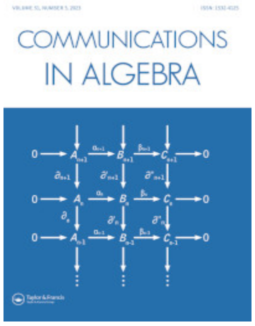

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Separation theorems in the commutative algebra of $\mathcal{C}^\infty$-rings and applications

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ABSTRACT
In this paper we state and prove *ad hoc* "Separation Theorems" of the so-called Smooth Commutative Algebra, the Commutative Algebra of $\mathcal{C}^\infty$-rings. These results are formally similar to the ones we find in (ordinary) Commutative Algebra. However, their proofs are not so straightforward, since they depend on the introduction of the key concept of "smooth saturation." As an application of these theorems we present an interesting result that sheds light on the natural connection between the smooth Zariski spectrum and smooth real spectrum of a $\mathcal{C}^\infty$-ring, the $\mathcal{C}^\infty$-analog of the real spectrum of a commutative unital ring.

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1. Introduction

It is a well-known fact that a smooth manifold M—which is a geometrical object—can be encoded by an algebraic entity, its $\mathbb{R}$-algebra of smooth real functions, $\mathcal{C}^\infty(M,\mathbb{R})$, since there is a canonical bijection (evaluation) $M \cong \mathrm{Hom}_{\mathbb{R}-\mathbf{Alg}}(\mathcal{C}^\infty(M,\mathbb{R}),\mathbb{R})$. This identification works even at the level of morphisms $\mathcal{C}^\infty(M,M') \cong \mathrm{Hom}_{\mathbb{R}-\mathbf{Alg}}(\mathcal{C}^\infty(M',\mathbb{R}),\mathcal{C}^\infty(M,\mathbb{R}))$. Moreover, geometric constructions over a manifold M, such as its tangent bundle, TM, remain algebraically represented: $TM \cong \mathrm{Hom}_{\mathbb{R}-\mathbf{Alg}}(\mathcal{C}^\infty(M,\mathbb{R}), \mathbb{R}[x]/(x^2))$.

The set $\mathcal{C}^\infty(M,\mathbb{R})$ supports a far richer structure than just the one of an $\mathbb{R}$-algebra: it interprets not only the real polynomial functions but all smooth real functions $\mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$. Moreover, this extended interpretation also satisfies all the compositional identities that hold between the smooth real functions. Thus, $\mathcal{C}^\infty(M,\mathbb{R})$ is a natural instance of the algebraic structure called $\mathcal{C}^\infty$-ring.

A more systematic algebraic study of these rings of smooth functions was carried out in the mid 1960s and early 1970s. It was not until the decades of 1970s and 1980s that a study of the abstract (algebraic) theory of $\mathcal{C}^\infty$-rings was made, mainly in order to construct—out of the ideas of F. W. Lawvere—topos models for Synthetic Differential Geometry [12, 17–19]; In [10], the first steps toward an "algebraic geometry of $\mathcal{C}^\infty$-rings" were taken, through the definition of the $\mathcal{C}^\infty$-schemes.

The interest in $\mathcal{C}^\infty$-rings gained strength in recent years [2, 8, 9, 13, 20]. In [13], D. Joyce presents the foundations of a version of Algebraic Geometry in which the role of rings is replaced by $\mathcal{C}^\infty$-rings, with the goal to apply these new notions and results to Differential Geometry: this can be considered as a part of a larger (and ambitious) program, also pursued by D. I. Spivak (see [20]), of extending/transferring Jacob Lurie's program of Derived Algebraic Geometry to Derived Differential Geometry.

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In [8], Kremnizer and Borisov give a detailed account of six notions of radicals of an ideal of a $\mathcal{C}^\infty$-ring, among which we find the ∞-radical of an ideal of a $\mathcal{C}^\infty$-ring. Here we focus on this concept, that we call " the $\mathcal{C}^\infty$-radical" of an ideal: this concept first appeared in [17], and it is germane to the theory of $\mathcal{C}^\infty$-rings, carrying some differences with respect to the usual notion of radical in ordinary Commutative Algebra—which only makes use of powers of elements.

The difference between the notions of "radical" and "$\mathcal{C}^\infty$-radical" ideals brings us, alone, a whole new study of some important concepts, such as $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-rings and the "smooth Zariski spectrum." This "smooth" depiction of Zariski's spectrum presents some crucial differences when compared to the ordinary Zariski spectrum, both in its topological features and in its functorial and sheaf-theoretic features (see [3, 15]).

This paper also provides some sketches for proofs of results scattered across literature, filling in some gaps. The main reference on Category Theory is [14]. For Commutative Algebra we refer the reader to [1].

Overview of the Paper:

In the Section 2 we provide some preliminaries on the main subject on which we build this version of Commutative Algebra, the category $\mathcal{C}^\infty$-rings, presenting some definitions and some of their fundamental constructions (the main definitions can be found in [19], and a more detailed account of $\mathcal{C}^\infty$-rings can be found, for example, in [4]).

In the Section 3 we present the $\mathcal{C}^\infty$ parallels of the ordinary Commutative Algebraic construction of the ring of fractions, radicals and the concept of saturation. In order to prove the existence of the $\mathcal{C}^\infty$-ring of fractions we make use of the notion of the $\mathcal{C}^\infty$-ring of $\mathcal{C}^\infty$-polynomials, studied in [4], and then we define the concept of "smooth saturation" (Definition 3.3 of Section 3.1), pointing some of its relationships with the ordinary concept of saturation (Theorem 3.7). We state and prove various results about this concept.

