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Content Knowledge for Teaching:

What Makes It Special?

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Abstract

This article reports on efforts to develop a practice-based theory of "content knowledge for teaching," building on Shulman's (1986) notion of pedagogical content knowledge (PCK). Although the concept of PCK quickly became popular, it was in need of theoretical development, analytic clarification, and empirical testing. To contribute to the development of a theory of professionally oriented subject matter knowledge, we decided to study mathematics teaching practice itself and to identify the mathematical knowledge and skill entailed by the work. In conjunction, we developed measures of *mathematical knowledge for teaching*. These lines of research indicate at least two empirically discernable sub-domains within PCK (*knowledge of content and students* and *knowledge of content and teaching*) and an important sub-domain of "pure" content knowledge unique to the work of teaching, *specialized content knowledge,* which is distinct from the *common content knowledge* also needed by others. The article concludes with a discussion of the next steps needed in developing a useful theory of content knowledge for teaching.

Content Knowledge for Teaching:

What Makes it Special?

A teacher's own understanding of the content matters for teaching. Still, although most agree with this basic principle, what constitutes "understanding of the content" is not commonly defined. Traditionally, "knowing content" was equated with holding a degree in the subject or completing a particular set of courses. The content of "teacher content knowledge" was taken for granted. In the mid-1980's, a major breakthrough initiated a new wave of interest in the conceptualization of teacher content knowledge. In his 1985 AERA presidential address, Lee Shulman identified a special domain of teacher knowledge, which he termed *pedagogical content knowledge* (1986). He distinguished knowing content "for oneself" as different from the special amalgam of content and pedagogy needed to teach the subject. These ideas refocused attention on the foundational importance of content knowledge in teaching. What most captured the interest of researchers and educators was the proposal that there was a type of content knowledge uniquely needed by teachers—a subject-matter-based form of professional knowledge. The continuing appeal of the notion of pedagogical content knowledge is that it bridges content knowledge and the practice of teaching, assuring that discussions of content are relevant to teaching and that discussions of teaching retain attention to content. However, after two decades of work, the nature of this bridge between knowledge and practice remains inadequately understood and the "coherent theoretical framework," called for by Shulman (1986, p. 9), remains underdeveloped. While the ideas are so compelling that they continue to shape contemporary research, it is often unclear what it is that distinguishes content knowledge from pedagogical content knowledge and what distinguishes both from teaching in general.

In this paper, we argue that, while the term *pedagogical content knowledge* is widely used, its potential remains insufficiently exploited. Its essential conceptual core has too often been taken for granted, and hence remains poorly differentiated and specified. In particular, the ways in which the ideas introduced by Shulman and his colleagues have been developed and used by the field have lacked definition and empirical foundation, severely limiting their practical utility.

Throughout the past twenty years, for example, researchers have used the term "pedagogical content knowledge" to refer to a wide range of aspects of subject matter knowledge and the teaching of subject matter, and indeed, have used it differently across—and even within—subject areas. Besides differences in the breadth of what is included, there have been significant differences in *how* the term is used to relate content knowledge to the practice of teaching. Furthermore, while the work of Shulman and his colleagues was developed from extensive observation of classroom teaching, most subsequent research takes particular domains of knowledge, such as pedagogical content knowledge, as given or uses logical arguments to substantiate claims about the existence and the role of these domains. Few studies test whether there are, indeed, distinct bodies of identifiable content knowledge that matter for teaching. In particular, the field has done little to develop measures of such knowledge and to use these measures to test definitions and understanding of the nature and the effects of content knowledge.

Without this empirical testing, the ideas are bound to play a limited role in improving teaching and learning—in revamping the curriculum for teacher content preparation, in informing policies about certification, professional development, and in furthering our understanding of the relationships among teacher knowledge, teaching, and student learning. Without this empirical testing, the ideas remain, as they were twenty years ago, promising hypotheses based on logical and ad hoc arguments about the content people *think* teachers need.

For the last fifteen years, the work of the Mathematics Teaching and Learning to Teach Project and of the Learning Mathematics for Teaching Project has focused both on the *teaching* of mathematics and on the mathematics *used* in teaching. Although the aim has been to inform discussions about what teachers *need* to know, researchers began by asking what effective teaching itself demands. Instead of reasoning from the school curriculum to a list of topics teachers must know, we developed an empirical approach to understanding the content knowledge needed for teaching. The first project focused on the *work* teachers do in teaching mathematics. Researchers used studies of teaching practice to analyze the mathematical demands of teaching and, based on these analyses, developed a set of testable hypotheses about the nature of mathematical knowledge for teaching. In a related line of work, the second project developed survey measures of content knowledge for teaching mathematics. The measures provide a way to investigate the nature, the role, and the importance of different types of mathematical knowledge for teaching.

In particular, these studies have led us to hypothesize some refinements to the popular concept of *pedagogical content knowledge* and to the broader concept of *content knowledge for teaching*. In this paper, we use a focus on the work of teaching to frame our conceptualization of the mathematical knowledge needed by teachers. We identify and define two empirically discernable sub-domains of pedagogical content knowledge. In addition, and to our surprise, we have begun to uncover and articulate a less recognized domain of content knowledge for teaching that is *not* contained in pedagogical content knowledge, but yet—we hypothesize—essential to effective teaching. We refer to this as *specialized content knowledge*. These possible refinements to the map of teacher content knowledge are the subject of this paper.

Because their work was so influential, the article begins by reviewing the problem that Shulman and his colleagues framed and addressed beginning in the mid-1980s. We assess the progress made on identifying the special nature of content knowledge needed for teaching and the questions that remain unanswered. We use this discussion to clarify the problems of definition, empirical foundation, and practical utility that our work addresses. The article turns to *mathematics* in particular, and describes work on the problem of identifying mathematical knowledge for teaching. We report on refinements to the categories of mathematical knowledge for teaching evident from our analyses of mathematics teaching and from our attempts to measure it. The article concludes with an appraisal of next steps needed to continue to develop a useful theory of "content knowledge for teaching."

Content Knowledge and its Role in Defining Teaching as a Profession

A central contribution of the work of Shulman and his colleagues was to reframe the study of teacher knowledge in ways that included direct attention to the role of *content* in teaching. This was a radical departure from the research of the day, which had focused almost exclusively on general aspects of teaching such as classroom management, time allocation, or teacher planning. Prevailing process-product studies (see Brophy & Good, 1986; Rosenshine & Stevens, 1986), teacher-thinking studies (see Clark & Peterson, 1986), and production-function studies (see Hanushek, 1986; Greenwald, Hedges, & Laine, 1996) sought to link teacher characteristics, teaching behaviors, or features of teacher reasoning to student achievement gains. Subject matter was little more than context: Although these studies were conducted in classrooms where mathematics, or reading, or other subjects were taught, attention to the subject itself and to the role it played in teaching or teacher thinking was "missing." In fact, so little attention was devoted to examining content and its role in instruction that Shulman dubbed this the "missing paradigm" in research on teaching and teacher knowledge (1986). In the few cases where researchers did consider the role of teacher content knowledge, they used numbers of content courses or college grades as proxies of teacher content knowledge.

A second contribution of the work of Shulman and his colleagues was to represent content understanding as a special technical knowledge key to the profession of teaching. In the late 1980s, they conducted case studies of beginning high school teachers as part of their research in the Knowledge Growth in Teaching project. Participants were recent graduates with strong subject matter preparation in mathematics, science, English literature, and history. By examining these novices in the process of leaning to teach, the group sought to investigate how strong subject matter preparation translated into the knowledge needed for teaching that subject. Deliberately working across subjects provided a comparative basis for examining more general characteristics of the knowledge the teachers used in their practice.

