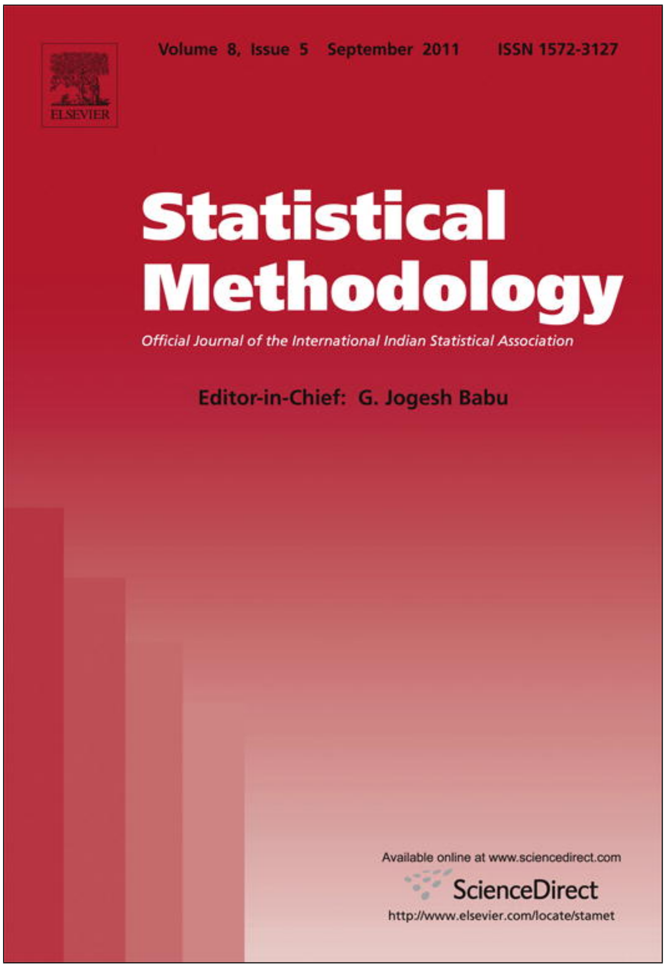
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The Kumaraswamy generalized gamma distribution with application in survival analysis

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ABSTRACT

We introduce and study the so-called Kumaraswamy generalized gamma distribution that is capable of modeling bathtub-shaped hazard rate functions. The beauty and importance of this distribution lies in its ability to model monotone and non-monotone failure rate functions, which are quite common in lifetime data analysis and reliability. The new distribution has a large number of well-known lifetime special sub-models such as the exponentiated generalized gamma, exponentiated Weibull, exponentiated generalized half-normal, exponentiated gamma, generalized Rayleigh, among others. Some structural properties of the new distribution are studied. We obtain two infinite sum representations for the moments and an expansion for the generating function. We calculate the density function of the order statistics and an expansion for their moments. The method of maximum likelihood and a Bayesian procedure are adopted for estimating the model parameters. The usefulness of the new distribution is illustrated in two real data sets.

1. Introduction

The gamma distribution is the most popular model for analyzing skewed data. The generalized gamma distribution (GG) was introduced by Stacy [43] and includes as special sub-models: the exponential, Weibull, gamma and Rayleigh distributions, among others. It is suitable for modeling data with different forms of hazard rate function: increasing, decreasing, in the form of a bathtub

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and unimodal. This characteristic is useful for estimating individual hazard rate functions and both relative hazards and relative times [12]. The GG distribution has been used in several research areas such as engineering, hydrology and survival analysis. Ortega et al. [35] discussed influence diagnostics in GG regression models; Nadarajah and Gupta [34] used this distribution with application to drought data; Cox et al. [13] presented a parametric survival analysis and taxonomy of the GG hazard rate functions and Ali et al. [3] derived the exact distributions of the product X_1X_2 and the quotient X_1/X_2, when X_1 and X_2 are independent GG random variables providing applications of their results to drought data from Nebraska. Further, Gomes et al. [18] focused on the parameter estimation; Ortega et al. [37] compared three types of residuals based on the deviance component in GG regression models under censored observations; Cox [12] discussed and compared the F-generalized family with the GG model; Almpanidis and Kotropoulos [4] presented a text-independent automatic phone segmentation algorithm based on the GG distribution and Nadarajah [32] analyzed some incorrect references with respect to the use of this distribution in electrical and electronic engineering. More recently, Barkauskas et al. [5] modeled the noise part of a spectrum as an autoregressive moving average (ARMA) model with innovations having the GG distribution; Malhotra et al. [28] provided a unified analysis for wireless systems over generalized fading channels that is modeled by a two-parameter GG model and Xie and Liu [47] analyzed three-moment auto conversion parametrization based on this distribution. Further, Ortega et al. [36] proposed a modified GG regression model to allow the possibility that long-term survivors may be presented in the data and Cordeiro et al. [10] proposed the exponentiated generalized gamma (EGG) distribution.

In the last decade, several authors have proposed new classes of distributions, which are based on modifications (in different ways) of the Weibull distribution to provide hazard rate functions having the form of U. Among these, we mention the Weibull, exponentiated Weibull [30], which also exhibits unimodal hazard rate function, the additive Weibull [46] and the extended Weibull [48] distributions. More recently, Carrasco et al. [7] presented a four-parameter generalized modified Weibull (GMW) distribution, Gusmão et al. [22] studied a three-parameter generalized inverse Weibull distribution with decreasing and unimodal failure rate and Pescim et al. [39]proposed the four-parameter generalized half-normal distribution.

The distribution by Kumaraswamy (denoted with the prefix "KumW" for short) [24] is not very common among statisticians and has been little explored in the literature. Its cumulative distribution function (cdf) (for $0 < x < 1$) is $F(x) = 1 - (1 - x^{\lambda})^{\varphi}$, where $\lambda > 0$ and $\varphi > 0$ are shape parameters. The Kum probability density function (pdf) has a simple form $f(x) = \lambda\varphi x^{\lambda-1}(1 - x^{\lambda})^{\varphi-1}$, which can be unimodal, increasing, decreasing or constant, depending on the values of its parameters. This distribution does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before, nor has its relative interchangeability with the beta distribution been widely appreciated. However, in a very recent paper, Jones [23] explored the background and genesis of the Kum distribution and, more importantly, made clear some similarities and differences between the beta and Kum distributions.

If $G(x)$ is the baseline cdf of a random variable, Cordeiro and de Castro [9] defined the cdf of the Kumaraswamy-G (Kum-G) distribution by

$$F(x) = 1 - [1 - G(x)^{\lambda}]^{\varphi}, \tag{1}$$

where $\lambda > 0$ and $\varphi > 0$ are two additional parameters to the G distribution. Their role is to govern skewness and generate a distribution with heavier tails. The density function corresponding to (1) is

$$f(x) = \lambda\varphi g(x)G(x)^{\lambda-1}[1 - G(x)^{\lambda}]^{\varphi-1}, \tag{2}$$

where $g(x) = \mathrm{d}G(x)/\mathrm{d}x$. The density (2) does not involve any special function, such as the incomplete beta function as is the case of the beta-G distribution [14]. This generalization contains distributions with unimodal and bathtub shaped hazard rate functions. It also contemplates a broad class of models with monotone risk functions. Some structural properties of the Kum-G distribution derived by Cordeiro and de Castro [9] are usually much simpler than those properties of the beta-G distribution.

In this note, we combine the works of Kumaraswamy [24], Cordeiro et al. [10] and Cordeiro and de Castro [9] to study the mathematical properties of a new model, the so-called Kumaraswamy

Table 1
Some GG distributions.

Distribution	τ	α	k
Gamma	1	α	k
Chi-square	1	2	$\frac{n}{2}$
Exponential	1	α	1
Weibull	c	α	1
Rayleigh	2	α	1
Maxwell	2	α	$\frac{3}{2}$
Folded normal	2	$\sqrt{2}$	$\frac{1}{2}$

generalized gamma (KumGG) distribution. The rest of the article is organized as follows. Section 2 introduces the KumGG distribution. Several important special models are presented in Section 3. In Section 4, we demonstrate that the KumGG density function can be written as a mixture of GG density functions. Section 5 provides two explicit expansions for the moments and an expansion for the moment generating function (mgf). In Section 6, we obtain expansions for the moments of the order statistics. Maximum likelihood estimation is investigated in Section 7. In Section 8, a Bayesian methodology is applied to estimate the model parameters. Two real lifetime data sets are used in Section 9 to illustrate the usefulness of the KumGG model. Concluding comments are given in Section 10.

2. The Kumaraswamy-generalized gamma distribution

The cdf of the GG(α, τ, k) distribution [43] is

$$G_{\alpha,\tau,k}(t) = \frac{\gamma(k, (t/\alpha)^{\tau})}{\Gamma(k)},$$

where $\alpha > 0$, $\tau > 0$, $k > 0$, $\gamma(k, x) = \int_0^x w^{k-1} e^{-w} dw$ is the incomplete gamma function and $\Gamma(.)$ is the gamma function. Basic properties of the GG distribution are given by Stacy and Mihram [44] and Lawless [26,27]. Some important special sub-models of the GG distribution are listed in Table 1.

The KumGG cumulative distribution (for $t > 0$) is defined by substituting $G_{\alpha,\tau,k}(t)$ into Eq. (1). Hence, the associated density function with five positive parameters α, τ, k, λ and φ has the form

$$f(t) = \frac{\lambda\varphi\tau}{\alpha\Gamma(k)} \left(\frac{t}{\alpha}\right)^{\tau k-1} \exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right] \times \left\{\gamma_1\left[k, \left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1} \left(1 - \left\{\gamma_1\left[k, \left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi-1}, \quad (3)$$

where $\gamma_1(k, x) = \gamma(k, x)/\Gamma(k)$ is the incomplete gamma ratio function, α is a scale parameter and the other positive parameters τ, k, φ and λ are shape parameters. One major benefit of (3) is its ability of fitting skewed data that cannot be properly fitted by existing distributions. The KumGG density allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology.

The Weibull and GG distributions are the most important sub-models of (3) for $\varphi = \lambda = k = 1$ and $\varphi = \lambda = 1$, respectively. The KumGG distribution approaches the log-normal (LN) distribution when $\varphi = \lambda = 1$ and $k \to \infty$. Other sub-models can be immediately defined from Table 1: Kum–Gamma, Kum–Chi-Square, Kum–Exponential, Kum–Weibull, Kum–Rayleigh, Kum–Maxwell and Kum–Folded normal with 4, 3, 3, 4, 3, 3 and 2 parameters, respectively.