We start by recalling the ordinary Commutative Algebraic construction of the ring of fractions, radical of an ideal and the concept of saturation, motivating the introduction of their $\mathcal{C}^\infty$ parallels. In order to prove the existence of the $\mathcal{C}^\infty$-ring of fractions we make use of the notion of the $\mathcal{C}^\infty$-ring of $\mathcal{C}^\infty$-polynomials (see [4]). We study the concept of "$\mathcal{C}^\infty$-radical" introduced in [17], relating it to concept of "smooth saturation," introduced in [2], pointing some of its relationships with the ordinary concept of saturation in Commutative Algebra. We present many results about these three concepts. In Section 4 we present distinguished classes of $\mathcal{C}^\infty$-rings (i.e., $\mathcal{C}^\infty$-rings which satisfy some further axioms), as $\mathcal{C}^\infty$-fields, $\mathcal{C}^\infty$-domains and local $\mathcal{C}^\infty$-rings. We prove some results connecting the filter of closed subsets of $\mathbb{R}^n$ and the set of $\mathcal{C}^\infty$-radical ideals of $\mathcal{C}^\infty(\mathbb{R}^n)$ (in fact, a Galois connection [Proposition 4.10]), as well as various results about them—including the fact that the $\mathcal{C}^\infty$-radical of any ideal is again an ideal (Proposition 4.6). The authors know of no such proof in the current literature. From the preceding results on smooth saturation and on smooth radical ideals, we present in Section 5 a similar version of the Separation Theorems (Theorem 5.1) one finds in ordinary Commutative Algebra.

In Section 6 we present some order-theoretic aspects of $\mathcal{C}^\infty$-rings, defining fundamental concepts as the "real $\mathcal{C}^\infty$-spectrum" of a $\mathcal{C}^\infty$-ring—a concept introduced in [7]—and its topology (the "Harrison smooth topology"), establishing that every $\mathcal{C}^\infty$-ring is semi-real (Proposition 6.4) and, as an application of the Separation Theorems, we prove an important result which establishes a spectral bijection from the real $\mathcal{C}^\infty$-spectrum of a $\mathcal{C}^\infty$-ring to its smooth Zariski spectrum (Theorem 6.22): comparing this smooth algebraic scenario with the usual commutative algebraic setting, this is a surprising result.

2. Preliminaries

We provide here the main preliminary notions on $\mathcal{C}^\infty$-rings, with respect to their universal algebra (cf. [4]).

In order to formulate and study the concept of $\mathcal{C}^\infty$-ring, we use a first order language, $\mathcal{L}$, with a denumerable set of variables ($\mathrm{Var}(\mathcal{L}) = \{x_1, x_2, \ldots, x_n, \ldots\}$), whose nonlogical symbols are the symbols of $\mathcal{C}^\infty$-functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, with $m, n \in \mathbb{N}$, i.e., the non-logical symbols consist only of function symbols, described as follows:

For each $n \in \mathbb{N}$, the n-ary **function symbols** of the set $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$, i.e., $\mathcal{F}_{(n)} = \{f^{(n)} | f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})\}$. Thus, the set of function symbols of our language is given by:

$$\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_{(n)} = \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n).$$

Note that our set of constants is identified with the set of all 0-ary function symbols, i.e., $\mathcal{C} = \mathcal{F}_{(0)} = \mathcal{C}^\infty(\mathbb{R}^0) \cong \mathcal{C}^\infty(\{*\})$.

The terms of this language are defined, in the usual way, as the smallest set which comprises the individual variables, constant symbols and n-ary function symbols followed by n terms ($n \in \mathbb{N}$).

Functorially, a (set-theoretic) $\mathcal{C}^\infty$-ring is a finite product preserving functor from the category $\mathcal{C}^\infty$, whose objects are of the form $\mathbb{R}^n$, $n \in \mathbb{N}$, and whose morphisms are the smooth functions between them, i.e., a finite product preserving functor:

$$A : \mathcal{C}^\infty \to \mathbf{Set}$$

Apart from the functorial definition and the "first-order language" definition we just gave, there are many equivalent descriptions. We focus, first, on the universal-algebraic description of a $\mathcal{C}^\infty$-ring in **Set**, given in the following:

Definition 2.1. A $\mathcal{C}^\infty$-**structure** on a set A is a pair $\mathfrak{A} = (A, \Phi)$, where:

$$\begin{array}{rccc} \Phi : & \bigcup_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}) & \to & \bigcup_{n \in \mathbb{N}} \mathrm{Func}\,(A^n; A) \\ & (f : \mathbb{R}^n \stackrel{\mathcal{C}^\infty}{\to} \mathbb{R}) & \mapsto & \Phi(f) := (f^A : A^n \to A) \end{array},$$

that is, Φ interprets the **symbols**[1] of all smooth real functions of n variables as n-ary function symbols on A.

We call a $\mathcal{C}^\infty$-structure $\mathfrak{A} = (A, \Phi)$ a $\mathcal{C}^\infty$-**ring** whenever it preserves projections and all equations between smooth functions. More precisely, we have the following:

Definition 2.2. Let $\mathfrak{A} = (A, \Phi)$ be a $\mathcal{C}^\infty$-structure. We say that $\mathfrak{A}$ (or, when there is no danger of confusion, A) is a $\mathcal{C}^\infty$-**ring** if the following is true:

- Given any $n, k \in \mathbb{N}$ and any projection $p_k : \mathbb{R}^n \to \mathbb{R}$, we have:

$$\mathfrak{A} \models (\forall x_1) \cdots (\forall x_n)(p_k(x_1, \ldots, x_n) = x_k).$$

- For every $f, g_1, \ldots g_n \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R})$ with $m, n \in \mathbb{N}$, and every $h \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ such that $f = h \circ (g_1, \ldots, g_n)$, one has:

$$\mathfrak{A} \models (\forall x_1) \cdots (\forall x_m)(f(x_1, \ldots, x_m) = h(g(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))).$$

Definition 2.3. Let (A, Φ) and (B, Ψ) be two $\mathcal{C}^\infty$-rings. A function $\varphi : A \to B$ is called a **morphism of** $\mathcal{C}^\infty$-**rings** or $\mathcal{C}^\infty$-**homomorphism** if for any $n \in \mathbb{N}$ and any $f : \mathbb{R}^n \stackrel{\mathcal{C}^\infty}{\to} \mathbb{R}$, one has $\Psi(f) \circ \varphi^{(n)} = \varphi \circ \Phi(f)$, where $\varphi^{(n)} = (\varphi, \ldots, \varphi) : A^n \to B^n$.