A closely related purpose was to draw from these categories of teacher knowledge to inform the development of a National Board system for the certification of teachers that would "focus upon the teacher's ability to reason about teaching and to teach specific topics, and to base his or her actions on premises that can bear the scrutiny of the professional community" (Shulman, 1987, p. 20). Attention to certification was deliberately geared toward informing debates around what constituted professional expertise in teaching and related implications for teacher preparation and policy. In particular, Shulman was concerned with the current conceptions of teacher competency, which focused on generic teaching behaviors. He argued that "the currently incomplete and trivial definitions of teaching held by the policy community comprise a far greater danger to good education than does a more serious attempt to formulate the knowledge base" (Shulman, 1987, p. 20). Implicit in such comments is the argument that high quality instruction requires a sophisticated professional knowledge that goes beyond simple rules such as how long to wait for students to respond.

To characterize professional knowledge for teaching, Shulman and his colleagues developed typologies. Although the specific boundaries and names of categories varied across publications, one of the more complete articulations is reproduced below in Figure 1.

- General pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter
- Knowledge of learners and their characteristics
- Knowledge of educational contexts, ranging from workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures
- Knowledge of educational ends, purposes, and values, and their philosophical and historical grounds
- Content knowledge
- Curriculum knowledge, with particular grasp of the materials and programs that serve as "tools of the trade" for teachers
- Pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding

(Shulman, 1987, p. 8)

Figure 1. Shulman's major categories of teacher knowledge

These categories were meant to highlight the important role of content knowledge and to situate content-based knowledge in the larger landscape of professional knowledge for teaching. The first four categories address general dimensions of teacher knowledge that were the mainstay of teacher education programs at the time. They were not the main focus of Shulman's work. Instead, they functioned as placeholders in a broader conception of teacher knowledge that emphasized content knowledge. At the same time, however, Shulman made clear that these general categories were crucial and that an emphasis placed on content dimensions of teacher knowledge was not intended to denigrate the importance of pedagogical understanding and skill: Shulman (1986) argued that "mere content knowledge is likely to be as useless pedagogically as content-free skill" (p. 8).

The remaining three categories define content-specific dimensions and together comprise what Shulman referred to as the missing paradigm in research on teaching—"a blind spot with respect to content that characterizes most research on teaching, and as a consequence, most of our state-level programs of teacher evaluation and teacher certification" (1986, p. 7-8).

The first, *content knowledge*, includes knowledge of the subject and its organizing structures (Grossman, Wilson, & Shulman, 1989; Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987). Shulman (1986) argued that:

> . . . . to think properly about content knowledge requires going beyond knowledge of the facts or concepts of a domain. It requires understanding the structure of the subject matter in the manner defined by such scholars as Joseph Schwab. (p. 9)

Shulman (1986) continues:

> For Schwab, the structures of a subject include both the substantive and the syntactic structures. The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its

> facts. The syntactic structure of a discipline is the ways in which truth or falsehood, validity or invalidity are established. (p. 9)

In this view, teaching a subject is more than knowing the facts and concepts presented in a field. Although this knowledge is clearly important, teachers also need to understand the organizing principals and structures and the rules for establishing what is legitimate to do and say in a field.

Shulman (1986) went on to suggest that that:

> . . . . the subject matter content understanding of the teacher [needs to] be at least equal to that of his or her lay colleague, the mere subject matter major. The teacher need not only understand *that* something is so; the teacher must further understand *why* it is so, on what grounds its warrant can be asserted, and under what circumstances our belief in its justification can be weakened or denied. Moreover, we expect the teacher to understand why a particular topic is particularly central to a discipline whereas another may be somewhat peripheral (p. 9).

The second category, *curricular knowledge*, is "represented by the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances" (Shulman, 1986, p. 10). In addition, Shulman points to two other dimensions of curricular knowledge that are important for teaching, what he refers to as lateral curriculum knowledge and vertical curriculum knowledge. Lateral knowledge relates knowledge of the curriculum being taught to the curriculum that students are learning in other classes (in other subject areas). Vertical knowledge includes "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them" (Shulman, 1986, p. 10).

The last, and arguably most influential, of the three content-related categories was the new concept of *pedagogical content knowledge*. Shulman (1986) defined pedagogical content knowledge as comprising:

> . . . . the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the most useful ways of representing and formulating the subject that make it comprehensible to others…. Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons (p. 9).

The claim for pedagogical content knowledge was founded on observations that effective teachers in the Knowledge Growth in Teaching study represented key ideas using metaphors, diagrams, and explanations that were at once attuned to students' learning and to the integrity of the subject matter (Carlsen, 1988; Grossman, 1990; Marks, 1990; Shulman, 1986; Wilson et al., 1987; Wilson, 1988; Wineburg, 1990). Some representations are especially powerful; others, although technically correct, do not open the ideas effectively to learners.

Representation, as it is used here, refers to the range of ways that content can be organized or formulated in classroom instruction to present key ideas and concepts to students. Different from knowing the subject for oneself, knowledge of representations is one aspect of knowing a subject in ways suitable for presenting it to learners (Grossman, 1989; Grossman, 1991; Wilson et al., 1987; Wilson & Wineberg, 1988). In their analysis of the knowledge and reasoning of history teachers, Wineberg and Wilson (1991) suggest that:

> . . . . creating a representation is an act of pedagogical reasoning. Teachers must first turn inward to comprehend the key ideas, events, concepts, and interpretations of their discipline. But in fashioning representations teachers must also turn outward. They must try, as it were, to think themselves into the minds of students who lack the depth of understanding they, as teachers, possess. (p. 332-333)

A second important idea is that representations of the subject are informed by content specific knowledge of student *conceptions*. A focus on conceptions, and in many cases a particular interest in student misconceptions, acknowledges that accounting for how students understand a content domain is a key feature of the work of teaching that content. Grossman (1990) points out that these ideas:

> . . . . are inherent in Dewey's admonition that teachers must learn to 'psychologize' their subject matter for teaching, to rethink disciplinary topics to make them more accessible to students. . . . Teachers must draw upon both their knowledge of subject matter to select appropriate topics and their knowledge of student's prior knowledge and conceptions to formulate appropriate and provocative representations of the content to be learned. (p. 8)

As a concept, pedagogical content knowledge, with its focus on representations and conceptions/misconceptions, broadened ideas about how knowledge might matter to teaching, suggesting that it is not only knowledge of content, on the one hand, and knowledge of pedagogy, on the other hand, but also a kind of amalgam of knowledge of *content-and-pedagogy* that is central to the knowledge needed for teaching. This amalgam resonated with the commonsense view that the practical content knowledge needed for teaching does not come in clean disciplinary categories but requires an understanding of the interactions that occur at the boundaries between theoretical knowledge domains and in relationship to their use. In Shulman's (1987) words, "pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from the pedagogue" (p. 8).

Over the course of their work, the categories for teacher knowledge underwent a number of revisions. Modifications to the framework suggest that these ideas were not fully formed by Shulman and the larger research group. Indeed, the research group itself was clear that they saw their understanding of teacher knowledge as incomplete and distinctions and labels as provisional. Shulman (1987) noted that:

> . . . . practitioners simply know a great deal that they have never tried to articulate . . . . Much, if not most, of the proposed knowledge base [for teaching] remains to be discovered, invented, and refined . . . . Our current "blueprint" for the knowledge base for teaching had many cells or categories with only the most rudimentary place-holders, much like the chemists' periodic table of a century ago (p. 12).

More than a fixed framework, the research group appears to have seen the value in these distinctions as heuristic, as a tool for helping the field to identify distinctions in teacher knowledge that could matter for effective teaching.

Shulman and his colleagues did not seek to build a list or catalogue of what teachers need to know in any particular subject area. Instead, their work sought to provide a conceptual orientation and a set of analytic distinctions that would focus the attention of the research and policy community on the nature and types of knowledge needed for teaching a subject. In drawing attention to the missing paradigm, or the virtual absence of research focused directly on teacher content knowledge, Shulman and his colleagues defined a perspective that highlighted the content-intensive nature of teaching. However, they also were deliberate in identifying the ways in which content knowledge for teaching is distinct from disciplinary content knowledge. This had important implications for informing an emerging argument that teaching is professional work with its own unique professional knowledge base.