If T is a random variable with density function (3), we write $T \sim$ KumGG$(\alpha, \tau, k, \lambda, \varphi)$. The survival and hazard rate functions corresponding to (3) are

$$S(t) = 1 - F(t) = \left(1 - \left\{\gamma_1\left[k, \left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi} \quad (4)$$

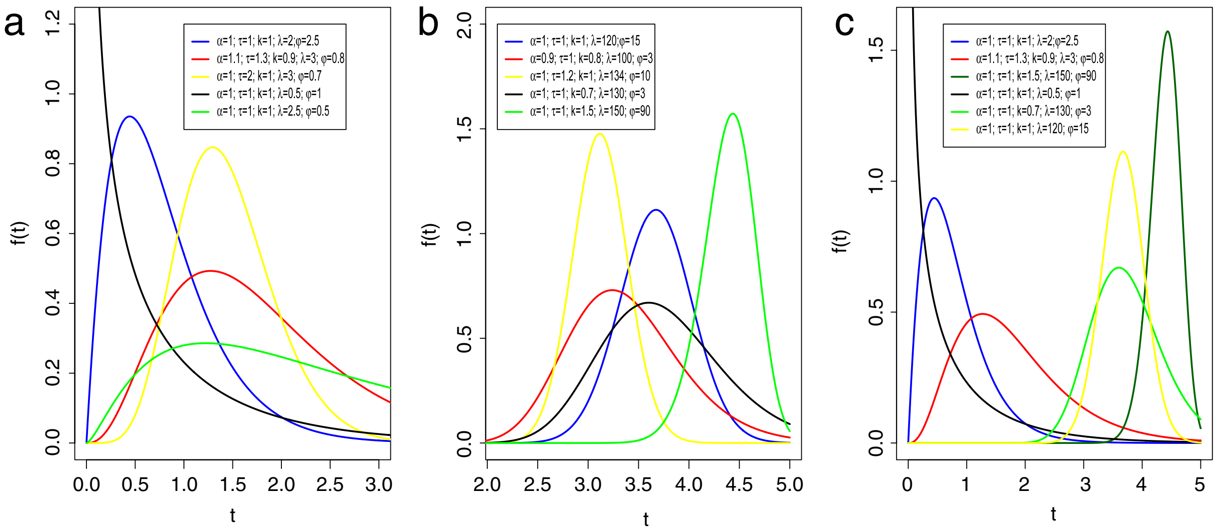

Fig. 1. Plots of the KumGG density function for some parameter values.

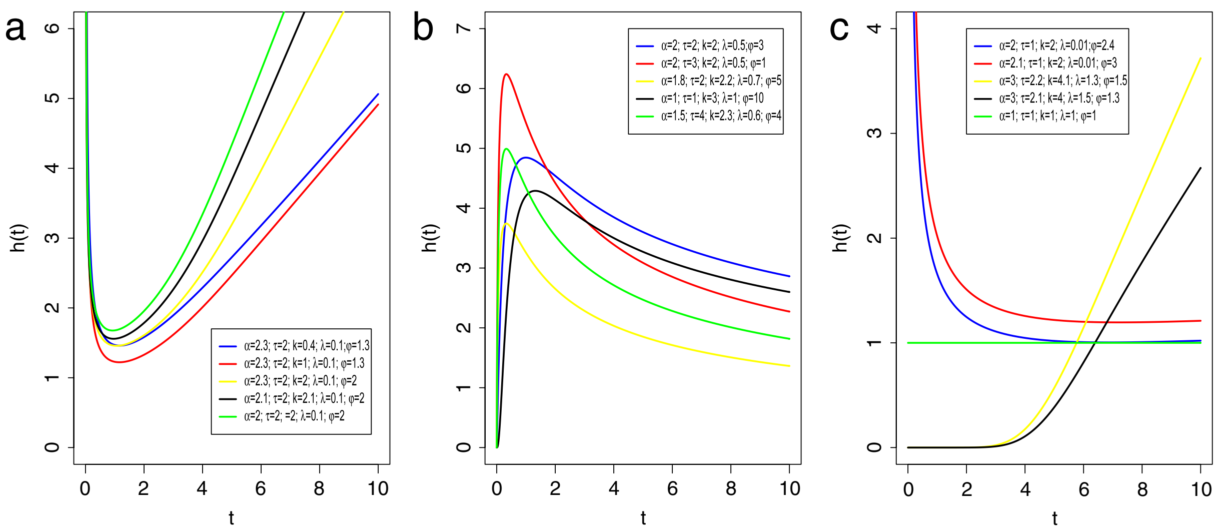

Fig. 2. The KumGG hazard rate function. (a) A bathtub hazard rate function. (b) An unimodal hazard rate function. (c) Increasing, decreasing and constant hazard rate function.

and

$$h(t) = \frac{\lambda\varphi\tau}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right]\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1} \times\left(1-\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{-1}, \tag{5}$$

respectively. Plots of the KumGG density function for selected parameter values are given in Fig. 1. The hazard rate function (5) is quite flexible for modeling survival data. See the plots for selected parameter values given in Fig. 2.

We can simulate the KumGG distribution by solving the nonlinear equation

$$\left(1-u^{1/\varphi}\right)^{1/\lambda} - \gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right] = 0, \tag{6}$$

where u has the uniform $U(0, 1)$ distribution. Some properties of the KumGG distribution are:

$$\text{If } T \sim \text{KumGG}(\alpha, \tau, k, \lambda, \varphi) \Rightarrow bT \sim \text{KumGG}(b\alpha, \tau, k, \lambda, \varphi), \quad \forall b > 0$$
$$\text{If } T \sim \text{KumGG}(\alpha, \tau, k, \lambda, \varphi) \Rightarrow T^m \sim \text{KumGG}(\alpha^m, \tau/m, k, \lambda, \varphi), \quad \forall m \neq 0.$$

So, the new distribution is closed under power transformation.

A physical interpretation of the KumGG distribution (for λ and φ positive integers) is as follows. Suppose a system is made of φ independent components and that each component is made up of λ independent subcomponents. Suppose the system fails if any of the φ components fails and that each component fails if all of the λ subcomponents fail. Let $X_{j1}, \ldots, X_{j\lambda}$ denote the lifetimes of the subcomponents within the jth component, $j = 1, \ldots, \varphi$ having a common GG distribution. Let X_j denote the lifetime of the jth component, $j = 1, \ldots, \varphi$ and let X denote the time to failure distribution of the entire system. The cdf of X is

$$\Pr(X \leq x) = 1 - \Pr\left(X_1 > x, \ldots, X_\varphi > x\right) = 1 - \{1 - \Pr(X_1 \leq x)\}^\varphi$$

and then

$$\Pr(X \leq x) = 1 - \{1 - \Pr(X_{11} \leq x, \ldots, X_{1\lambda} \leq x)\}^\varphi = 1 - \{1 - \Pr^\lambda(X_{11} \leq x)\}^\varphi.$$

So, X has precisely the KumGG distribution given by (3).

3. Special sub-models

The following well known and new distributions are special sub-models of the KumGG distribution.

- *Exponentiated Generalized Gamma distribution.*
  If $\varphi = 1$, the KumGG distribution reduces to the exponentiated generalized gamma (EGG) density introduced by Cordeiro et al. [10]. If $\tau = \varphi = 1$ in addition to $k = 1$, the special case corresponds to the exponentiated exponential (EE) distribution [20,21]. If $\tau = 2$ in addition to $k = \varphi = 1$, it becomes the generalized Rayleigh (GR) distribution [25].
- *Kum–Weibull distribution* [9].
  For $k = 1$, Eq. (3) yields the Kum–Weibull (KumW) distribution. If $\varphi = k = 1$, it reduces to the exponentiated Weibull (EW) distribution (see, [30,31]). If $\varphi = \lambda = k = 1$, (3) becomes the Weibull distribution. If $\tau = 2$ and $k = 1$, we obtain the Kum–Rayleigh (KumR) distribution. If $k = \tau = 1$, it gives the Kum–exponential (KumE) distribution. If $\varphi = \lambda = k = 1$, it yields two important special sub-models: the exponential ($\tau = 1$) and Rayleigh ($\tau = 2$) distributions, respectively.
- *Kum–Gamma distribution* [9].
  For $\tau = 1$, the KumGG distribution reduces to the four-parameter Kum–Gamma (KumG_4) distribution. If $\varphi = \tau = 1$, we obtain the exponentiated gamma (EG_3) distribution with three parameters. If $\varphi = \tau = \alpha = 1$, it gives to the exponentiated gamma (EG_2) distribution with two parameters. Further, if $k = 1$, we obtain the Kum–Gamma distribution with one parameter. If $\varphi = \lambda = \tau = 1$, it produces the two-parameter gamma distribution. In addition, if $k = 1$, we obtain the one-parameter gamma distribution.
- *Kum–Chi-Square distribution (new).*
  For $\tau = 2, \alpha = 2$ and $k = p/2$, it becomes the Kum–Chi-Square (KumChiSq) distribution. If $\varphi = 1, \alpha = \tau = 2$ and $k = p/2$, it gives the exponentiated-chi-square (EChiSq) distribution. If $\varphi = \lambda = 1$, in addition to $\alpha = \tau = 2$ and $k = p/2$, we obtain the well-known chi-square distribution.
- *Kum–Scaled Chi-Square distribution (new).*
  For $\tau = 1, \alpha = \sqrt{2}\sigma$ and $k = p/2$, it becomes the Kum–Scaled Chi-Square (KumSChiSq) distribution. For $\varphi = \tau = 1, \alpha = \sqrt{2}\sigma$ and $k = p/2$, it gives the exponentiated scaled chi-square (ESChiSq) distribution. If $\varphi = \lambda = 1$, in addition to $\alpha = \sqrt{2}\sigma, \tau = 1$ and $k = p/2$, the special case coincides with the scaled chi-square (SChiSq) distribution.

- *Kum–Maxwell distribution (new).*
For $\tau = 2, \alpha = \sqrt{\theta}$ and $k = 3/2$, the KumGG distribution reduces to the Kum–Maxwell (KumMa) distribution. For $\varphi = 1, \tau = 2, \alpha = \sqrt{\theta}$ and $k = 3/2$, we obtain the exponentiated Maxwell (EM) distribution. If $\varphi = \lambda = 1$ in addition to $\alpha = \sqrt{\theta}, \tau = 2$ and $k = 3/2$, it reduces to the Maxwell (Ma) distribution (see, for example, [6]).
- *Kum–Nakagami distribution (new).*
For $\tau = 2, \alpha = \sqrt{w/\mu}$ and $k = \mu$, it becomes the Kum–Nakagami (KumNa) distribution. For $\varphi = 1, \tau = 2, \alpha = \sqrt{w/\mu}$ and $k = \mu$, we obtain the exponentiated Nakagami (EM) distribution. If $\varphi = \lambda = 1$, in addition to $\alpha = \sqrt{w/\mu}, \tau = 2$ and $k = \mu$, it corresponds to the Nakagami (Na) distribution (see, for example, [41]).
- *Kum–generalized half-normal distribution (new).*
If $\tau = 2\gamma, \alpha = 2^{\frac{1}{2\gamma}}\theta$ and $k = 1/2$, the special case is referred to as the Kum–generalized half-normal (KumGHN) distribution. For $\varphi = 1, \tau = 2\gamma, \alpha = 2^{\frac{1}{2\gamma}}\theta$ and $k = 1/2$, it gives the exponentiated generalized half-normal (EGHN) distribution. For $\alpha = 2^{\frac{1}{2}}\theta, \tau = k = 2$, we obtain the Kum–half-normal (KumHN) distribution. If $\varphi = 1, \alpha = 2^{\frac{1}{2}}\theta, \tau = 2$ and $k = 1/2$, the reduced model is called the exponentiated half-normal (EHN) distribution. If $\varphi = \lambda = 1$, in addition to $\alpha = 2^{\frac{1}{2\gamma}}\theta, \tau = 2\gamma, k = 1/2$, it becomes the generalized half-normal (GHN) distribution [8]. Further, if $\varphi = \lambda = 1$ in addition to $\alpha = 2^{\frac{1}{2}}\theta, \tau = 2$ and $k = 1/2$, it gives the well-known half-normal (HN) distribution.