Remark 2.4. (on universal algebraic constructions) It is not difficult to see that $\mathcal{C}^\infty$-structures, together with their morphisms (which we call $\mathcal{C}^\infty$-morphisms) compose a category (see Theorems 1 and 2 of [4]), that we denote by $\mathcal{C}^\infty$**Str**, and that $\mathcal{C}^\infty$-rings, together with all the $\mathcal{C}^\infty$-morphisms between $\mathcal{C}^\infty$-rings (which we call $\mathcal{C}^\infty$-homomorphisms) compose a full subcategory, $\mathcal{C}^\infty$**Ring**. In particular, since

[1] Here considered simply as syntactic symbols rather than functions.

$\mathcal{C}^\infty$**Ring** is a "variety of algebras" (see Remark 5 of [4]), i.e., it is a class of $\mathcal{C}^\infty$-structures which satisfies a given set of equations, (or equivalently, by **Birkhoff's HSP Theorem**) it is closed under substructures (Definition 8 of [4]), homomorphic images (Proposition 7 of [4]) and products (Definition 10 of [4]). Moreover:

• $\mathcal{C}^\infty$**Ring** is a concrete category and the forgetful functor, $U : \mathcal{C}^\infty\mathbf{Ring} \to \mathbf{Set}$ creates directed inductive colimits (see p. 22 of [19]);

• Each set X freely generates a $\mathcal{C}^\infty$-ring (see Section 3 of [4]). In particular, the free $\mathcal{C}^\infty$-ring on n generators is $\mathcal{C}^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$ (see Proposition 1.1 of [19]);

• Every $\mathcal{C}^\infty$-ring is the homomorphic image of some free $\mathcal{C}^\infty$-ring determined by some set, being isomorphic to the quotient of a free $\mathcal{C}^\infty$-ring by some congruence;

• The congruences of $\mathcal{C}^\infty$-rings are classified by their "ring-theoretical" ideals (Proposition 17 of [4]);

• In $\mathcal{C}^\infty$**Ring** one defines "the $\mathcal{C}^\infty$-coproduct" between two $\mathcal{C}^\infty$-rings $\mathfrak{A} = (A, \Phi)$ and $\mathfrak{B} = (B, \Psi)$, denoted by $A \otimes_\infty B$ (see Section 4.2 of [4]);

• Using free $\mathcal{C}^\infty$-rings and the $\mathcal{C}^\infty$-coproduct, one gets the "$\mathcal{C}^\infty$-ring of polynomials" on any set S of variables with coefficients in A, given by $A\{x_s \mid s \in S\} = A \otimes_\infty \mathcal{C}^\infty(\mathbb{R}^S)$ (see Section 4.3 of [4]).

3. Smooth rings of fractions

We begin by giving a description of the fundamental concept of "smooth ring of fractions", presenting a slight modification of the axioms given in [17]. In order to show that the $\mathcal{C}^\infty$-ring of fractions exists in the category of $\mathcal{C}^\infty$-rings, we use the $\mathcal{C}^\infty$-ring of $\mathcal{C}^\infty$-polynomials. The definitions and results we state here may be found with more detail in [5].

We turn to the discussion of how to obtain the ring of fractions of a $\mathcal{C}^\infty$-ring A with respect to some of its subsets, $S \subseteq U(A, \Phi)$.

For any commutative unital ring R, one characterizes $S^{-1}R$ by the following universal property (cf. Proposition 3.1 of [1]):

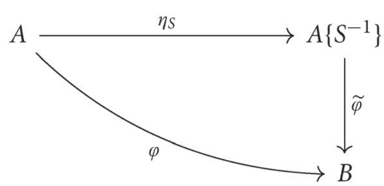
Proposition 3.1. *Given a ring homomorphism $g : R \to B$ such that $(\forall s \in S)(g(s) \in B^\times)$, there is a unique ring homomorphism $\widetilde{g} : S^{-1}R \to B$ such that $\widetilde{g} \circ \eta_S = g$.*

In order to extend this notion to the category $\mathcal{C}^\infty$**Ring** we make use of the universal property described in Proposition 3.1, as we see in the following:

Definition 3.2. Let A be a $\mathcal{C}^\infty$-ring and $S \subseteq A$ one of its subsets. The $\mathcal{C}^\infty$-**ring of fractions** of A with respect to S is a $\mathcal{C}^\infty$-ring, $A\{S^{-1}\}$, together with a $\mathcal{C}^\infty$-homomorphism $\eta_S : A \to A\{S^{-1}\}$ satisfying the following properties:

(1) $(\forall s \in S)(\eta_S(s) \in (A\{S^{-1}\})^\times)$

(2) If $\varphi : A \to B$ is any $\mathcal{C}^\infty$-homomorphism such that for every $s \in S$ we have $\varphi(s) \in B^\times$, then there is a unique $\mathcal{C}^\infty$-homomorphism $\widetilde{\varphi} : A\{S^{-1}\} \to B$ such that the following triangle commutes:



By this universal property, the $\mathcal{C}^\infty$-ring of fractions is unique, up to (unique) isomorphisms.

The existence of such a $\mathcal{C}^\infty$-ring of fractions is proved making use of the constructions available within the category $\mathcal{C}^\infty$**Ring**, such as the free $\mathcal{C}^\infty$-ring on a set of generators, their coproduct, their quotients and others described in [4].

3.1. Smooth saturation

A key concept for Commutative Algebra was introduced by A. Grothendieck and J. Dieudonné in [11], namely the concept of "a saturated (multiplicative) set": given a commutative unital ring, A, and $S \subseteq A$, the saturation of S was defined by $S^{\mathrm{sat}} = \{a \in A | (\exists d \in A)(a \cdot d \in \langle S \rangle)\}$, where $\langle S \rangle$ denotes the multiplicative submonoid of A generated by S. In other words, the saturation of a set S is the set of all divisors of elements in $\langle S \rangle$.