Testing Shulman's Hypothesis about Content Knowledge and Pedagogical Content Knowledge

There was immediate and widespread interest in the ideas presented by Shulman and his colleagues. In the two decades since these ideas were first presented, Shulman's presidential address (1986) and the related *Harvard Education Review* article (1987) have been cited in over 1200, refereed journal articles. This interest has been sustained with no less than 50 citations to these two articles in every year since 1990. Perhaps most remarkable is the reach of this work, with citations appearing in 125 different journals representing subjects as diverse as mathematics, religion, and special education; professions ranging from law to nursing to business; and knowledge for teaching students preschool through doctoral studies. Much of the interest has focused directly on pedagogical content knowledge. Thousands of articles, book chapters, and reports make use of or claim to study the notion of pedagogical content knowledge, in a wide variety of subject areas: science, mathematics, social studies, English, physical education, communication, religion, chemistry, engineering, music, special education, English language learning, higher education, and others. And, such studies show no signs of abating. Rarely does an idea—or a term—catch on at such a scale.

But, how has the field taken up the idea? What have we learned, and what do we yet need to understand?

Much of the work that followed in the wake of Shulman's proposals showed how teachers' orientations to content influenced the ways in which they taught that content. Grossman (1990) showed how teachers' orientations to literature shaped the ways in which they approached texts with their students. Wilson and Wineburg (1988) described how social studies teachers' disciplinary backgrounds—in political science, anthropology, sociology—shaped the ways in which they represented historical knowledge for high school students. And, Ball (1990) introduced the phrase "knowledge *about* mathematics" to contrast with "knowledge *of* mathematics" and to highlight the nature of knowledge in the discipline—where it comes from, how it changes, and how truth is established. In science education, study of the "nature of science" showed that specific orientations aligned with distinct sub-disciplines and significantly influenced the teaching carried out in classrooms. For instance, teachers trained in biology teach physics courses differently from teachers trained in physics, or in chemistry.

A second line of work has contributed to our understanding of the knowledge teachers need about common conceptions and misconceptions that students bring to the classroom, or develop as they learn a subject. For instance, Wineburg's (1990) analysis of students' natural efforts to understand motives and explanations for past events can be at cross-purposes with the special nature of historical understanding. And Smith and Anderson (1984) showed that children's conceptions of food and eating persistently interfered with their learning about the process of photosynthesis as the means by which plants make their own food. Likewise, in the Cognitively Guided Instruction project, researchers found that students over-generalize from experiences with problems in which the equals sign acts as a signal to compute (as it does in many programming languages (Carpenter, Franke, & Levi, 2003; Carpenter & Levi, 2000). In other words, given the problem, 5 + 7 = __ + 8, students are likely to answer 12 or 20, where the equal sign is interpreted as a signal to add. Fueled by developments in cognitive science and by increased attention to the role of prior knowledge in theories of learning, the investigation into what teachers need to know about students' conceptions and misconceptions of particular subject matter have flourished. This line of research elaborates the concept of pedagogical content knowledge by showing the special ways in which teaching demands a simultaneous integration of key ideas in the content with ways in which students apprehend them.

In another line of work spawned by Shulman's call to attend to content, researchers documented the lack of teachers' content and pedagogical content knowledge. In mathematics, Ball (1988) developed interview questions that revealed, on the one hand, the inadequacies of teachers' and prospective teachers' knowledge of important mathematics needed for teaching and, on the other hand, how much there was to understand. The tasks used by Ball in her studies were used by other researchers to elaborate more fully the special nature of the content knowledge needed for teaching that is beyond simply "knowing" the content. Finding the perimeter of a rectangle is different from analyzing a student's unanticipated generalization about the relationship between perimeter and area. The first requires only knowing how to calculate perimeter; the second requires an ability to think flexibly about perimeter to analyze another's claim. Borko, et al. (1992) described the case of a middle school student teacher, Ms. Daniels, who was asked by a child to explain why the invert-and-multiply algorithm for dividing fractions works. Despite having taken two years of calculus, a course in proof, a course in modern algebra, and four computer science courses, *and* being able to divide fractions herself, Ms. Daniels was nonetheless unable to provide a correct representation for division of fractions or to explain why the invert-and-multiply algorithm works. In addition, examination of the instances when Ms. Daniels did successfully teach for conceptual understanding revealed the central importance of using appropriate representations that made the content comprehensible to students.

Pedagogical content knowledge has permeated scholarship on teaching and teacher education, but unevenly across fields. Interestingly, our survey of the literature shows that roughly one fourth of the articles about pedagogical content knowledge are in science education, with slightly fewer in mathematics education. However, it is the breadth of literature on pedagogical content knowledge that highlights the term's heuristic value as a way of conceptualizing teacher knowledge. In physical education, the term helps to distinguish a teacher's own proficiency in a skill area (e.g., throwing a ball, dribbling) from the explicit knowledge of the skill that is needed in order to teach it to students (Chen, 2002; Rovegno, Chen, & Todorovich, 2003). In reading, there is a growing recognition that teaching reading requires a detailed knowledge of text, language, and reading process that goes substantially beyond just being able to decode and comprehend text proficiently (Hapgood, Palincsar, Kucan, Gelpi-Lomangino, & Khasnabis, 2005; Moats, 1999; Phelps, 2005; Phelps & Schilling, 2004).

Still, however, the field has made little progress on Shulman's initial charge: to develop a "coherent theoretical framework" for content knowledge for teaching. The ideas remain theoretically scattered, lacking clear definition. Because researchers tend to specialize in a single subject, much of the work has unfolded in roughly parallel but independent strands. Often it is unclear how ideas in one subject area relate to another, or even whether findings within the same subject take similar or different views of teacher subject matter knowledge. Somewhat ironically, nearly one third of the articles that cite pedagogical content knowledge do so without direct attention to a specific content area, instead making general claims about teacher knowledge, teacher education, or policy. Overall, scholars have used the concept of pedagogical content knowledge as though its theoretical foundations, conceptual distinctions and empirical testing were already well defined and universally understood.

Particularly striking is the lack of definition of key terms. Pedagogical content knowledge is often not clearly distinguished from other forms of teacher knowledge, sometimes referring to something that is simply content knowledge and sometimes to something that is largely pedagogical skill. Most definitions are perfunctory and often broadly conceived. This appears to be the case across all subject areas. For example, pedagogical content knowledge has been defined as "the intersection of knowledge of the subject with knowledge of teaching and learning" (Niess, 2005, p. 510), or as "that domain of teachers' knowledge that combines subject matter knowledge and knowledge of pedagogy" (Lowery, 2002, p. 69). It is "the amalgam of content knowledge and teaching knowledge that makes that content better able to be understood through the particular approach adopted" (Loughran, Milroy, Berry, Gunstone, & Mulhall, 2001, p. 289). An, Kulm, and Wu (2004) define pedagogical content knowledge "as the knowledge of effective teaching which includes three components, knowledge of content, knowledge of curriculum, and knowledge of teaching" (p. 146). In even broader terms pedagogical content knowledge is defined simply as "the product of transforming subject matter into a form that will facilitate student learning" (de Berg & Greive, 1999, p. 20). While these and a host of other short definitions capture the general idea of pedagogical content knowledge as a domain that combines the subject with teaching, they are broad enough to include nearly any package of teacher knowledge and beliefs.

A definition's brevity, however, is not the only factor that contributes to a lack of clarity over what might count as pedagogical content knowledge. More careful and detailed definitions still leave unclear where the boundary is between pedagogical content knowledge and other forms of teacher knowledge. For example, Magnusson, et. al. (1999), define the construct as follows.

> Pedagogical content knowledge is a teacher's understanding of how to help students understand specific subject matter. It includes knowledge of how particular subject matter topics, problems, and issues can be organized, represented and adapted to the diverse interests and abilities of learners, and then presented for instruction . . . . The defining feature of pedagogical content knowledge is its conceptualization as the result of a *transformation* of knowledge from other domains. (p. 96)

When defined these ways, pedagogical content knowledge begins to look as though it includes most everything a teacher might know in teaching a particular topic obscuring distinctions between teacher actions, reasoning, beliefs, and knowledge.