4. Expansion for the density function

Let T follow the KumGG$(\alpha, \tau, k, \lambda, \varphi)$ distribution. The density function of T is straightforward to compute using any statistical software with numerical facilities. The density function of the GG(α, τ, k) distribution in given by

$$g_{\alpha,\tau,k}(t) = \frac{\tau}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right], \quad t > 0.$$

From Eq. (3) and using the expansion

$$(1-z)^{b-1} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(b)}{\Gamma(b-j)j!}z^j,$$

which holds for $|z| < 1$ and $b > 0$ real non-integer, the density function of T can be rewritten as

$$f(t) = \frac{\lambda\varphi\tau}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right]\left\{\gamma_1\left[k;\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1}$$
$$\times\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\varphi)}{\Gamma(\varphi-j)j!}\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda j}.$$

Using Eq. (19) (given in Appendix A), we obtain

$$f(t) = \frac{\lambda\varphi\tau}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right]\sum_{m,j=0}^{\infty}\frac{(-1)^j\Gamma(\varphi)s_m(\lambda)}{\Gamma(\varphi-j)j!}\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda j+m}, \qquad (7)$$

where the quantities $s_m(\lambda)$ are calculated from (20). Further, if $\lambda j + m$ is a real non-integer, we have

$$\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda j+m} = \sum_{l=0}^{\infty}(-1)^l\binom{\lambda j+m}{l}\left\{1-\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^l.$$

Using the binomial expansion in the above expression, (7) can be rewritten as

$$f(t) = \frac{\lambda\varphi\tau}{\alpha\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right]\sum_{j,l,m=0}^{\infty}\sum_{q=0}^{l}\frac{(-1)^{j+l+q}\Gamma(\varphi)s_m(\lambda)}{\Gamma(\varphi-j)j!}$$
$$\times\binom{\lambda j+m}{l}\binom{l}{q}\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{q}.$$

Now, by Eq. (23) (given in Appendix A), $f(t)$ admits the mixture representation

$$f(t) = \sum_{d,j,l,m=0}^{\infty}\sum_{q=0}^{l} w(d,j,l,m,q)g_{\alpha,\tau,k(1+q)+d}(t), \quad t > 0, \tag{8}$$

whose weighted coefficients are

$$w(d,j,l,m,q) = \frac{(-1)^{j+l+q}\lambda\varphi\Gamma(\varphi)\Gamma[k(1+q)+d]s_m(\lambda)c_{q,d}}{\Gamma(k)^{q+1}\Gamma(\varphi-j)j!}\binom{\lambda j+m}{l}\binom{l}{q}.$$

The coefficients satisfy $\sum_{m,i=0}^{\infty} w(d,j,l,m,q) = 1$ and the quantities $s_m(\lambda)$ and $c_{q,d}$ are determined from (20) and from the recurrence relation (22), respectively.

Eq. (8) shows that the KumGG density function is a mixture of GG density functions. Hence, some of their mathematical properties (such as the ordinary, inverse and factorial moments, mgf and characteristic function) can follow directly from those properties of the GG distribution.

5. Moments and generating function

Let T be a random variable having the KumGG$(\alpha, \tau, k, \lambda, \varphi)$ density function (3). In this section, we provide two different expansions for determining the *rth* ordinary moment of T, say $\mu_r' = E(T^r)$. First, we derive μ_r' as infinite sums from the mixture representation (8). The rth moment of the GG(α, β, k) distribution is $\mu_{r,\mathrm{GG}}' = \alpha^r\,\Gamma(k+r/\beta)/\Gamma(k)$ and then Eq. (8) yields

$$\mu_r' = \lambda\varphi\alpha^r\Gamma(\varphi)\sum_{d,j,l,m=0}^{\infty}\sum_{q=0}^{l}\frac{(-1)^{j+l+q}\Gamma[k(1+q)+d+r/\tau]s_m(\lambda)c_{q,d}}{\Gamma(k)^{q+1}\Gamma(\varphi-j)j!}\binom{\lambda j+m}{l}\binom{l}{q}. \tag{9}$$

Eq. (9) depends on the quantities $c_{q,d}$ that can be computed recursively from (22).

Now, we derive another infinite sum representation for μ_r' by computing the moment directly

$$\mu_r' = \frac{\lambda\varphi\tau\alpha^{r-1}}{\Gamma(k)}\int_0^{+\infty}\left(\frac{t}{\alpha}\right)^{\tau k+r-1}\exp\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right]\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1}$$
$$\times\left(1-\left\{\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi-1}\mathrm{d}t.$$

Setting $x = (t/\alpha)^{\tau}$ in the last equation yields

$$\mu_r' = \frac{\lambda\varphi\alpha^r}{\Gamma(k)}\int_0^{+\infty}x^{k+\frac{r}{\tau}-1}\exp(-x)\,\gamma_1(k,x)^{\lambda-1}\left[1-\gamma_1(k,x)^{\lambda}\right]^{\varphi-1}\mathrm{d}x. \tag{10}$$

For the $\varphi > 0$ real non-integer, we can write

$$\left[1-\gamma_1(k,x)^{\lambda}\right]^{\varphi-1} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\varphi)}{\Gamma(\varphi-j)j!}\gamma_1(k,x)^{\lambda j}$$

and then

$$\mu_r' = \frac{\lambda\varphi\alpha^r}{\Gamma(k)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\varphi)}{\Gamma(\varphi - j) j!} \int_0^{+\infty} x^{k+\frac{r}{\tau}-1} \gamma_1(k, x)^{\lambda(1+j)-1} \exp(-x) dx.$$

Applying Eq. (19) (given in Appendix A), $\gamma_1\,(k, x)^{\lambda(1+j)-1}$ can be expanded as

$$\gamma_1\,(k, x)^{\lambda(1+j)-1} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} (-1)^{l+m} \binom{\lambda(1+j)-1}{l} \binom{l}{m} \gamma_1\,(k, x)^m$$

and then μ_r' reduces to

$$\mu_r' = \frac{\lambda\varphi\alpha^r}{\Gamma(k)} \sum_{j,l=0}^{\infty} \sum_{m=0}^{l} v_{j,l,m} I\left(k + \frac{r}{\tau}, m\right), \tag{11}$$

where

$$v_{j,l,m} = \frac{(-1)^{j+l+m}\Gamma(\varphi)}{\Gamma(\varphi - j) j!} \binom{\lambda(1+j)-1}{l} \binom{l}{m}$$

and

$$I\left(k + \frac{r}{\tau}, m\right) = \int_0^{\infty} x^{k+\frac{r}{\tau}-1} \gamma_1(k, x)^m \exp(-x) dx.$$

For $\varphi = 1$, we obtain the same result by Cordeiro et al. [10]. The series expansion for the incomplete gamma function yields

$$I\left(k + \frac{r}{\tau}, m\right) = \int_0^{\infty} x^{k+\frac{r}{\tau}-1} \left[x^k \sum_{p=0}^{\infty} \frac{(-x)^p}{(k+p)p!} \right]^m \exp(-x) dx.$$

This integral can be determined from Eqs. (24) and (25) of Nadarajah [33] in terms of the Lauricella function of type A [15,2] defined by

$$F_A^{(n)}\,(a; b_1, \ldots, b_n; c_1, \ldots, c_n; x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b_1)_{m_1} \ldots (b_n)_{m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \frac{x_1^{m_1} \ldots x_n^{m_n}}{m_1! \ldots m_n!},$$

where $(a)_i == a(a+1)\ldots(a+i-1)$ is the ascending factorial (with the convention that $(a)_0 = 1$). Numerical routines for the direct computation of the Lauricella function of type A are available, see [15] and Mathematica [45]. We obtain

$$I\left(k + \frac{r}{\tau}, m\right) = k^{-m}\Gamma\big(r/\tau + k(m+1)\big) \times F_A^{(m)}\big(r/\tau + k(m+1); k, \ldots, k; k+1, \ldots, k+1; -1, \ldots, -1\big). \tag{12}$$

The moments of the KumGG distribution can be obtained from (9) or from the alternative equations (11) and (12). Graphical representations of the skewness and kurtosis when $\alpha = 0.5$, $\tau = 0.08$ and $k = 3$, as a function of λ for selected values of φ, and as a function of φ for some choices of λ, are given in Figs. 3 and 4, respectively.

Further, we derive the mgf of the GG(α, τ, k) distribution as

$$M_{\alpha,\tau,k}(s) = \frac{1}{\Gamma(k)} \sum_{m=0}^{\infty} \Gamma\left(\frac{m}{\tau} + k\right) \frac{(\alpha s)^m}{m!}.$$

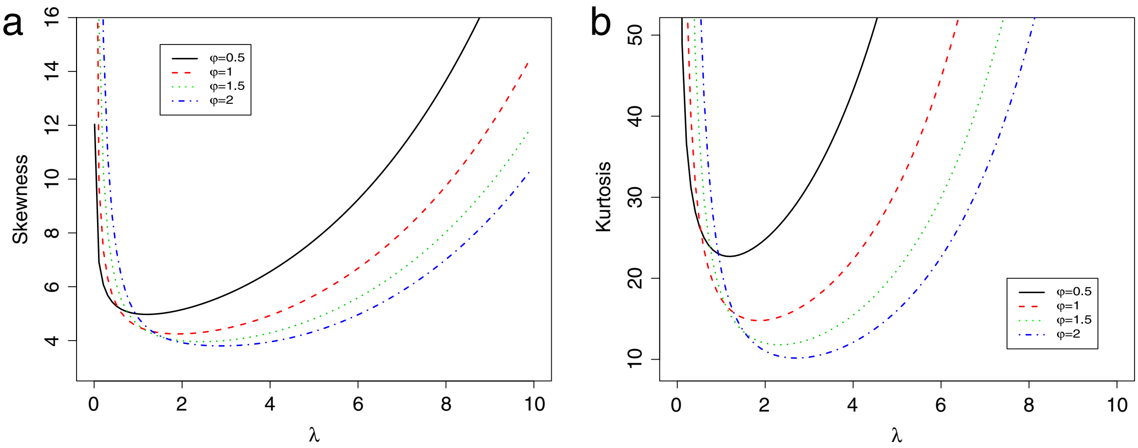

Fig. 3. Skewness and kurtosis of the KumGG distribution as a function of the parameter λ for selected values of φ.