One easily checks that the saturation of a subset S of a commutative ring A equals the pre-image of the set of all invertible elements of $A[S^{-1}]$ by the canonical map $\eta_S : A \to A\{S^{-1}\}$, i.e., $S^{\mathrm{sat}} = \eta_S^{\dashv}[(A[S^{-1}])^{\times}]$. We use this specific characterization in order to introduce the concept of "the *smooth* saturation of a subset S of a $\mathcal{C}^{\infty}$-ring A," that we are going to denote by $S^{\infty-\mathrm{sat}}$.

Definition 3.3. Let A be a $\mathcal{C}^{\infty}$-ring, $S \subseteq A$ and (F, σ) be a ring of fractions of A with respect to S. The **smooth saturation** of S in A is:

$$S^{\infty-\mathrm{sat}} := \{a \in A | \sigma(a) \in F^{\times}\}.$$

It is straightforward to check that the smooth saturation of a subset of A does not depend on any particular choice of the "inverting" $\mathcal{C}^{\infty}$-homomorphisms, that is, if both (F, σ) and (F', σ') satisfy the Definition 3.2, then $\sigma^{\dashv}[F^{\times}] = \sigma'^{\dashv}[F'^{\times}]$ (for a proof, see Proposition 3.6, p. 14 of [5]).

Remark 3.4. Since for every $s \in S$, $\eta_S(s) \in (A\{S^{-1}\})^{\times}$, from now on we use the more suggestive notation:

$$(\forall s \in S) \left(1/\eta_S(s) \doteq \eta_S(s)^{-1}\right),$$

For any $a \in A$ and $s \in S$ we write:

$$\frac{\eta_S(a)}{\eta_S(s)} \doteq \eta_S(a) \cdot \eta_S(s)^{-1}.$$

Alternatively, but equivalently, we have:

Definition 3.5. Let A be a $\mathcal{C}^{\infty}$-ring and let $S \subseteq A^{\times}$ be any subset. The **smooth saturation of** S is $S^{\infty-\mathrm{sat}} = \eta_S{}^{\dashv}[A\{S^{-1}\}^{\times}]$, where $\eta_S : A \to A\{S^{-1}\}$ is the canonical map of the ring of fractions of A with respect to S.

Remark 3.6. Let A be a $\mathcal{C}^{\infty}$-ring, $S \subseteq A$ and consider the forgetful functor, $\mathcal{U} : \mathcal{C}^{\infty}\mathbf{Ring} \to \mathbf{CRing}$. We have always $S^{\mathrm{sat}} \subseteq S^{\infty-\mathrm{sat}}$.

Theorem 3.7 (Theorem 3.11 of [5]). *Since $\eta_S^{\infty} : A \to A\{S^{-1}\}$ is such that $\eta_S^{\infty}[S] \subseteq (A\{S^{-1}\})^{\times}$, then $(\eta_S^{\infty})[S] \subseteq (\mathcal{U}(A\{S^{1}\}))^{\times}$, so by the universal property of the ring of fractions, $\eta_S : \mathcal{U}(A) \to \mathcal{U}(A)[S^{-1}]$, there is a unique $\mathbb{R}$-algebras homomorphism* Can $: \mathcal{U}(A)[S^{-1}] \to \mathcal{U}(A\{S^{-1}\})$ *such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\eta_S} & \mathcal{U}(A)[S^{-1}] \\ & \searrow^{\mathcal{U}(\eta_S^{\infty})} & \downarrow \mathrm{Can} \\ & & \mathcal{U}(A\{S^{-1}\}) \end{array}$$

In these settings, the following assertions are equivalent:
(1) $S^{\mathrm{sat}} = S^{\infty-\mathrm{sat}}$;
(2) Can *is a ring-isomorphism.*

Now we give some properties relating the inclusion relation among the subsets of a $\mathcal{C}^{\infty}$-ring and their smooth saturations.

Proposition 3.8 (Proposition 3.12 of [5]). *Let A be a $\mathcal{C}^{\infty}$-ring and $T, S \subseteq A$ be any two of its subsets. Then:*
(i) $A^{\times} \subseteq S^{\infty-\mathrm{sat}}$
(ii) $S \subseteq S^{\infty-\mathrm{sat}}$
(iii) $S \subseteq T$ *implies* $S^{\infty-\mathrm{sat}} \subseteq T^{\infty-\mathrm{sat}}$ *and*
(iv) $S^{\infty-\mathrm{sat}} = \langle S\rangle^{\infty-\mathrm{sat}}$, *where $\langle S\rangle$ is the submonoid generated by S.*

Some necessary and sufficient conditions for the $\mathcal{C}^{\infty}$-homomorphism $\eta_S : A \to A\{S^{-1}\}$ to be a $\mathcal{C}^{\infty}$-isomorphism are given below:

Proposition 3.9. *The following assertions are equivalent:*
(i) $\eta_S : A \to A\{S^{-1}\}$ *is a $\mathcal{C}^{\infty}$-isomorphism;*
(ii) $S^{\infty-\mathrm{sat}} \subseteq A^{\times}$;
(iii) $S^{\infty-\mathrm{sat}} = A^{\times}$

Proof. Naturally $(i) \Rightarrow (ii)$, for if η_S is an isomorphism, both η_S^{-1} and η_S preserve invertible elements and $A^{\times} = \eta_S^{-1}[(A\{S^{-1}\})^{\times}] = \eta_S^{-1}[(A\{S^{-1}\})^{\times}] = S^{\mathrm{sat}}$. Since we always have $A^{\times} \subseteq S^{\infty-\mathrm{sat}}$, (ii) implies that $A^{\times} = S^{\infty-\mathrm{sat}}$, so $(ii) \Rightarrow (iii)$.

Finally, note that $\mathrm{id}_A : A \to A$ has the universal property of the ring of fractions of A with respect to $A^{\times}$, so it follows that $(\mathrm{id}_A : A \to A) \cong (\eta_{A^{\times}} : A \to A\{(A^{\times})^{-1}\})$. Thus $\eta_{A^{\times}}$ must be the composition of a $\mathcal{C}^{\infty}$-isomorphism with id_A, thus a $\mathcal{C}^{\infty}$-isomorphism.