We argue that the power of the idea, launched by Shulman and his colleagues, that teaching requires a special kind of content knowledge, is worth our collective investment and cultivation. That teaching demands content knowledge is obvious; policymakers are eager to set requirements based on the commonsense notion. The contribution of scholars can be to provide specification about the nature of the content knowledge needed; to provide this specification demands that we employ greater precision about the concepts and methods involved. Our aim in this paper is to describe how we have approached this problem and what we are learning about the nature of the content knowledge needed for teaching.

Our Approach to Studying Mathematical Knowledge for Teaching

In the past, a focus on what teachers need to know has led to a set of positions, each related to principled arguments about what teachers should know. The prevailing view is that teachers need to know whatever mathematics is in the curriculum plus some additional number of years of further study in college mathematics. A second hypothesis is that teachers need to know the curriculum, but "deeper," plus some amount of pedagogical content knowledge. In both cases, it is unclear what exactly it is that makes up the extra knowledge of mathematics.

A more focused question is: What do teachers need to know and be able to do to carry out the work of teaching effectively? This is a centrally important question that could be investigated in numerous ways—by examining the curriculum and standards for which teachers are responsible (or the tests their students must be prepared to pass); by asking expert mathematicians and mathematics educators to identify the core mathematical ideas and skills that teachers should have (CBMS, 2000); or by reviewing research on students' learning to ascertain those aspects of mathematics with which learners have difficulty (Stylianides & Ball, 2004). Our research group chose a different approach, one that might be characterized as working "bottom up," beginning with practice. Because it seemed obvious that teachers need to know the topics and procedures that they teach—primes, equivalent fractions, functions, translations and rotations, factoring, and so on—we decided to focus on *how* teachers need to know that content. In addition, we wanted to ask *what else* do teachers need to know about mathematics and how and where might teachers *use* such mathematical knowledge in practice?

Hence, we decided to focus on the "work of teaching." What do teachers *do* in teaching mathematics, and how does what they do demand mathematical reasoning, insight, understanding, and skill? Instead of starting with the curriculum, or with standards for student learning, we study teachers' work. We seek to unearth the ways in which mathematics is involved in contending with the regular day-to-day, moment-to-moment demands of teaching. Our analyses lay the foundation for a *practice-based theory of mathematical knowledge for teaching* (Ball & Bass, 2003b). We see this approach as a kind of job analysis, similar to analyses done of other mathematically intensive occupations that range from nursing and engineering physics (Hoyles, Noss, & Pozzi, 2001; Noss, Healy, & Hoyles, 1997) to carpentry and waiting tables.

By "mathematical knowledge for teaching," we mean the mathematical knowledge *needed to carry out the work of teaching mathematics.* Important to note here is that our definition begins with teaching, not teachers. It is concerned with the tasks involved in teaching and the mathematical demands of these tasks. Because teaching involves showing students how to solve problems, answering students' questions, and checking students' work, it demands an understanding of the content of the school curriculum. Beyond these obvious tasks, we seek to identify other aspects of the work and to analyze what these reveal about the content demands of teaching.

We continue to approach the problem in two ways. First, we conduct extensive qualitative analyses of teaching practice. Second, we design measures of mathematical knowledge for teaching based on hypotheses formulated from our qualitative studies. We briefly describe these two lines of work, and their intersection.

The questions that guide our qualitative analyses research are:

1) What are the recurrent tasks and problems of teaching mathematics? What do teachers *do* as they teach mathematics?
2) What mathematical knowledge, skills, and sensibilities are required to manage these tasks?

By "teaching," we mean everything that teachers do to support the learning of their students. Clearly we mean the interactive work of teaching lessons in classrooms, and all the tasks that arise in the course of that work. But we also mean planning for those lessons, evaluating students' work, writing and grading assessments, explaining the class' work to parents, making and managing homework, attending to concerns for equity, and dealing with the building principal who has strong views about the math curriculum. Each of these tasks, and many others as well, involve knowledge of mathematical ideas, skills of mathematical reasoning, fluency with examples and terms, and thoughtfulness about the nature of mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001).

Central to the qualitative work has been a large longitudinal NSF-funded database, documenting an entire year of the mathematics teaching in a third grade public school classroom during 1989-90. The records collected across that year include videotapes and audiotapes of the classroom lessons, transcripts, copies of students' written class work, homework, and quizzes, as well as the teacher's plans, notes, and reflections. A second major resource has been the fact that our research group comprises individuals from a wide range of different disciplines and experience. By analyzing these detailed records of practice, with different perspectives and knowledge, we seek to develop a theory of mathematical knowledge as it is entailed by and used in teaching (Ball, 1999). We have been studying not only specific episodes but also instruction over time, considering the work of developing both mathematics and students across the school year (Ball & Bass, 2000; Ball & Bass, 2003a). What sort of larger picture of a mathematical topic and its associated practices is needed for teaching over time? How do students' ideas and practices develop and what does this imply about the mathematical work of teachers? In addition to this extensive set of records, we work also with other collections we have assembled over the last decade. These collections, like our original one, typically include videotapes of classroom teaching, copies of student work and of teachers' notes, and curriculum materials from which the teacher is teaching.

As a complement to our ongoing qualitative analyses of teaching, and as a fortuitous result of our engagement in a large study of comprehensive school reform models (the Study of Instructional Improvement; www.sii.soe.umich.edu), we began to develop and validate survey measures of mathematical knowledge for teaching (Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2004, Hill, Rowan, & Ball, 2005). To do so, we again engage multi-disciplinary teams to draft, refine, and critique questions (Bass & Lewis, 2005). This measures-development work has helped to articulate, test, and refine categories and subcategories of the mathematical knowledge needed for teaching. In this work, we seek to write items in these different categories and to test the questions with large groups of teachers (Hill & Ball, 2004). Analytic techniques, such as factor analysis, provide a basis for testing our assumptions about the structure of mathematical knowledge for teaching, and help us to refine the categories and the measures. Although these analyses are ongoing, we see persuasive evidence that the mathematical knowledge needed for teaching is multi-dimensional. That is, general mathematical ability does not fully account for the knowledge and skills entailed in teaching mathematics (Hill, Ball, & Schilling, 2004).

Approaching the problem by analyzing teaching practice and developing instruments to test the ideas, we are able to do the kind of discovery and refinement called for by Shulman in 1987—we are able to fill in some of the rudimentary "periodic table" of teacher knowledge.

In a subsequent section, we describe our research group's current hypotheses about the structure and domains of mathematical knowledge needed for teaching and summarize evidence for these domains. We then return to the notion of pedagogical content knowledge, and discuss the relationship of our work to that body of research. First, though, we provide an illustration of the kind of knowledge that has surfaced from our analyses of teaching. Perhaps most interesting to us has been evidence that teaching may require a specialized form of pure subject matter knowledge—"pure" because it is not mixed with knowledge of students or pedagogy and is thus distinct from the pedagogical content knowledge identified by Shulman and his colleagues; "specialized" because it is not needed or used in settings other than mathematics teaching. It is this formulation in terms of the work of teaching that makes this content knowledge special.

An Example of What Makes Mathematical Knowledge for Teaching Special

Our analyses of teachers' practice reveal that the mathematical demands of teaching are substantial. The mathematical knowledge needed for teaching is not less than that needed by other adults. In fact, knowledge for teaching must be detailed in ways unnecessary for everyday functioning. In short, a teacher needs to know more, and different, mathematics—not less. To better understand what we mean by this, we offer an example based on a simple subtraction computation:

$$\begin{array}{r} 307 \\ -\,168 \\ \hline \end{array}$$

Most readers will know an algorithm to produce the answer 139, such as:

$$\begin{array}{r} \overset{2}{\cancel{3}}\,{}^{1}\overset{9}{\cancel{0}}\,{}^{1}7 \\ -\,1\;6\;8 \\ \hline 1\;3\;9 \end{array}$$

We start with this “pure” computational task because teachers who teach subtraction must be able to perform this calculation themselves. This is mathematical knowledge that others commonly hold, because this knowledge is used in a wide range of settings. However, being able to carry out this procedure is necessary, but not sufficient, for teaching it. We next take the example further into the work of teaching.