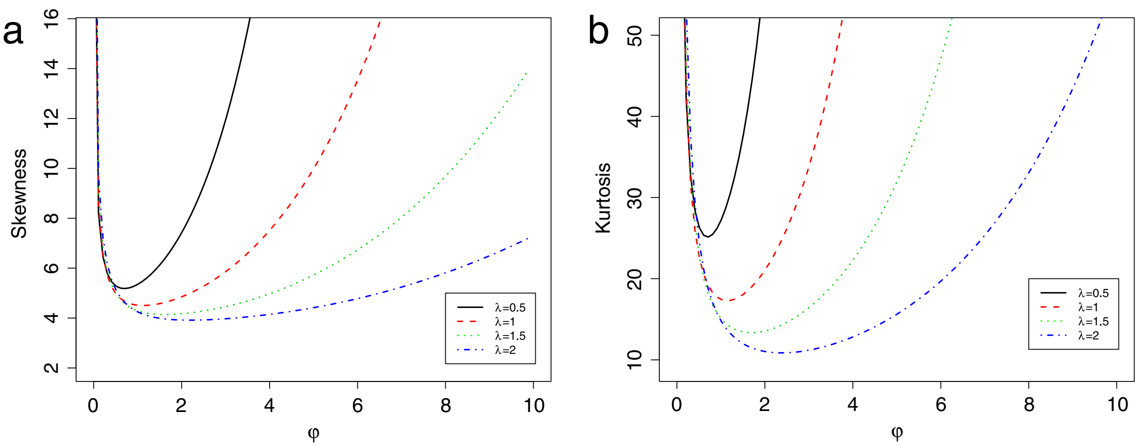

Fig. 4. Skewness and kurtosis of the KumGG distribution as a function of the parameter φ for selected values of λ.

Consider the Wright generalized hypergeometric function defined by

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p) \\ (\beta_1, B_1), \ldots, (\beta_q, B_q);\end{matrix} x\right] = \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j m)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j m)} \frac{x^m}{m!}. $$

By combining the last two equations, we can write

$$ M_{\alpha,\tau,k}(s) = \frac{1}{\Gamma(k)}\, {}_1\Psi_0\left[\begin{matrix}(k, \tau^{-1}) \\ -;\end{matrix} \alpha s\right], \tag{13} $$

provided that $\tau > 1$. Clearly, special formulas for the mgf of the distributions listed in Table 1 follow immediately from Eq. (13) by simple substitution of known parameters.

The KumGG generating function follows by combining Eqs. (8) and (13). For $\tau > 1$, we have

$$ M(s) = \sum_{d,j,l,m=0}^{\infty} \sum_{q=0}^{l} \frac{w(d, j, l, m, q)}{\Gamma(k(1+q)+d)}\, {}_1\Psi_0\left[\begin{matrix}(k(1+q)+d, \tau^{-1}) \\ -;\end{matrix} \alpha s\right]. \tag{14} $$

Eq. (14) is the main result of this section. The mgf of any KumGG sub-model, as those discussed in Section 3, can be determined immediately from (14) by substitution of known parameters.

6. Order statistics

The density function $f_{i:n}(t)$ of the ith order statistic, for $i = 1, \ldots, n$, from random variables $T_1, \ldots, T_n$ having KumGG density (3), is given by

$$f_{i:n}(t) = \frac{1}{B(i, n-i+1)} f(t) F(t)^{i-1} \{1 - F(t)\}^{n-i},$$

where $B(\cdot, \cdot)$ denotes the beta function. Using the binomial expansion in the above equation, we readily obtain

$$f_{i:n}(t) = \frac{1}{B(i, n-i+1)} f(t) \sum_{j_1=0}^{n-i} \binom{n-i}{j_1} (-1)^{j_1} F(t)^{i+j_1-1}.$$

We now present an expression for the density of the KumGG order statistics as a function of the baseline density multiplied by infinite weighted sums of powers of $G_{\alpha,\tau,k}(t)$. This result enables us to derive the ordinary moments of the KumGG order statistics as infinite weighted sums of the probability weighted moments (PWMs) of the GG distribution. Following Cordeiro and de Castro [9], we can write

$$F(t)^{i+j_1-1} = \sum_{r=0}^{\infty} p_{r,i+j_1-1} G_{\alpha,\tau,k}(t)^r,$$

where the coefficients $p_{r,u} = p_{r,u}(\lambda, \varphi)$ can be determined as

$$p_{r,u} = \sum_{l_1=0}^{u} (-1)^{l_1} \binom{u}{l_1} \sum_{m_1=0}^{\infty} \sum_{l_2=r}^{\infty} (-1)^{m_1+r+l_2} \binom{l_1\varphi}{m_1} \binom{m_1\lambda}{l_2} \binom{l_2}{r}$$

for $r, u = 0, 1, \ldots$ Hence, $f_{i:n}(t)$ can be further reduced to

$$f_{i:n}(t) = \frac{1}{B(i, n-i+1)} \sum_{d,j,l,m,r=0}^{\infty} \sum_{q=0}^{l} \sum_{j_1=0}^{n-i} (-1)^{j_1} \binom{n-i}{j_1} w(d, j, l, m, q) p_{r,i+j_1-1} \times \gamma_1 \left[k, \left(\frac{t}{\alpha}\right)^{\tau}\right]^r g_{\alpha,\tau,k(1+q)+d}(t). \tag{15}$$

The (s, r)th probability weighted moment (PWM) of a random variable Y having the GG(α, τ, k) distribution, say $\delta_{s,r}$, is formally defined by

$$\delta_{s,r} = E\{Y^s G_{\alpha,\tau,k}(Y)^r\} = \int_0^{\infty} y^s G_{\alpha,\tau,k}(y)^r g_{\alpha,\tau,k}(y) \mathrm{d}y.$$

Hence, Eq. (15) can be rewritten as

$$f_{i:n}(t) = \sum_{d,j,l,m,r=0}^{\infty} \sum_{q=0}^{l} \sum_{j_1=0}^{n-i} t(d, j, j_1, l, m, q) t^{\tau(kq+d)} \gamma_1 \left[k, \left(\frac{t}{\alpha}\right)^{\tau}\right]^r g_{\alpha,\tau,k}(t),$$

where

$$t(d, j, j_1, l, m, q) = \lambda\varphi\Gamma(\varphi) \frac{(-1)^{j+j_1+l+q} s_m(\lambda) c_{q,d}}{B(i, n-i+1)\alpha^{\tau(kq+d)}\Gamma(k)^q \Gamma(\varphi - j) j!} \binom{n-i}{j_1} \binom{\lambda j + m}{l} \binom{l}{q}.$$

It is important to point out that in the infinite summations, the indices can usually stop after a large number of summands. Finally, the moments of the KumGG order statistics can be expressed as

$$E\left(T_{i:n}^s\right) = \sum_{d,j,l,m,r=0}^{\infty} \sum_{q=0}^{l} \sum_{j_1=0}^{n-i} t(d, j, j_1, l, m, q) p_{r,i+j_1-1} \delta_{s+\tau(kq+d),r}.$$

7. Maximum likelihood estimation

Let T_i be a random variable following (3) with the vector of parameters $\boldsymbol{\theta} = (\alpha, \tau, k, \lambda, \varphi)^T$. The data encountered in survival analysis and reliability studies are often censored. A very simple random censoring mechanism that is often realistic is one in which each individual i is assumed to have a lifetime T_i and a censoring time C_i, where T_i and C_i are independent random variables. Suppose that the data consist of n independent observations $t_i = \min(T_i, C_i)$ for $i = 1, \ldots, n$. The distribution of C_i does not depend on any of the unknown parameters of T_i. Parametric inference for such data are usually based on likelihood methods and their asymptotic theory. The censored log-likelihood $l(\boldsymbol{\theta})$ for the model parameters is

$$\begin{aligned} l(\boldsymbol{\theta}) = {} & r \log\left[\frac{\lambda\varphi\tau}{\alpha\Gamma(k)}\right] - \sum_{i\in F}\left(\frac{t_i}{\alpha}\right)^{\tau} + (\tau k - 1)\sum_{i\in F}\log\left(\frac{t_i}{\alpha}\right) \\ & + (\lambda - 1)\sum_{i\in F}\log\left\{\gamma_1\left[k, \left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\} \\ & + (\varphi - 1)\sum_{i\in F}\log\left(1 - \left\{\gamma_1\left[k, \left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right) \\ & + \varphi\sum_{i\in C}\log\left(1 - \left\{\gamma_1\left[k, \left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right), \end{aligned} \tag{16}$$

where r is the number of failures and F and C denote the uncensored and censored sets of observations, respectively.

The score components corresponding to the parameters in $\boldsymbol{\theta}$ are:

$$U_\alpha(\boldsymbol{\theta}) = -\frac{r\tau k}{\alpha} + \frac{\tau}{\alpha}\sum_{i\in F} u_i - \frac{\tau}{\alpha}\sum_{i\in F} v_i s_i + \frac{\lambda\tau(\varphi - 1)}{\alpha\Gamma(k)}\sum_{i\in F} u_i p_i + \frac{\lambda\tau\varphi}{\alpha\Gamma(k)}\sum_{i\in C} u_i p_i,$$

$$\begin{aligned} U_\tau(\boldsymbol{\theta}) = {} & \frac{r}{\tau} - \frac{1}{\tau}\sum_{i\in F} u_i\log(u_i) + \frac{k}{\tau}\sum_{i\in F}\log(u_i) + \frac{1}{\tau}\sum_{i\in F} v_i s_i\log(u_i) \\ & - \frac{\lambda(\varphi - 1)}{\tau}\sum_{i\in F} v_i p_i\log(u_i) - \frac{\lambda\varphi}{\tau}\sum_{i\in C} v_i p_i\log(u_i), \end{aligned}$$

$$\begin{aligned} U_k(\boldsymbol{\theta}) = {} & -r\lambda\psi(k) + \sum_{i\in F}\log(u_i) + \sum_{i\in F} s_i q_i + (\varphi - 1)[r\lambda\psi(k)]\sum_{i\in F} p_i\gamma_1(k, u_i) \\ & - \lambda(\varphi - 1)\sum_{i\in F} p_i q_i + \lambda\varphi\psi(k)(n - r - 1)\sum_{i\in C} p_i\gamma_1(k, u_i) - \lambda\varphi\sum_{i\in F} p_i q_i, \end{aligned}$$

$$U_\lambda(\boldsymbol{\theta}) = \frac{r}{\lambda} + \sum_{i\in F}\log[\gamma_1(k, u_i)] - (\varphi - 1)\sum_{i\in F} b_i[\gamma_1(k, u_i)]^{\lambda} - \varphi\sum_{i\in C} b_i[\gamma_1(k, u_i)]^{\lambda}$$

and

$$U_\varphi(\boldsymbol{\theta}) = \frac{r}{\varphi} + \sum_{i=1}^{n}\log(\omega_i),$$

where

$$u_i = \left(\frac{t_i}{\alpha}\right)^{\tau}, \qquad g_i = u_i^k\exp(-u_i), \qquad \omega_i = 1 - \gamma_1(k, u_i)^{\lambda},$$

$$v_i = \frac{g_i}{\Gamma(k)}, \qquad s_i = \frac{(\lambda - 1)}{\gamma_1(k, u_i)},$$

$$[\dot{\gamma}(k,u_i)]_k = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} J(u_i, k+n-1, 1), \qquad p_i = \frac{\gamma_1(k,u_i)^{\lambda-1}}{\omega_i},$$

$$q_i = \frac{[\dot{\gamma}(k,u_i)]_k}{\Gamma(k)}, \qquad b_i = \frac{\log[\gamma_1(k,u_i)]}{\omega_i},$$

$\psi(.)$ is the digamma function and $J(u_i, k+n-1, 1)$ is defined in Appendix B.

The MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ is obtained numerically from the nonlinear equations $U_\alpha(\boldsymbol{\theta}) = U_\tau(\boldsymbol{\theta}) = U_k(\boldsymbol{\theta}) = U_\lambda(\boldsymbol{\theta}) = U_\varphi(\boldsymbol{\theta}) = 0$. For interval estimation and hypothesis tests on the model parameters, we require the 5×5 unit observed information matrix $J = J(\boldsymbol{\theta})$ whose elements are given in Appendix B. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $N_5(0, I(\boldsymbol{\theta})^{-1})$, where $I(\boldsymbol{\theta})$ is the expected information matrix. This matrix can be replaced by $J(\widehat{\boldsymbol{\theta}})$, i.e., the observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$. The multivariate normal $N_5(0, J(\widehat{\boldsymbol{\theta}})^{-1})$ distribution can be used to construct approximate confidence intervals for the individual parameters. We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct LR statistics for testing some sub-models (see Section 3) of the KumGG distribution. For example, we may use LR statistics to check if the fit using the KumGG distribution is statistically "superior" to the fits using the KumGHN, KumSChiSq, GG and KumW distributions for a given data set.

8. A Bayesian analysis

As an alternative analysis, we use the Bayesian method which allows for the incorporation of previous knowledge of the parameters through informative priori density functions. When this information is not available, we can consider a noninformative prior. In the Bayesian approach, the information referring to the model parameters is obtained through a posterior marginal distribution. In this way, two difficulties usually arise. The first refers to attaining marginal posterior distribution, and the second to the calculation of the moments of interest. Both cases require numerical integration that, many times, do not present an analytical solution. Here, we use the simulation method of Markov Chain Monte Carlo (MCMC), such as the Gibbs sampler and Metropolis–Hastings algorithm.

Since we have no prior information from historical data or from previous experiment, we assign conjugate but weakly informative prior distributions to the parameters. Since we assumed informative (but weakly) prior distribution, the posterior distribution is a well-defined proper distribution. Here, we assume the elements of the parameter vector to be independent and consider that the joint prior distribution of all unknown parameters has a density function given by

$$\pi(\alpha, \tau, k, \lambda, \varphi) \propto \pi(\alpha) \times \pi(\tau) \times \pi(k) \times \pi(\lambda) \times \pi(\varphi). \tag{17}$$

Here, $\alpha \sim \Gamma(a_1, b_1)$, $\tau \sim \Gamma(a_2, b_2)$, $k \sim \Gamma(a_3, b_3)$, $\lambda \sim \Gamma(a_4, b_4)$ and $\varphi \sim \Gamma(a_5, b_5)$, where $\Gamma(a_i, b_i)$ denotes a gamma distribution with mean a_i/b_i, variance a_i/b_i^2 and density function given by

$$f(v; a_i, b_i) = \frac{b_i^{a_i} v^{a_i - 1} \exp(-v b_i)}{\Gamma(a_i)},$$

where $v > 0$, $a_i > 0$ and $b_i > 0$. All hyper-parameters are specified. Combining the likelihood function (16) and the prior distribution (17), the joint posterior distribution for α, τ, k, λ and φ reduces to

$$\begin{aligned}\pi(\alpha, \tau, k, \lambda, \varphi | t) \propto{}& \left(\frac{\lambda\varphi\tau}{\alpha^{\tau k}\Gamma(k)}\right)^r \exp\left[-\sum_{i\in F}\left(\frac{t_i}{\alpha}\right)^\tau\right] \prod_{i\in F} t_i^{\tau k-1} \left\{\gamma_1\left[k, \left(\frac{t_i}{\alpha}\right)^\tau\right]\right\}^{\lambda-1} \\ &\times \prod_{i\in F}\left(1 - \left\{\gamma_1\left[k, \left(\frac{t_i}{\alpha}\right)^\tau\right]\right\}^\lambda\right)^{\varphi-1} \prod_{i\in C}\left(1 - \left\{\gamma_1\left[k, \left(\frac{t_i}{\alpha}\right)^\tau\right]\right\}^\lambda\right)^\varphi \\ &\times \pi(\alpha, \tau, k, \lambda, \varphi).\end{aligned} \tag{18}$$

The joint posterior density (18) is analytically intractable because the integration of the joint posterior density is not easy to perform. So, the inference can be based on MCMC simulation methods such as the Gibbs sampler and Metropolis–Hastings algorithm, which can be used to draw samples, from which features of the marginal distributions of interest can be inferred. In this direction, we first obtain the full conditional distributions of each unknown quantity, which are given by

$$\begin{aligned}
\pi(\alpha|t,\tau,k,\lambda,\varphi) \propto\ & (\alpha^{\tau k})^{-r}\exp\left[-\sum_{i\in F}\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\prod_{i\in F}\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1} \\
& \times\prod_{i\in F}\left(1-\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi-1}\prod_{i\in C}\left(1-\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi} \\
& \times\pi(\alpha), \\
\pi(\tau|t,\alpha,k,\lambda,\varphi) \propto\ & \left(\frac{\tau}{\alpha^{\tau k}}\right)^{r}\exp\left[-\sum_{i\in F}\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\prod_{i\in F}t_i^{\tau k}\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1} \\
& \times\prod_{i\in F}\left(1-\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi-1}\prod_{i\in C}\left(1-\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi} \\
& \times\pi(\tau), \\
\pi(k|t,\alpha,\tau,\lambda,\varphi) \propto\ & [\alpha^{\tau k}\Gamma(k)]^{-r}\prod_{i\in F}t_i^{\tau k}\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda-1} \\
& \times\prod_{i\in F}\left(1-\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi-1}\prod_{i\in C}\left(1-\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi} \\
& \times\pi(k), \\
\pi(\lambda|t,\alpha,\tau,k,\varphi) \propto\ & (\lambda)^{r}\prod_{i\in F}\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\left(1-\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi-1} \\
& \times\prod_{i\in C}\left(1-\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi}\times\pi(\lambda)
\end{aligned}$$

and

$$\begin{aligned}
\pi(\varphi|t,\alpha,\tau,k,\lambda) \propto\ & (\varphi)^{r}\prod_{i\in F}\left(1-\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi}\prod_{i\in C}\left(1-\left\{\gamma_1\left[k,\left(\frac{t_i}{\alpha}\right)^{\tau}\right]\right\}^{\lambda}\right)^{\varphi} \\
& \times\pi(\varphi).
\end{aligned}$$

Since the full conditional distributions for α, τ, k, λ and φ do not have a closed form, we require the use of the Metropolis–Hastings algorithm. The MCMC computations were implemented in the statistical software package R.

9. Applications

In this section, the usefulness of the KumGG distribution is illustrated in two real data sets.

9.1. Aarset data-uncensored

We show the superiority of the KumGG distribution as compared to some of its sub-models and also to the following non-nested models: the exponentiated generalized gamma (EGG) and beta Weibull

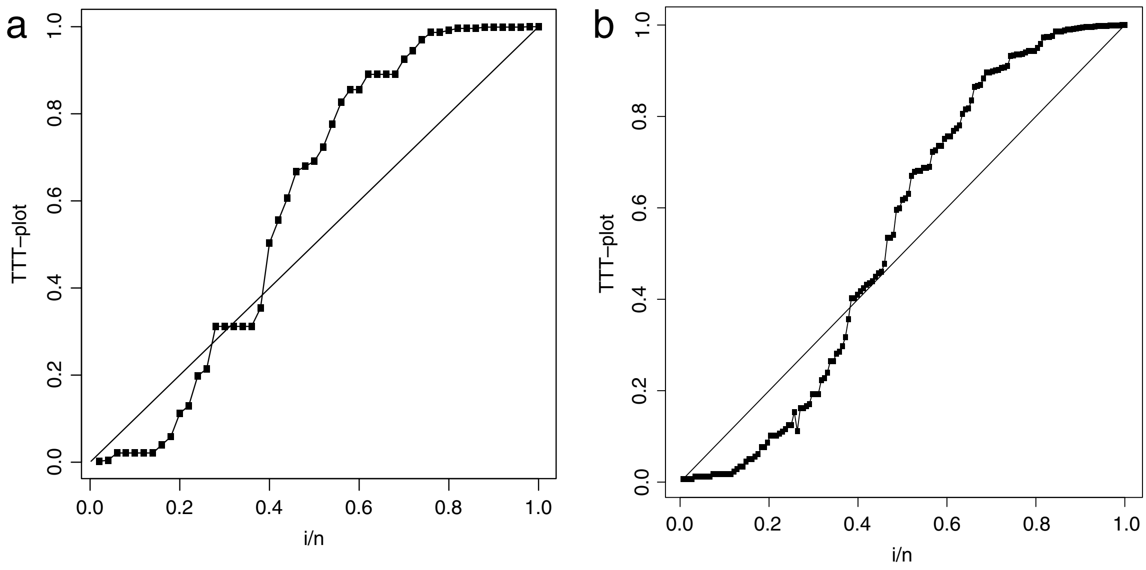

Fig. 5. (a) TTT plot for the Aarset data. (b) TTT plot for the Serum reversal data.

Table 2
MLEs of the model parameters for Aarset data, the corresponding SEs (given in parentheses) and the statistics AIC, BIC and CAIC.