□

Observe that, by Proposition 3.2, we have $A^{\times} = \eta_{A^{\times}}[(A\{(A^{\times})^{-1}\})^{\times}] = (A^{\times})^{\infty-\mathrm{sat}}$ – that is, $A^{\times}$ is a "$\mathcal{C}^{\infty}$-saturated set."

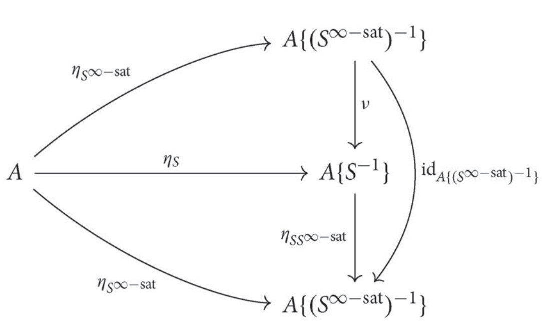
Next we give some properties of the smooth saturation of a set.

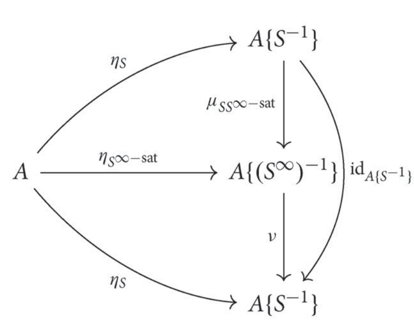
Proposition 3.10. *Let A be a $\mathcal{C}^{\infty}$-ring and $S \subseteq A$ be any of its subsets. Then*

$$(S^{\infty-\mathrm{sat}})^{\infty-\mathrm{sat}} = S^{\infty-\mathrm{sat}}$$

Proof. Since $S \subseteq S^{\infty-\mathrm{sat}}$, there exists a unique morphism $\mu_{SS^{\infty-\mathrm{sat}}} : A\{S^{-1}\} \to A\{(S^{\infty-\mathrm{sat}})^{-1}\}$ such that $\mu_{SS^{\infty-\mathrm{sat}}} \circ \eta_S = \eta_{S^{\infty-\mathrm{sat}}}$. By the definition of $S^{\infty-\mathrm{sat}}$, we have $\eta_S[S^{\infty-\mathrm{sat}}] \subseteq (A\{S^{-1}\})^{\times}$, and by the universal property of $\eta_{S^{\infty-\mathrm{sat}}} : A \to A\{(S^{\infty-\mathrm{sat}})^{-1}\}$, there must exist a unique $\nu : A\{(S^{\infty-\mathrm{sat}})^{-1}\} \to A\{S^{-1}\}$ such that $\nu \circ \eta_{S^{\infty-\mathrm{sat}}} = \eta_S$.

Thus, we get two commuting diagrams:





so $(\mu_{SS^{\infty-\mathrm{sat}}})^{-1} = \nu$, and $\mu_{SS^{\infty-\mathrm{sat}}}$ is an isomorphism. Hence $A\{S^{-1}\} \cong A\{(S^{\infty-\mathrm{sat}})^{-1}\}$, and by Definition 3.2, $(S^{\infty-\mathrm{sat}})^{\infty-\mathrm{sat}} = S^{\infty-\mathrm{sat}}$. □

Proposition 3.11. *If $g, h : A\{S^{-1}\} \to B$ are two morphisms such that $g \circ \eta_S = h \circ \eta_S$ then $g = h$. In other words, $\eta_S : A \to A\{S^{-1}\}$ is an epimorphism.*

Proof. Since $g \circ \eta_S$ is such that $(g \circ \eta_S)[S] \subseteq B^\times$, there exists a unique morphism $\widetilde{t} : A\{S^{-1}\} \to B$ such that $\widetilde{t} \circ \eta_S = g \circ \eta_S$. By hypothesis we have $h \circ \eta_S = g \circ \eta_S$, so g has the property which determines $\widetilde{t}$ and $g = \widetilde{t} = h$. □

Proposition 3.12. *Let A be a $\mathcal{C}^\infty$-ring and $S, T \subseteq A$ two of its subsets. The following assertions are equivalent:*
(i) $S^{\infty-\mathrm{sat}} \subseteq T^{\infty-\mathrm{sat}}$
(ii) There is a unique morphism $\mu : A\{S^{-1}\} \to A\{T^{-1}\}$ such that $\mu \circ \eta_S = \eta_T$.

Proof. See Proposition 3.16 of [5]. □

Corollary 3.13. *The following assertions are equivalent:*
(i) $S^{\infty-\mathrm{sat}} = T^{\infty-\mathrm{sat}}$
(ii) There is an isomorphism $\mu : A\{S^{-1}\} \to A\{T^{-1}\}$ such that $\mu \circ \eta_S = \eta_T$;
(iii) There is a <u>unique</u> isomorphism $\mu : A\{S^{-1}\} \to A\{T^{-1}\}$ such that $\mu \circ \eta_S = \eta_T$.

Proof. See Corollary 3.17 of [5]. □

Proposition 3.14. *Let A be a $\mathcal{C}^\infty$-ring and S, T two of its subsets such that $S \subseteq T$. The following assertions are equivalent:*
(i) $\mu_{ST} : A\{S^{-1}\} \to A\{T^{-1}\}$ is a $\mathcal{C}^\infty$-isomorphism;
(ii) $S \subseteq T \subseteq S^{\infty-\mathrm{sat}}$;
(iii) $T^{\infty-\mathrm{sat}} = S^{\infty-\mathrm{sat}}$.