Many third graders struggle with the subtraction algorithm, often making errors. One common error is:

$$\begin{array}{r} 307 \\ -\,168 \\ \hline 261 \end{array}$$

A teacher needs to be able to spot that 261 is incorrect. This does not require any special knowledge to do: Anyone who can solve the problem above can readily see this. However, teaching involves more than identifying an incorrect answer. Skillful teaching requires being able to size up the source of a mathematical error. Moreover, this is work that teachers must do rapidly, often on the fly, since, in a classroom, students cannot wait as a teacher puzzles over the mathematics himself. Here, for example, a student has, in each column, calculated the difference between the two digits, or subtracted the smaller digit from the larger one. A teacher who is mystified about what could have produced 261 as an answer will arguably move more slowly and with less precision to help correct the student's problem. Consider next another error that teachers may confront when teaching this subtraction problem:

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 169 \end{array}$$

What line of thinking would produce this error? In this case, in contrast to the first example, the student has "borrowed" one from the hundreds column, "carried the one" to the ones place, and subtracted 8 from 17, yielding 9. The thinking might continue by "bringing down" the 6 and subtracting 2 – 1 = 1. Teachers need to be able to perform this kind of mathematical error analysis efficiently and fluently. Error analysis is of course a common practice among mathematicians in the course of their own work; the task in teaching differs only in that it focuses on the errors produced by learners.

These two errors stem from different difficulties with the algorithm for subtracting multi-digit numbers. In the first, the student considered the difference between digits with no thought to the relationships among columns. In the second, the student attempts to regroup the number, but without careful consideration of the value of the places and the conservation of the value of the number. Seeing both answers as simply wrong does not equip a teacher with the detailed mathematical understanding required for a skillful treatment of the problems these students face.

Analysis such as this are characteristic of the distinctive work teachers do and they require a kind of mathematical reasoning that most adults do not need to do on a regular basis. And although mathematicians engage in analyses of error, often of failed proofs, the analysis used to uncover a student error appears to be related to, but not the same as, other error analysis in the discipline. Further, there is no demand on mathematicians to conduct their work quickly as students wait for guidance.

It is also common in instruction for students to produce non-standard approaches that are unfamiliar to the teacher. For instance, what mathematical issues confront a teacher if a student asserts that she would "take 8 away from both the top and the bottom," yielding the easier problem:

$$\begin{array}{r} 299 \\ -\ 160 \\ \hline \end{array}$$

Is it legitimate to do this? Why? Would it work in general? Is it easier for some numbers and harder for others? How might you describe the method the student is using and how would you justify it mathematically? Being able to engage in this sort of mathematical "inner dialogue," and to provide mathematically sound answers to these questions, is a crucial foundation for determining what to *do* in teaching this mathematics.

Teachers confront all kinds of student solutions. They have to figure out what students have done, whether the thinking is mathematically correct for the problem, and whether the approach would work in general. Consider the following three executions of our original subtraction problem. What is going on mathematically in each case?

307	11 307	307
- 168	- 168 11	- 168
-1	139	2
-60		30
200		107
139		139

In fact, these examples are all correct and could be generalized in plausible ways, but figuring this out is not a straightforward task for those who only know how to do the subtraction as they themselves learned it in third grade.

Interpreting student error and evaluating alternative algorithms is not all that teachers do, however. Teaching also involves explaining procedures. For example, for the subtraction algorithm, one could give a set of procedural directions. In the above example a teacher might say, "cross out the three, put a two, put a one on top of the zero, cross out the one and the zero and put a nine, and then put a one by the seven; now subtract." However, this procedure is specific to this problem: it does not generalize to, for instance, 314 – 161, where one only "crosses out" and "puts" once, not twice. It also does nothing to show *how* the procedure works. Teachers need to know rationales for procedures, meanings for terms, and explanations for concepts. Teachers need effective ways of representing the *meaning* of the subtraction algorithm—not just to confirm the answer, but to show what the steps of the procedure mean, and why they make sense. Our point here is not about what teachers need to teach, but about what they themselves need to know and be able to do in order to carry out that teaching.

How might teachers explain the meaning of the subtraction algorithm to students? One possibility is to use money as a model. To represent 307 – 168, what money would be needed? First of all, a teacher needs to recognize that not all US coins come in denominations in the base-ten numeration system. In making change for 68 cents, you would be likely to use two quarters, a dime, a nickel, and three pennies. But the base-ten system does not use 25 or 5 as units; instead it uses the decimal units—100, 10, 1. Representing 68 cents with 6 dimes and 8 pennies is obviously possible in money, but not the most typical or efficient choice given the coins we have. Giving students 3 U.S. dollars and 7 pennies and asking them to take away $1.68 does not lead readily to regrouping $3.07 into 2 dollars, 9 dimes, and 17 pennies, which would be necessary in order to use money to represent the regrouping central to the conventional subtraction algorithm. Furthermore, to carry out the regrouping of $3.07 in a manner that fits the standard algorithm, requires 10 dimes, not 9. What might a different model make visible? For instance, money requires "trading" one dime for 10 pennies, while straws rubber banded into groups of ten can be used to model the processes of grouping ten ones into one ten and ungrouping one ten into ten ones.

Teaching also involves considering what numbers are strategic to use in an example. The numbers 307 and 168 may not be ideal choices to make visible the conceptual structure of the algorithm. Should the numerical examples require two regroupings, as in this case, or should examples be sequenced from ones requiring no regrouping to ones that require several? And what about the role of zeros at different points in the procedure? Should the example include zeros—or perhaps not at first? Questions such as these, as well as those posed in the discussion above, require mathematical reasoning and insight crucial to teaching, yet foreign to most well educated adults. This is what we mean by the special mathematical demands of teaching mathematics.

Our study of the mathematical demands of teaching has yielded a wealth of tasks that require mathematical knowledge and skill. What caught us by surprise, however, was how much purely *mathematical* knowledge was required, even in many everyday tasks of teaching—assigning student work, listening to student talk, grading or commenting on student work. Despite the fact that these tasks are done with and for students, close analysis revealed how intensively mathematical they were. We were surprised to see that many of the tasks of teaching require mathematical knowledge apart from knowledge of students or teaching. For instance, deciding whether a method or procedure would work in general requires mathematical knowledge and skill, not knowledge of students or teaching. It is a form of mathematical problem solving used in the work of teaching. Likewise, determining the validity of a mathematical argument, or selecting appropriate mathematical representations, requires mathematical knowledge and skill important for teaching yet not entailing knowledge of students or teaching. And, in our research we began to notice how rarely these mathematical demands were ones that could be addressed with mathematical knowledge learned in university mathematics courses. We began to hypothesize that there were aspects of subject matter knowledge—not pedagogical content knowledge—that need to be uncovered, mapped, organized, and included in mathematics courses for teachers.

Looking back across our subtraction example, many of the tasks, such as determining whether an alternative method will work in general, explaining the meaning of a procedure, or recognizing mathematical properties offered by different materials or models, involve deep and explicit knowledge of the subtraction algorithm—more than simply knowing how to perform the calculation. Yet, none of these requires knowledge of *students* or knowledge of *teaching*.

