} J_{pp} & J_{p\beta} & J_{p\alpha} \\ J_{p\beta} & J_{\beta\beta} & J_{\beta\alpha} \\ J_{p\alpha} & J_{\beta\alpha} & J_{\alpha\alpha} \end{pmatrix},$$

where

$$\begin{aligned}
-J_{pp} &= \frac{\partial^2 \ell}{\partial p^2} = 2\sum_{i=1}^{n} T^{(i)}_{0,0,2,2} - n(1-p)^{-2}, \\
-J_{p\alpha} &= \frac{\partial^2 \ell}{\partial p \partial \alpha} = -2\beta^{\alpha}\sum_{i=1}^{n}(p\,T^{(i)}_{1,1,2,2} + T^{(i)}_{1,1,1,1}), \\
-J_{p\beta} &= \frac{\partial^2 \ell}{\partial p \partial \beta} = -2\alpha\beta^{\alpha-1}\sum_{i=1}^{n}(p\,T^{(i)}_{1,0,2,2} + T^{(i)}_{1,0,1,1}), \\
-J_{\alpha\alpha} &= \frac{\partial^2 \ell}{\partial \alpha^2} = -n\alpha^{-2} + \sum_{i=1}^{n}(2p^2\beta^{2\alpha}T^{(i)}_{2,2,2,2} + 2p\beta^{2\alpha}T^{(i)}_{2,2,1,1} - \beta^{\alpha}T^{(i)}_{1,2,0,0} - \\
&\quad 2p\beta^{\alpha}T^{(i)}_{1,2,1,1}), \\
-J_{\beta\alpha} &= \frac{\partial^2 \ell}{\partial \alpha \partial \beta} = n\beta^{-1} - \beta^{\alpha-1}\sum_{i=1}^{n}(\alpha T^{(i)}_{1,1,0,0} + T^{(i)}_{1,0,0,0})(1 + 2pT^{(i)}_{0,0,1,1}) + \\
&\quad 2p\alpha\beta^{2\alpha-1}\sum_{i=1}^{n}(p\,T^{(i)}_{2,1,2,2} + T^{(i)}_{2,1,1,1}), \\
-J_{\beta\beta} &= \frac{\partial^2 \ell}{\partial \beta^2} = -n\alpha\beta^{-2} - \alpha(\alpha-1)\beta^{\alpha-2}\sum_{i=1}^{n}(T^{(i)}_{1,0,0,0} + 2pT^{(i)}_{1,0,1,1}) + \\
&\quad 2\alpha^2\beta^{2\alpha-2}p\sum_{i=1}^{n}(pT^{(i)}_{2,0,2,2} + T^{(i)}_{2,0,1,1}).
\end{aligned}$$

Here,

$$T^{(i)}_{j,k,l,m} = T^{(i)}_{j,k,l,m}(x_i, \theta) = x_i^{\alpha j}\{\log(\beta x_i)\}^k\{1 - p\,e^{-(\beta x_i)^{\alpha}}\}^{-l}e^{-m(\beta x_i)^{\alpha}},$$

for $(j,k,l,m) \in \{0,1,2\}$ and $i = 1,\ldots,n$.

Under conditions that are fulfilled for the parameter θ in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\widehat{\theta} - \theta)$ is multivariate normal $N_3(0, K(\theta)^{-1})$, where $K(\theta) = \lim_{n\to\infty} n^{-1}J_n(\theta)$ is the unit information matrix. This asymptotic behavior remains valid if $K(\theta)$ is replaced by the average observed information matrix evaluated at $\widehat{\theta}$, i.e., $n^{-1}J_n(\widehat{\theta})$. We can use the asymptotic multivariate normal $N_3(0, J_n(\widehat{\theta})^{-1})$ distribution of $\widehat{\theta}$ to construct approximate confidence regions for some parameters and for the hazard and survival functions. In fact, an asymptotic $100(1-\gamma)\%$ confidence interval for each parameter θ_r is given by

$$ACI_r = (\widehat{\theta}_r - z_{\gamma/2}\sqrt{\widehat{J}^{\theta_r\theta_r}}, \widehat{\theta}_r + z_{\gamma/2}\sqrt{\widehat{J}^{\theta_r\theta_r}}),$$