Proof. Ad (ii) → (iii): Since $S \subseteq T$ we have $S^{\infty-\mathrm{sat}} \subseteq T^{\infty-\mathrm{sat}}$, and since $T \subseteq S^{\infty-\mathrm{sat}}$ we have $T^{\infty-\mathrm{sat}} \subseteq (S^{\infty-\mathrm{sat}})^{\infty-\mathrm{sat}} = S^{\infty-\mathrm{sat}}$. Thus $S^{\infty-\mathrm{sat}} \subseteq T^{\infty-\mathrm{sat}} \subseteq S^{\infty-\mathrm{sat}}$ and $S^{\infty-\mathrm{sat}} = T^{\infty-\mathrm{sat}}$. Assuming (*iii*), note that we always have $T \subseteq T^{\mathrm{sat}}$, and since $T^{\mathrm{sat}} = S^{\mathrm{sat}}$, it follows that $T \subseteq S^{\mathrm{sat}}$ – so (*ii*) holds. Finally, the equivalence (*i*) $\iff$ (*iii*) was established in Corollary 3.13. □

Proposition 3.15. *Let A be a $\mathcal{C}^\infty$-ring and $S \subseteq A$. Whenever $\{S_i\}_{i \in I}$ is a directed system such that:*

$$S = \bigcup_{i \in I} S_i$$

we have:

$$S^{\infty-\mathrm{sat}} = \left(\bigcup_{i \in I} S_i\right)^{\infty-\mathrm{sat}} = \bigcup_{i \in I} {S_i}^{\infty-\mathrm{sat}}$$

Proof. It is clear that $\bigcup_{i \in I} {S_i}^{\infty-\mathrm{sat}} \subseteq S^{\infty-\mathrm{sat}}$. Note that $A\{S^{-1}\}$ is isomorphic to the vertex of the following directed colimit:

$$\begin{array}{ccc} & \varinjlim_{i\in I} A\{S_i{}^{-1}\} & \\ {}^{\alpha_i}\nearrow & & \nwarrow^{\alpha_j} \\ A\{S_i{}^{-1}\} & \xrightarrow{\quad\alpha_{ij}\quad} & A\{S_j{}^{-1}\} \end{array}$$

and that $\eta_S : A \to A\{S^{-1}\}$ is such that for any $i \in I$, $\eta_S[S_i] \subseteq \eta_S[S] \subseteq (A\{S^{-1}\})^\times$, so by the universal property of $\eta_{S_i} : A \to A\{(S_i)^{-1}\}$, there is a unique $\mathcal{C}^\infty$-rings homomorphism $\varphi_i : A\{(S_i)^{-1}\} \to A\{S^{-1}\}$ such that $\varphi_i \circ \eta_{S_i} = \eta_S$, so

$$\begin{array}{ccc} & A\{S^{-1}\} & \\ {}^{\varphi_i}\nearrow & & \nwarrow^{\varphi_j} \\ A\{S_i{}^{-1}\} & \xrightarrow{\quad\varphi_{ij}\quad} & A\{S_j{}^{-1}\} \end{array}$$

commutes for every $i, j \in I$ such that $i \leq j$ (for $\eta_{S_i} : A \to A\{S_i{}^{-1}\}$ is an epimorphism). By the universal property of the colimit, there is a unique $\mathcal{C}^\infty$-homomorphism $\varphi : \varinjlim_{i\in I} A\{S_i{}^{-1}\} \to A\{S^{-1}\}$ such that $(\forall i \in I)(\varphi \circ \alpha_i = \varphi_i)$.

On the other hand, given $s \in S = \bigcup_{i\in I} S_i$ there is $i \in I$ such that $s \in S_i$, so $\eta_{S_i}(s) \in A\{S_i{}^{-1}\}^\times$. Taking $\widetilde{\eta} := \alpha_i \circ \eta_{S_i} : A \to \varinjlim_{i\in I} A\{S_i{}^{-1}\}$, we have $\widetilde{\eta}(s) \in \left(\varinjlim_{i\in I} A\{S_i{}^{-1}\}\right)^\times$. By the universal property of $\eta_S : A \to A\{S^{-1}\}$, there is a unique $\mathcal{C}^\infty$-homomorphism $\psi : A\{S^{-1}\} \to \varinjlim_{i\in I} A\{S_i{}^{-1}\}$ such that $\psi \circ \eta_S = \widetilde{\eta}$. Now, it is easy to see, by the universal properties involved, that φ and ψ are inverse $\mathcal{C}^\infty$-isomorphisms and that $\widetilde{\eta}^{\dashv}\left[\left(\varinjlim_{i\in I} A\{S_i{}^{-1}\}\right)^\times\right] = \bigcup_{i\in I} S_i{}^{\infty-\mathrm{sat}}$., so $S = \bigcup_{i\in I} S_i{}^{\infty-\mathrm{sat}}$. □

Our next goal is to give a characterization of ring of fractions in $\mathcal{C}^\infty$**Ring** using a similar axiomatization one has in Commutative Algebra. In order to motivate it, we first present an important characterization of the ring of fractions in **CRing**.

Fact 3.16 (Theorem 3.23 of [5]). *Let A be a commutative ring with unity and $S \subseteq A$. Then $\varphi : A \to B$ is isomorphic to the localization map $\eta : A \to A[S^{-1}]$ if and only if:*
(i) $(\forall b \in B)(\exists c \in S)(\exists d \in A)(b \cdot \varphi(c) = \varphi(d))$
(ii) $(\forall b \in A)(\varphi(b) = 0 \to (\exists c \in S)(c \cdot b = 0))$
hold.

For $\mathcal{C}^\infty$-rings we have the analogous result, that generalizes Theorem 1.4 of [17], in the sense that it is an equivalence (an "if and only if" statement) and that S needs not to be a singleton:

Theorem 3.17. *Let A be a $\mathcal{C}^\infty$-ring $\Sigma \subseteq A$ a set. Then $\varphi : A \to B$ is isomorphic to the smooth localization $\eta_\Sigma : A \to A\{\Sigma^{-1}\}$ if and only if:*
(i) $(\forall b \in B)(\exists c \in \Sigma^{\infty-\mathrm{sat}})(\exists d \in A)(b \cdot \varphi(c) = \varphi(d))$
(ii) $(\forall b \in A)(\varphi(b) = 0 \to (\exists c \in \Sigma^{\infty-\mathrm{sat}})(c \cdot b = 0))$
hold.