Two points are central to our argument. First, much of the work of teaching is mathematical in nature, with significant mathematical demands. While the mathematical tasks we have been identifying would inform teachers' choices and moves with students, these tasks can also be seen as illustrating the special *mathematical* thinking that teachers need to themselves do or understand in order to teach the mathematics. They require significant mathematical knowledge, skill, habits of mind, and insight. While our examples are drawn from the context of teaching, the mathematical knowledge needed to engage them stands on its own as a domain of knowledge needed by teachers for their work. A second point is that the mathematical knowledge and skill we have been identifying here has a relevance to teaching often missing from discussions about the mathematics needed by teachers. By identifying mathematics in relation to specific tasks in which teachers engage, we establish its relevance to what teachers *do*. Finally, we suspect that many of these insights extend to the knowledge teachers need in other subjects as well.

In our analyses of the mathematical work involved in teaching mathematics, we noticed that the nature of that mathematical knowledge and skill seemed itself to be of different types. We hypothesized that teachers' opportunities to learn mathematics for teaching could be better tuned if we could identify those types more clearly. If mathematical knowledge required for teaching is indeed multidimensional, then professional education could be organized to help teachers learn the range of knowledge and skill they need in focused ways. If, however, it is basically all the same as general mathematical ability, then discriminating professional learning opportunities would be unnecessary. Based on our analysis of the mathematical demands of teaching, we hypothesized that Shulman's content knowledge could be subdivided into common content knowledge and specialized content knowledge, on the one hand, and his pedagogical content knowledge could be divided into knowledge of content and students and knowledge of content and teaching, on the other. Turning back to the results of our studies, in the next section we define and illustrate each of these sub-domains.

Mathematical Knowledge for Teaching and Its Structure

In analyzing the mathematical demands of teaching, we seek to identify mathematical knowledge that is demanded by the work teachers do. To pursue this, we define the mathematical knowledge we are studying as mathematical knowledge "entailed by teaching"—in other words, *mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students*. To avoid a strictly reductionist and utilitarian perspective, however, we seek a generous conception of "need" that allows for the perspective, habits of mind, and appreciation that matter for effective teaching of the discipline.

The first domain represents the first step in the example above: simply calculating an answer, or more generally, correctly solving mathematics problems. We call this *common content knowledge (CCK)* and define it as the mathematical knowledge and skill used in settings other than teaching. Teachers need to know the material they teach: They need to know when their students have answers wrong, or when the textbook gives an inaccurate definition. When they write on the board, they need to use terms and notation correctly. In short, they need to be able to *do the work* that they are assigning their students. But some of this requires mathematical knowledge and skill that others have as well—thus, it is not special to the work of teaching. This explains, too, how it is that others can "teach"—parents, for example.

In analyzing video of teaching, it was obvious that such knowledge is essential. When a teacher mispronounced terms, made calculation errors, or got stuck trying to solve a problem at the board, instruction suffered and valuable time was lost. In mapping out the mathematical knowledge needed by teachers, it was clear that such understanding of the mathematics in the student curriculum plays a critical role in planning and carrying out instruction.

Further evidence for common content knowledge comes from our work to develop instruments for measuring mathematical knowledge for teaching. We pose questions such as, "What is a number that lies between 1.1 and 1.11?" We ask questions that require knowing that a square is a rectangle, that 0/7 is 0, and that the diagonals of a parallelogram are not necessarily perpendicular. These are not specialized understanding, but are questions that would be commonly answerable by others who know mathematics. Often, as shown in Figure 2, we couch the problem in the context of teaching to point out where in the activity of teaching the use of such common knowledge might arise.

Ms. Dominguez was working with a new textbook and she noticed that it gave more attention to the number 0 than her old book did. She came across a page that asked students to determine if a few statements about 0 were true or false. Which statement(s) should she recognize as true?

	Yes	No
a) 0 is an even number.	1	2
b) 0 is not really a number. It is a placeholder in writing big numbers.	1	2
c) The number 8 can be written as 008.	1	2

Figure 2. Statements about 0

The activity of looking over textbooks requires, among other things, basic competence with the content. Knowing which statements are true in the above example is common mathematical knowledge that is not likely to be unique to teachers.

The second domain, *specialized content knowledge (SCK)*, is the mathematical knowledge and skill uniquely needed by teachers in the conduct of their work. This is the domain in which we have become particularly interested. Close examination reveals that it is mathematical knowledge not commonly needed for purposes other than teaching. In looking for patterns in student errors or in sizing up whether a non-standard approach would work in general, as in our subtraction example, teachers have to do a kind of mathematical work that others do not. Many of the everyday tasks of teaching are distinctive to this special work (Figure 3).

Presenting mathematical ideas
Responding to students' "why" questions
Finding an example to make a specific mathematical point
Recognizing what is involved in using a particular representation
Linking representations to underlying ideas and to other representations
Connecting a topic being taught to topics from prior or future years
Explaining mathematical goals and purposes to parents
Appraising and adapting the mathematical content of textbooks
Modifying tasks to be either easier or harder
Evaluating the plausibility of students' claims (often quickly)
Giving or evaluating mathematical explanations
Choosing and developing useable definitions
Using mathematical notation and language and critiquing its use
Asking productive mathematical questions
Selecting representations for particular purposes
Inspecting equivalencies

Figure 3. Mathematical tasks of teaching

Each of these is something teachers routinely do. Taken together, they demand unique mathematical understanding and reasoning. Teaching requires knowledge beyond that being taught to students. For instance, it requires understanding different interpretations of the operations in ways that students need not explicitly distinguish. And, it requires appreciating the difference between "take away" and "comparison" models of subtraction, and between "measurement" and "partitive" models of division. Consider, for instance, the problem in Figure 4 below.

Which of the following story problems can be used to represent $1\ ^1/_4$ divided by $^1/_2$?

	Yes	No
a) You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?	1	2
b) You have \$1.25 and may soon double your money. How much money would you end up with?	1	2
c) You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need?	1	2

Figure 4. Story problem that represents $1\ ^1/_4$ divided by $^1/_2$

The mathematics of this problem can be rather challenging: the first word problem is division by 2 rather than by 1/2; the second is multiplication by 2 rather than division by 1/2 (a subtle yet important point for teaching this content); and the third correctly fits the calculation—using a measurement meaning of division. The important point here, though, is that figuring out which story problems fit with which calculations, and vice versa, is a task engaged in teaching this content, not something done in solving problems with this content.

Teaching also involves the use of "decompressed" mathematical knowledge that might be taught to students but with the eventual curricular goal of developing, in students, compressed forms of that knowledge as it is understood by competent adults for various uses. Teaching involves making features of particular content visible to and learnable by students. Teaching about place value, for example, requires understanding the place-value system in a self-conscious way that goes beyond the kind of tacit understanding of place value expected of others. Teachers need to be able to talk explicitly about how mathematical language is used (e.g., how the mathematical meaning of "edge" is different from the everyday reference to the edge of a table), how to choose, make, and use mathematical representations effectively (e.g., recognizing advantages and disadvantages of using rectangles or circles to represent fractions), and how to explain and justify one's mathematical ideas (e.g., why you invert and multiply to divide fractions). All of these are examples of ways in which teachers work with mathematics in its decompressed or unpacked form.

The mathematical demands of teaching require specialized mathematical knowledge, needed by teachers and not needed by others. Accountants have to calculate and reconcile numbers and engineers have to mathematically model properties of materials, but neither group needs to explain why, when you multiply by ten, you "add a zero." In developing survey questions to measure such knowledge, we ask, for example, whether an unusual method proposed by a student would work in general, which statement best explains why we find common denominators when adding fractions, and which of a set of given drawings could be used to represent 2 divided by $^{2}/_{3}$. These and questions like them are the daily fare of teaching. The demands of the work of teaching mathematics create the need for such a body of mathematical knowledge specialized to teaching.

The third domain, *knowledge of content and students (KCS)*, is knowledge that combines knowing about students and knowing about mathematics. Teachers need to anticipate what students are likely to think and what they will find confusing. When choosing an example, they need to predict what students will find interesting and motivating. When assigning a task, they need to anticipate what students are likely to do with it and whether they will find it easy or hard. They must also be able to hear and interpret students' emerging and incomplete thinking as expressed in the ways that pupils use language. Each of these tasks requires an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking.