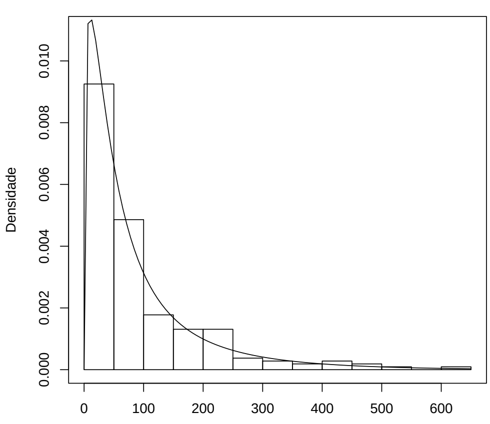
where $\widehat{J}^{\theta_r\theta_r}$ represents the (r, r)th diagonal element of $J_n(\widehat{\theta})^{-1}$ for $r = 1, 2, 3$ and $z_{\gamma/2}$ is the quantile $1 - \gamma/2$ of the standard normal distribution.

The asymptotic normality is also useful for testing goodness of fit of the three parameter WG distribution and for comparing this distribution with some of its special sub-models via the likelihood ratio (LR) statistic. We consider the partition $\theta = (\theta_1^T, \theta_2^T)^T$, where θ_1 is a subset of parameters of interest of the WG distribution and θ_2 is a subset of the remaining parameters. The LR statistic for testing the null hypothesis $H_0 : \theta_1 = \theta_1^{(0)}$ versus the alternative hypothesis $H_1 : \theta_1 \neq \theta_1^{(0)}$ is given by $w = 2\{\ell(\widehat{\theta}) - \ell(\widetilde{\theta})\}$, where $\widetilde{\theta}$ and $\widehat{\theta}$ are the MLEs under the null and alternative hypotheses, respectively. The statistic w is asymptotically (as $n \to \infty$) distributed as χ_k^2, where k is the dimension of the subset θ_1 of interest. For example, we can compare the EG model against the WG model by testing $H_0 : \alpha = 1$ versus $H_1 : \alpha \neq 1$ and the Weibull model against the WG model by testing $H_0 : \alpha = 1, p = 0$ versus $H_1 :$ H$_0$ is false.

6. Applications

In this section, we fit the WG models to two real data sets. The first data set consist of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes. The pooled data with 214 observations were first analyzed by Proschan [12] and discussed further by Dahiya and Gurland [4], Gleser [7], Adamidis and Loukas [2] and Kus [8]. The second data set is an uncensored data set from Nichols and Padgett [9] consisting of 100 observations on breaking stress of carbon fibres (in Gba).

For the first data set, the estimated parameters using an EM algorithm were $\hat{p} = 0.7841$, $\hat{\beta} = 0.0048$ and $\hat{\alpha} = 1.2246$. The fitted pdf and the estimated quantiles versus observed quantiles are given in Figures 4. This figure shows a good fit of the WG model for the first data set.



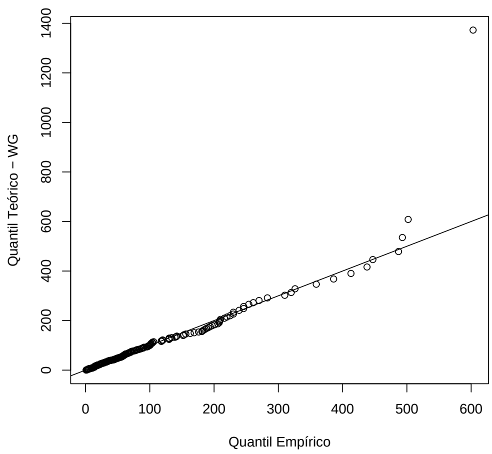


Figure 4. Plots of the fitted pdf and of the estimated quantiles versus observed quantiles for the first data set.

For the second data set, the estimates obtained using an EM algorithm are $\hat{p} = 0.3073$, $\hat{\beta} = 0.3148$ and $\hat{\alpha} = 3.0093$. The plot of the fitted pdf and the estimated quantiles versus observed quantiles in Figure 5 shows a good fit of the WG model.