We postpone the proof of this theorem, giving it right after Remark 3.24.

Theorem 3.18. *Let $A, \widetilde{A}$ be $\mathcal{C}^\infty$-rings, $\Sigma \subseteq A$ and let $\eta : A \to \widetilde{A}$ be a $\mathcal{C}^\infty$-rings homomorphism such that:*
(i) $(\forall d \in \widetilde{A})(\exists b \in A)(\exists c \in A)(\eta_\Sigma(c) \in \widetilde{A}^\times \& (d \cdot \eta(c) = \eta_\Sigma(b)))$;
(ii) $(\forall b \in A)((\eta_\Sigma(b) = 0_{\widetilde{A}}) \to (\exists c \in A)((\eta_\Sigma(c) \in \widetilde{A}^\times) \& (b \cdot c = 0_A)))$
Then $\eta_\Sigma : A \to \widetilde{A}$ is isomorphic to $\mathrm{Can}_{S_\eta} : A \to A\{S_\eta{}^{-1}\}$, where $S_\eta = \eta_\Sigma^{\dashv}[\widetilde{A}^\times]$.

Proof. First we show that $\eta_\Sigma : A \to \widetilde{A}$ has the universal property which characterize Can_{S_η}.

Let $f : A \to B$ be a $\mathcal{C}^\infty$-rings homomorphism such that $f[S_\eta] \subseteq B^\times$. We are going to show there is a unique $\mathcal{C}^\infty$-rings homomorphism $\widetilde{f} : \widetilde{A} \to B$ such that $\widetilde{f} \circ \eta_\Sigma = f$.

Note that $\eta_\Sigma[S_\eta] = \eta_\Sigma[\eta_\Sigma^{\dashv}[\widetilde{A}^\times]] \subseteq \widetilde{A}^\times$.

Candidate and Uniqueness of $\widetilde{f}$: Let $\widetilde{f_1}, \widetilde{f_2} : \widetilde{A} \to B$ be such that $\widetilde{f_1} \circ \eta_\Sigma = f = \widetilde{f_2} \circ \eta$.

From hypothesis (i), given any $d \in \widetilde{A}$ there must exist $b, c \in A$ with $\eta_\Sigma(c) \in \widetilde{A}^\times$, such that $d = \eta(b)/\eta(c)$, so:

$$\widetilde{f_1}(d) = \widetilde{f_1}(\eta(b) \cdot \eta(c)^{-1}) = \widetilde{f_1}(\eta(b)) \cdot \widetilde{f_1}(\eta(c))^{-1} =$$
$$= (\widetilde{f_1} \circ \eta)(b) \cdot ((\widetilde{f_1} \circ \eta)(c))^{-1} = f(b) \cdot f(c)^{-1} = (\widetilde{f_2} \circ \eta)(b) \cdot ((\widetilde{f_2} \circ \eta)(c))^{-1} =$$
$$= \widetilde{f_2}(\eta(b)) \cdot \widetilde{f_2}(\eta(c)^{-1}) = \widetilde{f_2}(\eta(b) \cdot \eta(c)^{-1}) = \widetilde{f_2}(d)$$

Thus $\widetilde{f_1} = \widetilde{f_2}$.

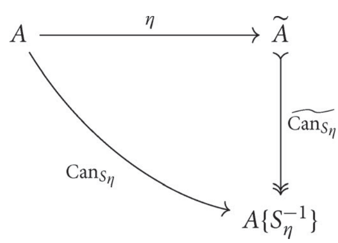
Existence of $\widetilde{f}$: We know that for every $d \in \widetilde{A}$ there are $b, c \in A$, $\eta(c) \in \widetilde{A}^\times$, such that $d = \eta(b) \cdot \eta(c)^{-1}$. We define the following relation: $\widetilde{f} = \{(d, f(b) \cdot f(c)^{-1}) | d \in \widetilde{A}\} \subseteq \widetilde{A} \times B$, which can be proved to be a function.

Therefore, there exists exactly one function $\widetilde{f} : \widetilde{A} \to B$ such that $\widetilde{f} \circ \eta = f$.

Note that, by the very definition of Can_{S_η}, $\mathrm{Can}_{S_\eta}[S_\eta] \subseteq A\{S_\eta^{-1}\}^\times$ so there is a unique function $\widetilde{\mathrm{Can}_{S_\eta}} : \widetilde{A} \to A\{S_\eta^{-1}\}$ such that $\widetilde{\mathrm{Can}_{S_\eta}} \circ \eta = \mathrm{Can}_{S_\eta}$. Now, from the universal property of Can_{S_η} there exists a unique $\mathcal{C}^\infty$-ring homomorphism, $\widehat{\eta} : A\{S_\eta^{-1}\} \to \widetilde{A}$, such that $\widehat{\eta} \circ \mathrm{Can}_{S_\eta} = \eta$.

We claim that $\widehat{\eta}$ is a bijection whose inverse is $\widetilde{\mathrm{Can}_{S_\eta}}$. In fact, $(\widehat{\eta} \circ \widetilde{\mathrm{Can}_{S_\eta}}) \circ \eta = \widehat{\eta} \circ (\widetilde{\mathrm{Can}_{S_\eta}} \circ \eta) = \widehat{\eta} \circ \mathrm{Can}_{S_\eta} = \eta = \mathrm{id}_{\widetilde{A}} \circ \eta$, so $(\widehat{\eta} \circ \widetilde{\mathrm{Can}_{S_\eta}}) \circ \eta = \mathrm{id}_{\widetilde{A}} \circ \eta$. We have seen, however, that there is exactly one function $\widetilde{\varphi}$ such that $\widetilde{\varphi} \circ \eta = \eta$, so it follows that $\mathrm{id}_{\widetilde{A}} = \widehat{\eta} \circ \widetilde{\mathrm{Can}_{S_\eta}}$.