Central to these tasks is knowledge of common student conceptions and misconceptions about particular mathematical content. For instance, in the subtraction example, knowing that students often "subtract up" when confronted with a problem such as 307 – 168 means that a teacher, who has seen this happen and knows that it is a common student response, is able to recognize it without extensive mathematical analysis or probing. In other words, recognizing a wrong answer is common content knowledge (CCK), while sizing up the nature of the error may be either specialized content knowledge (SCK) or knowledge of content and students (KCS) depending on whether a teacher draws predominantly from her knowledge of mathematics and her ability to carry out a kind of mathematical analysis or instead draws from familiarity with common student errors.

The demands of teaching require knowledge at the intersection of content and students. In developing an instrument to measure such knowledge, we ask questions, for example, about the kinds of shapes young students are likely to identify as triangles, the likelihood that that they may write 405 for forty-five, and problems where confusion between area and perimeter lead to erroneous answers. We also ask questions that require interpretation of students' emerging and inchoate thinking, or that reflect thinking or expression typical of particular learners, or that demand sensitivity to what is likely to be easy or challenging.

Many of our ideas in this area draw from the literature on student thinking: e.g., van Hiele's studies of levels of the development in representing two-dimensional figures (Burger & Shaughnessy, 1986; Crowley, 1987), CGI researchers' documentation of common misinterpretations of the equal sign (Carpenter, Franke, & Levi, 2003; Carpenter & Levi, 2000) or that subtraction problems involving comparison are harder for students than "take away" problems (Carpenter, et al., 1998), or Phillipps' observation that students misappropriate the subtraction language of "take away" when representing fractions, causing them to confound what is left with what is removed (2005). In each case, knowledge of students and content is an amalgam, involving a particular mathematical idea or procedure and familiarity with what students may think or do.

The last domain, *knowledge of content and teaching (KCT)*, is knowledge that combines knowing about teaching and knowing about mathematics. Many of the mathematical tasks of teaching require a mathematical knowledge of the design of instruction. Teachers need to sequence particular content for instruction, deciding which example to start with and which examples to use to take students deeper into the content. They need to evaluate the instructional advantages and disadvantages of representations used to teach a specific idea and identify what different methods and procedures afford instructionally. They also need to make instructional decisions about which student contributions to pursue and which to ignore or save for a later time. During a classroom discussion, they have to decide when to pause for more clarification, when to use a student's remark to make a mathematical point, and when to ask a new question or pose a new task to further students' learning. Each of these tasks requires an interaction between specific mathematical understanding and an understanding of pedagogical issues that affect student learning.

One example of KCT would be knowing different possible models for place value, what each can be used to make visible about the subtraction algorithm, as well as how to deploy them effectively. How is money different from coffee stirrers bundled with rubber bands, or base ten blocks from "unifix" cubes? Each of these *can* correctly represent subtraction of multi-digit numbers, but each also represents different aspects of the content that make a difference at different points in students' learning. Knowing how those differences matter for the development of the topic is part of what we call knowledge of content and teaching. Each model also requires different care in use in order to make the mathematical issues salient and usable by students (Cohen, 2005).

The demands of teaching require knowledge at the intersection of content and teaching. In developing an instrument to measure such knowledge, we ask questions about whether a tape measure would be good for teaching place value, about choosing examples for simplifying radicals for the purpose of discussing multiple strategies, or about sequencing subtraction problems with and without regrouping for instruction. We also ask questions about how language and metaphors can assist and confound student learning—the way language about "borrowing" or "canceling" may interfere with understanding of the mathematical principles underlying the subtraction algorithm or the solving of algebraic equations. In each of these examples, knowledge of teaching and content is an amalgam, involving a particular mathematical idea or procedure and familiarity with pedagogical principles for teaching that particular content.

Building a Map of Usable Professional Knowledge of Subject Matter

Several issues about our proposed categories are worth addressing—their relationship to pedagogical content knowledge, the "special" nature of specialized content knowledge, our use of teaching as a basis for defining the domains, and problems with the categories that need to be addressed.

From our definitions and examples it should be evident that this work may be understood as elaborating, not replacing, the construct of pedagogical content knowledge. For instance, the last two domains—knowledge of content and students and knowledge of content and teaching—coincide with the two central dimensions of pedagogical content knowledge identified by Shulman (1986):

- "the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 9), and;
- "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9).

However, we also see our work as developing in more detail the fundamentals of *subject matter knowledge for teaching* by elaborating sub-domains and by measuring and validating knowledge of those domains.

We have been most struck by the relatively uncharted arena of mathematical knowledge necessary for teaching the subject that is *not* intertwined with knowledge of pedagogy, students, curriculum, or other non-content domains. What distinguishes this sort of mathematical knowledge from other knowledge of mathematics is that it is subject matter knowledge needed by teachers for specific tasks of teaching, such as those in Figure 3, but still, clearly subject matter knowledge. These tasks of teaching depend on mathematical knowledge, and, significantly, they have aspects to them that do *not* depend on knowledge of students or of teaching. These tasks require knowing how knowledge is generated and structured in the discipline. They also require a host of other mathematical knowledge and skills, knowledge and skills not typically taught to teachers in the course of their formal mathematical preparation.

Where, for example, do teachers develop explicit and fluent use of mathematical notation? Where do they learn to inspect definitions and to establish the equivalence of alternative definitions for a given concept? Where do they learn definitions for fractions and compare their utility? Where do they learn what constitutes a good mathematical explanation? Do they learn why 1 is not considered prime, or how and why the long division algorithm works? Teachers must know these sorts of things, and engage in these mathematical practices themselves in order to teach and they must also learn to teach them *to* students. Explicit knowledge and skill in these areas is vital for teaching.

To represent our current hypotheses, we propose a diagram (Figure 5) as a refinement to Shulman's categories that includes a domain of specialized content knowledge as a form of subject matter knowledge that is specific to the work teachers do.