Model	α	τ	k	λ	φ	AIC	BIC	CAIC
KumGG	84.5056	79.5358	0.0080	0.5393	0.3431	423.1	432.7	424.5
	(0.2099)	(2.0929)	(0.0021)	(0.2387)	(0.0565)			
EGG	86.0359	28.0261	1.0398	0.0241	1	456.5	464.1	457.4
	(0.3373)	(0.0177)	(0.00007)	(0.0034)	(–)			
GG	86.9281	259.00	0.0028	1	1	446.7	452.4	447.2
	(1.2391)	(17.0524)	(0.0004)	(–)	(–)			
Weibull	44.9126	0.9490	1	1	1	486.0	489.8	486.3
	(6.6451)	(0.1196)	(–)	(–)	(–)			
	α	γ	a	b				
Beta Weibull	49.6326	5.9441	0.0783	0.0702	(–)	444.5	452.1	445.4
	(3.7606)	(0.1394)	(0.0166)	(0.0288)	(–)			

(BW) distributions. The BW density function [16] is given by

$$F(t) = \frac{1}{B(a,b)} \int_0^{\{1-\exp[-(t/\alpha)^{\gamma}]\}} w^{a-1}(1-w)^{b-1}\mathrm{d}w.$$

We consider the data set presented by Aarset [1]which describes the lifetimes of 50 industrial devices put on life test at time zero. These data have been used by Mudholkar and Srivastava [29] for illustrating the appropriateness of the exponentiated Weibull model to fit lifetime data. Fig. 5(a) shows that the TTT-plot for these data has first a convex shape and then a concave shape. It then indicates a bathtub-shaped hazard rate function. Hence, the KumGG distribution could be an appropriate model for fitting these data.

Table 2 lists the MLEs (and the corresponding standard errors in parentheses) of the model parameters and the values of the following statistics for some models: AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and CAIC (Consistent Akaike Information Criterion). The AIC and BIC values for the KumGG model are the smallest values among those of the five fitted models, and hence our new model can be chosen as the best model.

A comparison of the proposed distribution with some of its sub-models using LR statistics is performed in Table 3. The numbers in this table, specially the p-values, suggest that the KumGG model yields a better fit to these data than the other three distributions.

In order to assess if the model is appropriate, Fig. 6(a) plots the empirical survival function and the estimated survival function of the KumGG distribution. The proposed distribution is a very

Table 3
LR statistics for the Aarset data.

Model	Hypotheses	Statistics w	*P*-value
KumGG vs EGG	$H_0 : \varphi = 1$ vs $H_1 : H_0$ is false	35.4	<0.0001
KumGG vs GG	$H_0 : \varphi = \lambda = 1$ vs $H_1 : H_0$ is false	27.6	<0.0001
KumGG vs Weibull	$H_0 : \varphi = \lambda = k = 1$ vs $H_1 : H_0$ is false	68.9	<0.0001

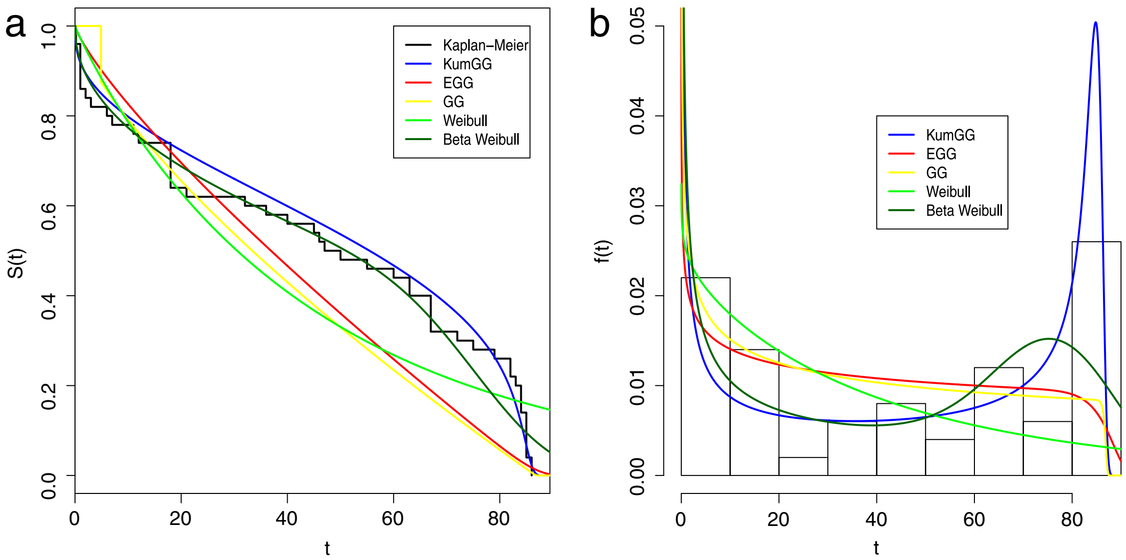

Fig. 6. (a) Estimated survival function by fitting the KumGG distribution and some other models and the empirical survival for the Aarset data. (b) Estimated densities of the KumGG, EGG, GG, Weibull and BW models for the Aarset data.

competitive model for describing the bathtub-shaped failure rate of the Aarset data. The plots of the estimated densities and the histogram of these data are given in Fig. 6(b). They show that the KumGG distribution produces a better fit than the other four models.

Bayesian analysis.

The following independent priors were considered to perform the Gibbs sampler:

$$\alpha \sim \Gamma(0.01, 0.01), \qquad \tau \sim \Gamma(0.01, 0.01),$$
$$k \sim \Gamma(0.01, 0.01), \qquad \lambda \sim \Gamma(0.01, 0.01) \quad \text{and} \quad \varphi \sim \Gamma(0.01, 0.01),$$

so that we have a vague prior distribution. Considering these prior density functions, we generated two parallel independent runs of the Gibbs sampler with size 50,000 for each parameter, disregarding the first 10,000 iterations to eliminate the effect of the initial values and, to avoid correlation problems, we considered a spacing of size 20, obtaining a sample of size 2000 from each chain. To monitor the convergence of the Gibbs sampler, we performed the methods suggested by Cowles and Carlin [11]. To monitor the convergence of the Gibbs samples, we used the between and within sequence information, following the approach developed in Gelman and Rubin [17] to obtain the potential scale reduction, $\hat{R}$. In all cases, these values were close to one, indicating the convergence of the chain. The approximate posterior marginal density functions for the parameters are presented in Fig. 7. In Table 4, we report posterior summaries for the parameters of the KumGG model. We note that the values for the means a posteriori (Table 4) are quite close (as expected) to the MLEs obtained for the KumGG model given in Table 2. SD represents the standard deviation from the posterior distributions of the parameters and HPD represents the 95% highest posterior density (HPD) intervals.

9.2. *Serum reversal data-censored*

Aids is a pathology that mobilizes its sufferers because of the implications for their interpersonal relationships and reproduction. Therapeutic advances have enabled seropositive women to bear

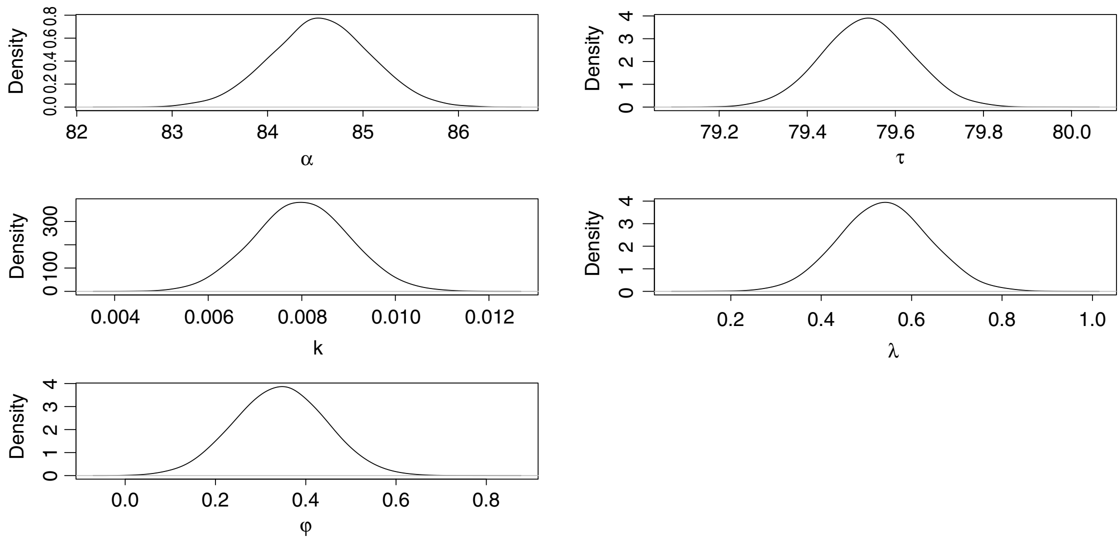

Fig. 7. Approximate posterior marginal densities for the parameters from the KumGG model for the Aarset data.

Table 4
Posterior summaries for the parameters from the KumGG model for the Aarset data.

Parameter	Mean	SD	HPD (95%)	$\hat{R}$
α	84.5551	0.5130	(83.5624; 85.5690)	1.0015
τ	79.5361	0.1007	(479.3344; 79.7280)	1.0002
k	0.0080	0.0010	(0.0061; 0.0099)	1.0003
λ	0.5397	0.0991	(0.3539; 0.7402)	0.9999
φ	0.3434	0.0999	(0.1550; 0.5452)	0.9999

children safely. In this respect, the pediatric immunology outpatient service and social service of Hospital das Clínicas have a special program for care of newborns of seropositive mothers, to provide orientation and support for antiretroviral therapy to allow these women and their babies to live as normally as possible. Here, we analyze a data set on the time to serum reversal of 148 children exposed to HIV by vertical transmission, born at Hospital das Clínicas (associated with the Ribeirão Preto School of Medicine) from 1995 to 2001, where the mothers were not treated [42,38]. Vertical HIV transmission can occur during gestation in around 35% of cases, during labor and birth itself in some 65% of cases, or during breast feeding, varying from 7% to 22% of cases. Serum reversal or serological reversal can occur in children of HIV-contaminated mothers. It is the process by which HIV antibodies disappear from the blood in an individual who tested positive for HIV infection. As the months pass, the maternal antibodies are eliminated and the child ceases to be HIV positive. The exposed newborns were monitored until definition of their serological condition, after administration of Zidovudin (AZT) in the first 24 h and for the following 6 weeks. We assume that the lifetimes are independently distributed, and also independent from the censoring mechanism. We assume right-censored lifetime data (censoring random). Fig. 5(b) shows that the TTT-plot for these data has first a convex shape and then a concave shape. It indicates a bathtub-shaped hazard rate function. Hence, the KumGG distribution could be an appropriate model for fitting the data.

Table 5 lists the MLEs (and the corresponding standard errors in parentheses) of the parameters and the values of the AIC, BIC and CAIC statistics. These results indicate that the KumGG model has the lowest AIC, BIC and CAIC values among those of all fitted models, and hence it could be chosen as the best model.