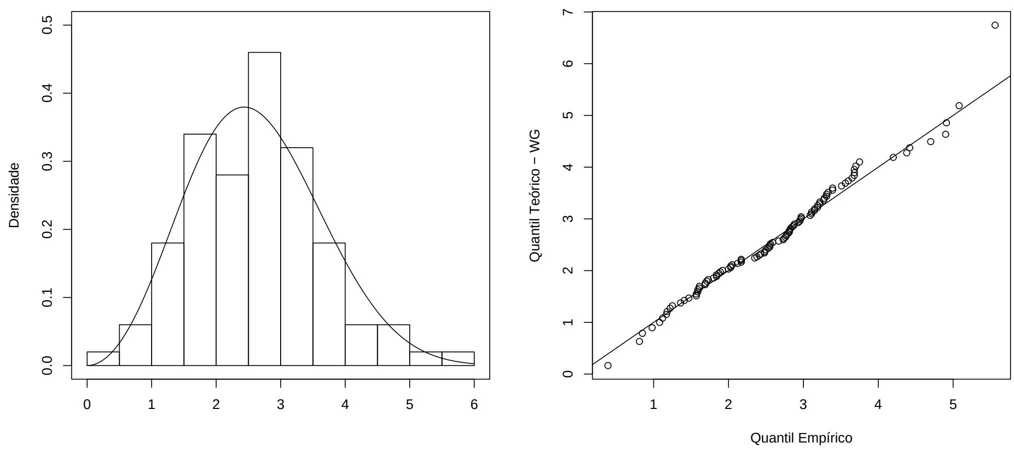


Figure 5. Plots of the fitted pdf and of the estimated quantiles versus observed quantiles for the second data set.

7. Conclusions

We define a new model so called the Weibull-geometric (WG) distribution that generalizes the exponential-geometric (EG) distribution introduced by Adamidis and Loukas [2]. Some mathematical properties are derived and plots of the pdf and hazard functions are presented to show the flexibility of the new distribution. We give closed form expressions for the moments of the distribution. We obtain the pdf of the order statistics and provide expansions for the moments of the order statistics. Estimation by maximum likelihood is discussed and an algorithm EM is proposed. We give asymptotic confidence intervals for the model parameters and present the use of the LR statistic to compare the fit of the WG model with special sub-models. Finally, we fit WG models to two real data sets to show the flexibility and the potentially of the new distribution.

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Appendix A.

We now show that the WG density is unimodal when $\alpha > 1$. Let

$$h(u) = u + pe^{-u}\left(u + \frac{\alpha - 1}{\alpha}\right).$$

When $u \to 0^+$, $h(u) \to p\frac{\alpha-1}{\alpha}$ and when $u \to \infty$, $h(u) \to \infty$. Thus, if $h(u)$ is an increasing function, the solution in (2) is unique and the WG distribution is unimodal. We have $h'(u) = 1 - pe^{-u}\left(u + \frac{\alpha-1}{\alpha}\right) + pe^{-u}$. Using the inequalities $-pe^{-u}\frac{\alpha-1}{\alpha} > -pe^{-u}$ and $-pe^{-u}u > e{-}1$, it follows that $h'(u) > 1 - e^{-1} > 0$, $\forall u > 0$. Hence, $h(u)$ is an increasing function and the WG distribution is unimodal if $\alpha > 1$.

Appendix B.

The following R function estimates the model parameters p, β and α through an EM algorithm.

```
fit.WG<-function(x,par,tol=1e-4,maxi=100){

# x       Numerical vector of data.
# par Vector of initial values for the parameters p, beta and alpha
# to be optimized over, on this exactly order.
# tol Convergence tolerance.
#maxi Upper end point of the interval to be searched.

p<-par[1]
beta<-par[2]
alpha<-par[3]
n<-length(x)
z.temp<-function(){
(1+p*exp(-(beta*x)^alpha))/(1-p*exp(-(beta*x)^alpha))
}
alpha.sc<-function(alpha){
n/alpha+sum(log(x))-n*sum(z*x^alpha*log(x))/sum(z*x^alpha)
}
test<-1
while(test>tol){
z<-z.temp()
alpha.new<-(alpha.sc,interval=c(0,maxi))$root
beta.new<-(n/sum(x^alpha.new*z))^(1/alpha.new)
p.new<-1-n/sum(z)
test<-max(abs(c(((alpha.new-alpha)),
```

```
                ((beta.new-beta)),
                ((p.new-p)))))
alpha<-alpha.new
beta<-beta.new
p<-p.new
}
c(p,beta,alpha)
}
```