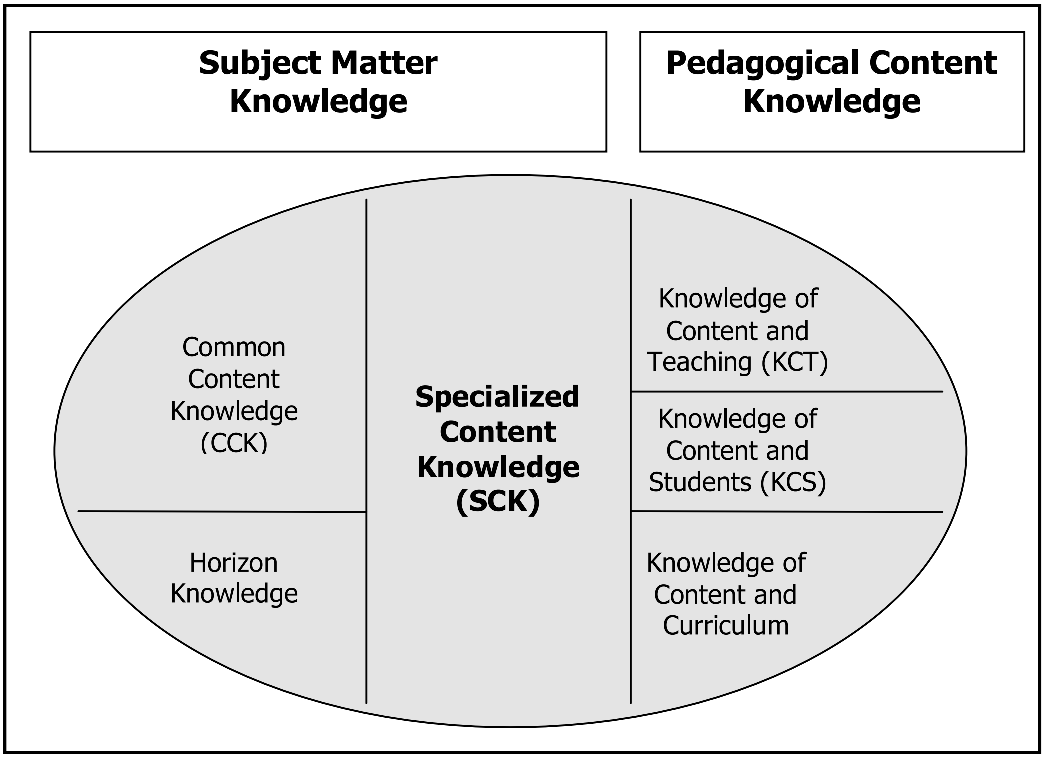

Figure 5. Domains of mathematical knowledge for teaching

Figure 5 shows the correspondence between our current map of the domain of content knowledge for teaching and Shulman's (1986) initial categories: subject matter knowledge and pedagogical content knowledge. We have provisionally placed Shulman's third category, curricular knowledge, within pedagogical content knowledge. This is consistent with later publications from members of Shulman's research team (Grossman, 1990). We are not yet sure whether this may be a part of our category of knowledge of content and teaching, or whether it may turn out to run across the several categories or be a category in its own right. We also provisionally include a second category within subject matter knowledge, what we call "horizon knowledge." Horizon knowledge is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum. First grade teachers, for example, may need to know how the mathematics they teach is related to the mathematics students will learn in third grade to be able to set the mathematical foundation for what will come later. Again we are not sure if this category belongs as part of subject knowledge, or whether it may run across the other categories. As our work progresses, we hope to explore these ideas theoretically, empirically, and also pragmatically as the ideas are used in teacher education or in the development of curriculum materials for use in professional development.

Our current empirical results, based on our factor analyses, suggest that it is likely that content knowledge for teaching is multidimensional (Hill, Ball, & Schilling, 2004; Schilling, in press). Whether these categories, as we propose them here, are the "right" ones is not most important. Likely they are not. Our current categories will continue to need refinement and work.

First, our theory is framed in relation to practice: This may be one of its greatest strengths, assuring that the knowledge identified is relevant to practice, yet also its greatest weakness, inheriting some of the natural messiness and variability of teaching and learning. We ask about the situations that arise in teaching that require teachers to *use* mathematics. Some situations, however, can be managed using different kinds of knowledge. Consider the example of analyzing a student error. A teacher might figure out what went wrong by analyzing the error mathematically. What steps were taken? What assumptions made? But another teacher might figure it out because she has seen students do this before with this particular type of problem. The first teacher is using specialized content knowledge, whereas the second is using knowledge of content and students.

Two other problems arise out of this first. Despite our expressed intention to focus on knowledge *use*, our categories seem static. Ultimately, we are interested in how teachers reason about and deploy mathematical ideas in their work. We are interested in skills, habits, sensibilities, and judgments as well as knowledge. The questions we pose in our measures of mathematical knowledge for teaching situate the knowledge we mean to measure in the context of its use, but how knowledge is actually used and what features of teacher thinking shape its use remain tacit and unexamined. How to capture the common and specialized aspects of teacher thinking, as well as how different categories of knowledge come into play in the course of teaching, needs to be addressed more effectively in this work.

Last is a boundary problem: It is not always easy to discern where one of our categories divides from the next and this affects the precision (or lack thereof) of our definitions. We define "common content knowledge" as the mathematical knowledge known in common with others who know and use mathematics, but we do not find that this term always communicates well what we mean. Consequently, although the distinction may be compelling as a heuristic, it can be difficult to discern common from specialized knowledge in particular cases. Take, for instance, the problem of what fraction represents the shaded portion of the *two* circles shaded in Figure 6:

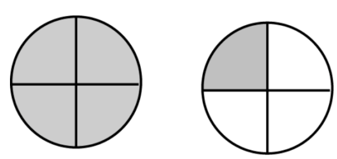

Figure 6. Representations of $^5/_8$ of 2

Is the knowledge that this is $\frac{5}{8}$ of 2 common? Or is it specialized? We tend to think that this kind of detailed knowledge of fractions and their correspondence to a particular representation as specialized knowledge; it is hard to think of others who use this knowledge in their day-to-day work. But this is not altogether clear as this may be known in common with others who use this kind of detailed knowledge of fractions. Similarly, it can be difficult at times to discriminate specialized content knowledge from knowledge of content and students—for example, consider what is involved in selecting a numerical example to investigate students' understanding. The shifts that occur across the four domains, for example, from ordering a list of decimals (CCK), to generating a list to be ordered that would reveal key mathematical issues (SCK), to recognizing which would cause students the most difficulty (KCS), to deciding what to do about their difficulties (KCT), are important yet subtle. That we are able to work empirically as well as conceptually helps us to refine our categories; still, we recognize the problems of definition and precision exhibited by our current formulation.

Conclusion

Teachers must know the subject they teach. Indeed, there may be nothing more foundational to teacher competency. The reason is simple. Teachers who do not themselves know a subject well are not likely to have the knowledge they need to help students learn this content. At the same time, however, just knowing a subject well may not be sufficient for teaching. One need only sit in a classroom for a few minutes to notice that the mathematics that teachers work with in instruction is not the same mathematics taught and learned in college classes. In addition, teachers need to know mathematics in ways useful for, among other things, making mathematical sense of student work and choosing powerful ways of representing the subject so that it is understandable to students. It seems unlikely that just knowing *more* advanced math will satisfy all of the content demands of teaching. What seems most important is knowing the mathematics actually used in teaching.

Unfortunately, though, subject matter courses in teacher preparation programs tend to be academic in both the best and worst sense of the word, scholarly and irrelevant, either way remote from classroom teaching. Disciplinary knowledge has the tendency to be oriented in directions other than teaching, toward the discipline—history courses toward knowledge and methods for doing history and science courses toward knowledge and methods for doing science. Although there are exceptions, the overwhelming majority of subject matter courses for teachers, and teacher education courses in general, are viewed by teachers, policymakers, and society at large as having little bearing on the day-to-day realities of teaching and little effect on the improvement of teaching and learning.

Although subject matter knowledge is essential for effective teaching, subject matter preparation tends to be irrelevant. This is the problem that Shulman and his colleagues were addressing in the late 1980s.

In this paper, we argue that the issues identified by Shulman and his colleagues more than two decades ago are key to research on teaching and teacher education. Content knowledge is immensely important to teaching and its improvement. Instead of taking pedagogical content knowledge as given, however, we argue that there is a need to carefully map it and measure it. This includes the need to better explicate how this knowledge is used in teaching effectively.

The work reported here takes this charge seriously. It is rooted in attention to the demands of practice to consider what mathematics arises in the work that teachers do. Our work tests these ideas by developing instruments to measure this knowledge, by using the results to inform our understanding of a map of teacher content knowledge, and by tying this knowledge to its use in practice. That there is a domain of content knowledge unique to the work of teaching is a hypothesis that has already developed; but it is a concept that can and needs to be taken further to understand the dimensions of teachers' professional knowledge. Doing so with care promises to have significant implications for understanding teaching and for improving the content preparation of teachers.

Why are new categories, or subcategories, needed? Three reasons capture our current thinking about the usefulness of refining the conceptual map of the content knowledge needed by teachers. First, in studying the relationships between teachers' content knowledge and their student achievement, it would be useful to ascertain whether there are aspects of teacher content knowledge that predict student achievement more than others. If, for instance, teachers' pedagogical content knowledge is the greatest predictor of students' achievement, this might direct our efforts in ways different than if advanced content knowledge has the largest effect. However, such studies are sorely missing. Second, it could be useful to study whether and how different approaches to teacher development have different impacts on particular aspects of teachers' pedagogical content knowledge. Third, and closely related, a clearer sense of the categories of content knowledge for teaching might inform the design of support materials for teachers, as well as teacher education and professional development. Indeed, it might clarify a curriculum for the content preparation of teachers that is professionally based—both distinctive and substantial, and fundamentally tied to professional practice and the knowledge and skill demanded by the work.

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