A comparison of the proposed distribution with some of its sub-models using LR statistics is performed in Table 6. The numbers in this table, specially the p-values, suggest that the KumGG model yields a better fit to these data than the other three distributions. In order to assess if the model is appropriate, plots of the estimated survival functions of the KumGG, EGG, GG, Weibull and BW

Table 5
MLEs of the model parameters for the serum reversal data, the corresponding SEs (given in parentheses) and the statistics AIC, BIC and CAIC.

Model	α	τ	k	λ	φ	AIC	BIC	CAIC
KumGG	350.05	49.8303	0.2176	0.1282	0.3424	770.7	785.7	771.1
	(1.5707)	(5.8895)	(0.0073)	(0.0236)	(0.0522)			
EGG	350.45	22.2991	1.0741	0.1072	1	798.1	810.1	798.3
	(2.4187)	(0.0375)	(0.0004)	(0.0113)	(–)			
GG	379.40	24.5312	0.0974	1	1	783.7	792.7	783.9
	(8.8211)	(10.3258)	(0.0402)	(–)	(–)			
Weibull	307.62	3.1132	1	1	1	808.0	814.0	808.1
	(12.3523)	(0.3250)	(–)	(–)	(–)			
	α	γ	a	b				
Beta Weibull	349.99	6.3895	0.3944	0.9273	(–)	797.9	809.9	798.2
	(23.0923)	(0.7657)	(0.0468)	(0.3361)	(–)			

Table 6
LR statistics for the serum reversal data.

Model	Hypotheses	Statistics w	P-value
KumGG vs EGG	$H_0 : \varphi = 1$ vs $H_1 : H_0$ is false	29.4	<0.0001
KumGG vs GG	$H_0 : \varphi = \lambda = 1$ vs $H_1 : H_0$ is false	17.0	0.0002
KumGG vs Weibull	$H_0 : \varphi = \lambda = k = 1$ vs $H_1 : H_0$ is false	43.3	<0.0001

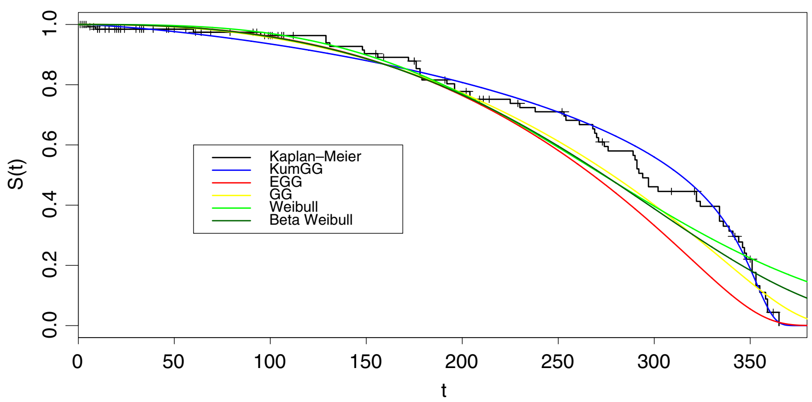

Fig. 8. Estimated survival function by fitting the KumGG distribution and some other models and the empirical survival for the serum reversal data.

distributions and the empirical survival function are given in Fig. 8. We conclude that the KumGG distribution provides a good fit for these data.

Bayesian analysis.

Now, for the serum reversal data, the following independent priors were considered to perform the Gibbs Sampler:

$$\alpha \sim \Gamma(0.01, 0.01), \qquad \tau \sim \Gamma(0.01, 0.01),$$
$$k \sim \Gamma(0.01, 0.01), \qquad \lambda \sim \Gamma(0.01, 0.01) \quad \text{and} \quad \varphi \sim \Gamma(0.01, 0.01),$$

so that we have a vague prior distribution. The histograms with the approximate posterior marginal density functions of the parameters are shown in Fig. 9. In Table 7, we report posterior summaries for the parameters of the KumGG model. We observe that the values for the means a posteriori (Table 7) are quite close (as expected) to the MLEs for the KumGG model listed in Table 5.

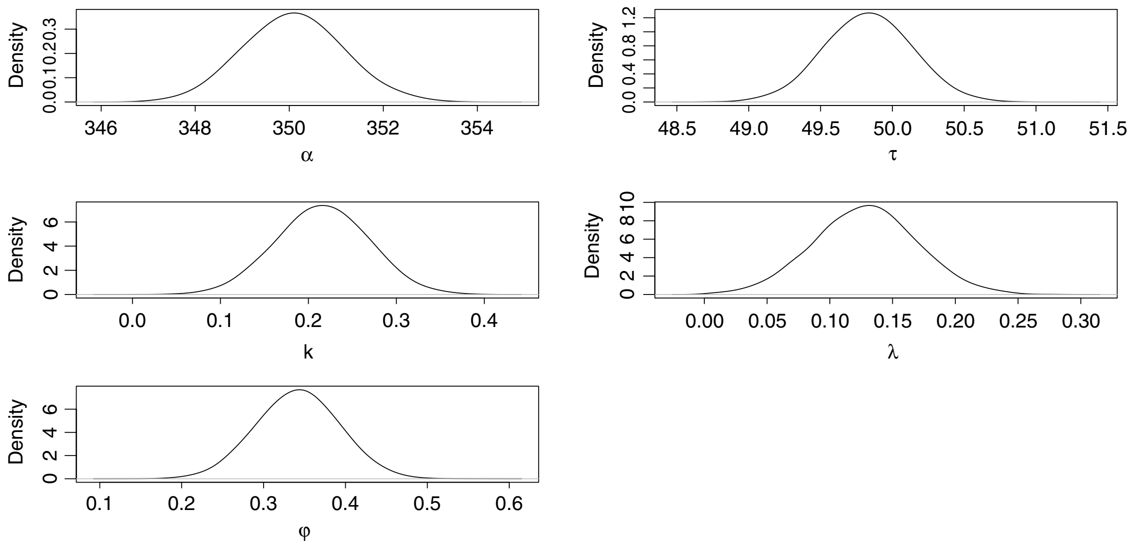

Fig. 9. Approximate posterior marginal densities for the parameters from the KumGG model for the serum reversal data.

Table 7
Posterior summaries for the parameters from the KumGG model for the serum reversal data.

Parameter	Mean	SD	HPD (95%)	$\hat{R}$
α	350.0872	1.0046	(348.0408; 351.9952)	1.0031
τ	49.8320	0.3021	(49.2220; 50.4063)	1.0027
k	0.2159	0.0519	(0.1166; 0.3173)	1.0018
λ	0.1283	0.0411	(0.0434; 0.2056)	1.0075
φ	0.3418	0.0501	(0.2422; 0.4373)	0.9998

10. Concluding comments

A four-parameter lifetime distribution, so-called "the Kumaraswamy generalized gamma (KumGG) distribution", is proposed as a simple extension of the generalized gamma (GG) distribution [43]. The new model extends several distributions widely used in the lifetime literature and it is more flexible than the GG, exponentiated GG, generalized half-normal, exponentiated Weibull, among several others distributions. The proposed distribution could have increasing, decreasing, bathtub and unimodal hazard rate functions. It is then very versatile to model lifetime data with a bathtub-shaped hazard rate function and also to model a variety of uncertainty situations. We provide a mathematical treatment of this distribution including the order statistics. Explicit expressions for the moments and moment generating function are provided which hold in generality for any parameter values. We obtain infinite weighted sums for the moments of the order statistics. The application of the proposed distribution is straightforward. The estimation of the parameters is approached by two different methods: maximum likelihood and a Bayesian approach. The KumGG distribution allows goodness-of-fit tests for some well-known distributions in reliability analysis by taking these distributions as sub-models. The practical relevance and applicability of the new model are demonstrated in two real data sets. The applications demonstrate the usefulness of the KumGG distribution, and with the use of modern computer resources with analytic and numerical capabilities, it can be an adequate tool comprising the arsenal of distributions for lifetime data analysis.

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Appendix A

Following Cordeiro and de Castro [9], we can write

$$\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]^{\lambda-1}=\sum_{m=0}^{\infty}s_m(\lambda)\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]^{m},\tag{19}$$

where

$$s_m(\lambda)=\sum_{j=m}^{\infty}(-1)^{j+m}\binom{\lambda-1}{j}\binom{j}{m}.\tag{20}$$

Now, we use the series expansion for the incomplete gamma ratio function given by

$$\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]=\frac{1}{\Gamma(k)}\left(\frac{t}{\alpha}\right)^{\tau k}\sum_{d=0}^{\infty}\left[-\left(\frac{t}{\alpha}\right)^{\tau}\right]^{d}\frac{1}{(k+d)d!}.$$

By application of an equation by Gradshteyn and Ryzhik [19, Section 0.314] for a power series raised to a positive integer q, we obtain

$$\left[\sum_{d=0}^{\infty}a_d\left(\frac{t}{\alpha}\right)^{\tau d}\right]^{q}=\sum_{d=0}^{\infty}c_{q,d}\left(\frac{t}{\alpha}\right)^{\tau d},\tag{21}$$

where the coefficients $c_{q,d}$ (for $d=1,2,\ldots$) are determined from the recurrence relation

$$c_{q,d}=(da_0)^{-1}\sum_{p=1}^{d}(qp-d+p)a_p c_{q,d-p},\tag{22}$$

$c_{q,0}=a_0^q$ and $a_p=(-1)^p/[(k+p)p!]$. Clearly, $c_{q,d}$ can be computed from $c_{q,0},\ldots,c_{q,d-1}$. It can be written explicitly as a function of the quantities $a_0,\ldots,a_d$, although it is not necessary for numerically programming our expansions. Further, using Eq. (21), we obtain

$$\gamma_1\left[k,\left(\frac{t}{\alpha}\right)^{\tau}\right]^{q}=\frac{1}{\Gamma(k)^q}\left(\frac{t}{\alpha}\right)^{\tau kq}\sum_{d=0}^{\infty}c_{q,d}\left(\frac{t}{\alpha}\right)^{\tau d},\tag{23}$$

whose quantities $c_{q,d}$ are obtained from (22).

Appendix B

By differentiating (16), the elements of the observed information matrix $J(\boldsymbol{\theta})$ for the parameters $(\alpha,\tau,k,\lambda,\varphi)$ are:

$$\begin{aligned}J_{\alpha\alpha}=&\frac{r\tau k}{\alpha^2}-\frac{\tau(1-\tau)}{\alpha^2}\sum_{i\in F}u_i-\frac{\tau}{\alpha^2}\sum_{i\in F}v_i s_i\left(-1+\frac{\tau}{\gamma_1(k,u_i)}\{\gamma_1(k,u_i)[-k+u_i]+v_i\}\right)\\&+\frac{\lambda\tau(\varphi-1)}{\alpha^2}\sum_{i\in F}v_i p_i\left(-1+\frac{\tau}{\omega_i}\{\omega_i(-k+u_i-v_i s_i)-\lambda v_i[\gamma_1(k,u_i)]^2\}\right)\\&+\frac{\lambda\tau\varphi}{\alpha^2}\sum_{i\in F}v_i p_i\left(-1+\frac{\tau}{\omega_i}\{\omega_i(-k+u_i-v_i s_i)-\lambda v_i[\gamma_1(k,u_i)]^2\}\right),\end{aligned}$$

$$
\begin{aligned}
J_{\alpha\tau} &= -\frac{rk}{\alpha} + \frac{1}{\alpha}\sum_{i\in F} u_i\,[1+\log(u_i)] - \frac{1}{\alpha}\sum_{i\in F} v_i s_i \\
&\quad \times \left(1 + \frac{\log(u_i)}{\gamma_1(k,u_i)}\,\{\gamma_1(k,u_i)[k-u_i] - v_i\}\right) \\
&\quad + \frac{\lambda(\varphi-1)}{\alpha}\sum_{i\in F} v_i p_i \left(1 + \frac{\log(u_i)}{\omega_i}\left\{\omega_i(k-u_i+v_i p_i) + \lambda v_i[\gamma_1(k,u_i)]^2\right\}\right) \\
&\quad + \frac{\lambda}{\alpha}\sum_{i\in C} v_i p_i \left(1 + \frac{\log(u_i)}{\omega_i}\left\{\omega_i(k-u_i+v_i p_i) + \lambda v_i[\gamma_1(k,u_i)]^2\right\}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_{\alpha k} &= -\frac{r\tau}{\alpha} - \frac{\tau}{\alpha}\sum_{i\in F} v_i s_i \left[\log(u_i) - \frac{q_i}{\gamma_1(k,u_i)}\right] + \frac{\lambda\tau(\varphi-1)}{\alpha}\sum_{i\in F} v_i p_i \\
&\quad \times \left(-\psi(k) + \frac{1}{\omega_i}\,(\omega_i\{\log(u_i) + s_i[-\psi(k)\gamma_1(k,u_i) + q_i]\}) \right.\\
&\quad \left. + \{\lambda\omega_i p_i[-\psi(k)\gamma_1(k,u_i) + q_i]\}\right) + \frac{\lambda\tau\varphi}{\alpha}\sum_{i\in C} v_i p_i \left(-\psi(k) + \frac{1}{\omega_i}(\omega_i\{\log(u_i) \right.\\
&\quad \left. + s_i[-\psi(k)\gamma_1(k,u_i) + q_i]\}) + \{\lambda\omega_i p_i[-\psi(k)\gamma_1(k,u_i) + q_i]\}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_{\alpha\lambda} &= -\frac{\tau}{\alpha}\sum_{i\in F}\frac{v_i}{\gamma_1(k,u_i)} + \frac{\tau(\varphi-1)}{\alpha}\sum_{i\in F} v_i p_i \left(1 + \lambda b_i\left\{\omega_i + [\gamma_1(k,u_i)]^{\lambda}\right\}\right) \\
&\quad + \frac{\tau\varphi}{\alpha}\sum_{i\in C} v_i p_i \left(1 + \lambda b_i\left\{\omega_i + [\gamma_1(k,u_i)]^{\lambda}\right\}\right),
\end{aligned}
$$

$$
J_{\alpha\varphi} = \frac{\lambda\tau}{\alpha}\sum_{i=1}^{n} v_i p_i, \qquad J_{\tau\varphi}(\boldsymbol{\theta}) = -\frac{\lambda}{\tau}\sum_{i=1}^{n} v_i p_i \log(u_i),
$$

$$
\begin{aligned}
J_{\tau\tau} &= -\frac{r}{\tau^2} - \frac{1}{\tau^2}\sum_{i\in F} u_i[\log(u_i)]^2 + \frac{1}{\tau^2}\sum_{i\in F}\frac{v_i s_i[\log(u_i)]^2}{\gamma_1(k,u_i)}[\gamma_1(k,u_i)(k-u_i) - v_i] \\
&\quad - \frac{\lambda(\varphi-1)}{\tau^2}\sum_{i\in F}\frac{v_i p_i[\log(u_i)]^2}{\omega_i}\{\omega_i[(k-u_i) + v_i s_i] + \lambda v_i[\gamma_1(k,u_i)]^2\} \\
&\quad - \frac{\lambda\varphi}{\tau^2}\sum_{i\in C}\frac{v_i p_i[\log(u_i)]^2}{\omega_i}\{\omega_i[(k-u_i) + v_i s_i] + \lambda v_i[\gamma_1(k,u_i)]^2\},
\end{aligned}
$$

$$
\begin{aligned}
J_{\tau k} &= \frac{1}{\tau}\sum_{i\in F}\log(u_i) + \frac{1}{\tau}\sum_{i\in F} v_i s_i \log(u_i)\left[\log(u_i) - \frac{q_i}{\gamma_1(k,u_i)}\right] - \frac{\lambda(\varphi-1)}{\tau} \\
&\quad \times \sum_{i\in F} v_i p_i \log(u_i)\left\{-\psi(k) + \log(u_i) + \left[-\psi(k) + \frac{q_i}{\gamma_1(k,u_i)}\right]\right.\\
&\quad \left.\times[\lambda - 1 + \lambda p_i\gamma_1(k,u_i)]\right\} - \frac{\lambda\varphi}{\tau}\sum_{i\in C} v_i p_i \log(u_i) \\
&\quad \times\left\{-\psi(k) + \log(u_i) + \left[-\psi(k) + \frac{q_i}{\gamma_1(k,u_i)}\right][\lambda - 1 + \lambda p_i\gamma_1(k,u_i)]\right\},
\end{aligned}
$$

$$
\begin{aligned}
J_{\tau\lambda} &= \frac{1}{\tau}\sum_{i\in F}\frac{v_i\log(u_i)}{\gamma_1(k,u_i)} - \frac{(\varphi-1)}{\tau}\sum_{i\in F} v_i p_i \log(u_i)\{1 + \lambda b_i\omega_i[1 + p_i\gamma_1(k,u_i)]\} \\
&\quad - \frac{\varphi}{\tau}\sum_{i\in C} v_i p_i \log(u_i)\{1 + \lambda b_i\omega_i[1 + p_i\gamma_1(k,u_i)]\},
\end{aligned}
$$

$$
\begin{aligned}
J_{kk} &= -r\lambda\psi'(k) + \frac{1}{\Gamma(k)}\sum_{i\in F} s_i\left\{[\ddot{\gamma}(k,u_i)]_{kk} - \frac{\left([\dot{\gamma}(k,u_i)]_k\right)^2}{\gamma_1(k,u_i)}\right\} + r\lambda(\varphi-1)\\
&\quad\times\sum_{i\in F} p_i\gamma_1(k,u_i)\left\{\psi'(k) + \frac{\lambda\psi(k)}{\gamma_1(k,u_i)}\left[-\frac{\psi(k)\gamma_1(k,u_i)}{\omega_i} + \frac{q_i}{\omega_i}\right]\right\} - \frac{\lambda(\varphi-1)}{\Gamma(k)}\\
&\quad\times\sum_{i\in F} p_i\left(-\psi(k)\,[\dot{\gamma}(k,u_i)]_k + \left\{q_i(\lambda-1)\left[-\psi(k)\gamma_1(k,u_i) + \frac{q_i}{\gamma_1(k,u_i)}\right]\right.\right.\\
&\quad\left.\left.+\frac{[\ddot{\gamma}(k,u_i)]_{kk}}{\omega_i}\right\}\right) + \lambda\varphi(n-r-1)\sum_{i\in C} p_i\gamma_1(k,u_i)\left\{\psi'(k) + \frac{\lambda\psi(k)}{\gamma_1(k,u_i)}\right.\\
&\quad\left.\times\left[-\frac{\psi(k)\gamma_1(k,u_i)}{\omega_i} + \frac{q_i}{\omega_i}\right]\right\} - \frac{\lambda\varphi}{\Gamma(k)}\sum_{i\in C} p_i\left(-\psi(k)\,[\dot{\gamma}(k,u_i)]_k\right.\\
&\quad\left.+\left\{q_i(\lambda-1)\left[-\psi(k)\gamma_1(k,u_i) + \frac{q_i}{\gamma_1(k,u_i)}\right] + \frac{[\ddot{\gamma}(k,u_i)]_{kk}}{\omega_i}\right\}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_{k\lambda} &= -r\psi(k) + \sum_{i\in F}\frac{q_i}{\gamma_1(k,u_i)} + r\psi(k)(\varphi-1)\sum_{i\in F} p_i\gamma_1(k,u_i)(1+\lambda b_i) - (\varphi-1)\\
&\quad\times\sum_{i\in F} p_i q_i(1+\lambda b_i) + \varphi\psi(k)(n-r-1)\sum_{i\in C} p_i\gamma_1(k,u_i)(1+\lambda b_i)\\
&\quad-\varphi\sum_{i\in C} p_i q_i(1+\lambda b_i),
\end{aligned}
$$

$$J_{k\varphi} = r\lambda\psi(k)\sum_{i\in F} p_i\gamma_1(k,u_i) + \lambda\psi(k)(n-r-1)\sum_{i\in C} p_i\gamma_1(k,u_i) - \lambda\sum_{i=1}^{n} p_i q_i,$$

$$J_{\lambda\lambda} = -\frac{r}{\lambda^2} - (\varphi-1)\sum_{i\in F} b_i[\gamma_1(k,u_i)]^{\lambda}\log[\gamma_1(k,u_i)] - \varphi\sum_{i\in C} b_i[\gamma_1(k,u_i)]^{\lambda}\log[\gamma_1(k,u_i)],$$

$$J_{\lambda\varphi} = -\sum_{i=1}^{n} b_i[\gamma_1(k,u_i)]^{\lambda}, \qquad J_{\varphi\varphi} = -\frac{r}{\varphi^2},$$

where

$$[\dot{\gamma}(k,u_i)]_k = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}J(u_i, k+n-1, 1),$$

$$[\ddot{\gamma}(k,u_i)]_{kk} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}J(u_i, k+n-1, 2),$$

$\psi'(.)$ is the derivative of the digamma function, u_i, g_i, ω_i, v_i, s_i, p_i, q_i and b_i are defined in Section 7. The $J(.,.,.)$ function can be easily calculated from the integral given by Prudnikov et al. [40, vol 1, Section 2.6.3, integral 1]

$$J(a,p,1) = \int_0^a x^p\log(x)\mathrm{d}x = \frac{a^{p+1}}{(p+1)^2}[(p+1)\log(a) - 1]$$

and

$$J(a,p,2) = \int_0^a x^p\log^2(x)\mathrm{d}x = \frac{a^{p+1}}{(p+1)^3}\{2-(p+1)\log(a)[2-(p+1)\log(a)]\}.$$

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