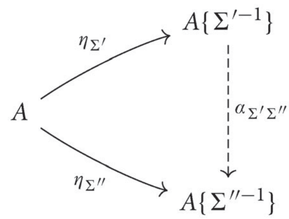
On the other hand, $\widetilde{\mathrm{Can}_{S_\eta}} \circ \widehat{\eta} \circ \mathrm{Can}_{S_\eta} = \widetilde{\mathrm{Can}_{S_\eta}} \circ \eta = \mathrm{Can}_{S_\eta} = \mathrm{id}_{A\{S_\eta^{-1}\}} \circ \mathrm{Can}_{S_\eta}$, so by the universal property of Can_{S_η} we have $\mathrm{id}_{A\{S_\eta^{-1}\}} = \widetilde{\mathrm{Can}_{S_\eta}} \circ \widehat{\eta}$. Hence $\widetilde{\mathrm{Can}_{S_\eta}}$ is the $\mathcal{C}^\infty$-rings isomorphism between η and Can_{S_η}, that is, it is a $\mathcal{C}^\infty$-rings isomorphism such that the following diagram commutes:



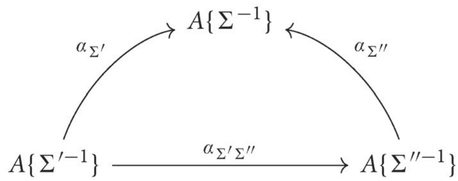
□

In order to smoothly invert larger subsets of $\mathcal{C}^\infty(\mathbb{R}^n)$ for some $n \in \mathbb{N}$, say $\Sigma \subseteq \mathcal{C}^\infty(\mathbb{R}^n)$, which is a set that contains possibly a non-countable amount of elements, we proceed as follows. First notice that we can obtain the $\mathcal{C}^\infty$-ring of fractions of $\mathcal{C}^\infty(\mathbb{R}^n)$ with respect to the singleton $\Sigma = \{f : \mathbb{R}^n \to \mathbb{R}\}$, provided that $f \not\equiv 0$ – as originally presented by I. Moerdijk and G. Reyes in [17]. Whenever $\Sigma = \{f_1, \ldots, f_k\}$ for some $k \in \mathbb{N}$, inverting Σ is equivalent to inverting $\prod \Sigma = f_1 \cdot f_2 \cdots f_{k-1} \cdot f_k$. Now, if Σ is infinite, we first decompose it as the union of its finite subsets $\Sigma = \bigcup_{\Sigma' \subseteq_{\mathrm{fin}} \Sigma} \Sigma'$.

Note that $\mathcal{S} = \{\Sigma' \subseteq \Sigma | \Sigma' \text{ is finite}\}$ is partially ordered by the inclusion relation. Also, whenever $\Sigma' \subseteq \Sigma''$, since $\eta_{\Sigma''}[\Sigma'] \subseteq \eta_{\Sigma''}[\Sigma''] \subseteq (A\{\Sigma''^{-1}\})^\times$, it follows from the universal property of $\eta_{\Sigma''} : A \to A\{\Sigma''^{-1}\}$, that there is a unique $\mathcal{C}^\infty$-homomorphism $\alpha_{\Sigma'\Sigma''} : A\{\Sigma'^{-1}\} \to A\{\Sigma''^{-1}\}$ such that the following diagram commutes:



It is simple to prove, using the "uniqueness part" of the $\mathcal{C}^\infty$-homomorphism obtained via universal property, that for any finite Σ' we have $\alpha_{\Sigma'\Sigma'} = \mathrm{id}_{A\{\Sigma'^{-1}\}}$, and that given any finite Σ', Σ'' and Σ''' such that $\Sigma' \subseteq \Sigma'' \subseteq \Sigma'''$, $\alpha_{\Sigma''\Sigma'''} \circ \alpha_{\Sigma'\Sigma''} = \alpha_{\Sigma'\Sigma'''}$. Thus we have an inductive system $\{\alpha_{\Sigma'\Sigma''} : A\{\Sigma'^{-1}\} \to A\{\Sigma''^{-1}\} | (\Sigma', \Sigma'' \in \mathcal{S}) \& (\Sigma' \subseteq \Sigma'')\}$. We take, then, the colimit of this system $A\{\Sigma^{-1}\} = \varinjlim_{\Sigma' \subseteq_{\mathrm{fin}} \Sigma} A\{\Sigma'^{-'}\} = \varinjlim_{\Sigma' \subseteq_{\mathrm{fin}} \Sigma} A\left\{\prod \Sigma'^{-1}\right\}$:



Thus, given any $\mathcal{C}^\infty$-ring A and any $\Sigma \subseteq A$, we take $\eta_\Sigma = \alpha_{\Sigma'} \circ \eta_{\Sigma'} : A \to A\{\Sigma^{-1}\}$, which can be easily proved to have the universal property that characterizes the $\mathcal{C}^\infty$-ring of fractions of A with respect to Σ.

The concept of $\mathcal{C}^\infty$-saturation of a subset of a $\mathcal{C}^\infty$-ring enables us to state the Proposition 1.2 of [17] in simple terms, as follows:

Remark 3.19. In the case that $A = \mathcal{C}^\infty(\mathbb{R}^n)$ and $\Sigma = \{f : \mathbb{R}^n \to \mathbb{R}\}$, we have $\Sigma^{\infty-\mathrm{sat}} = \{g \in \mathcal{C}^\infty(\mathbb{R}^n) | U_f \subseteq U_g\} = \{g \in \mathcal{C}^\infty(\mathbb{R}^n) | Z(g) \subseteq Z(f)\}$.

I. Moerdijk and G. Reyes show (for example, in [17]) that given $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$, denoting $U_\varphi = \{x \in \mathbb{R}^n | \varphi(x) \neq 0\} = \mathrm{Coz}(\varphi)$ we have $\mathcal{C}^\infty(U_\varphi) \cong \mathcal{C}^\infty(\mathbb{R}^{n+1})/\langle\{y \cdot \varphi(x) - 1\}\rangle$.

We now state a result, credited to Ortega and Muñoz by I. Moerdijk and G. Reyes in [17], that shall be used in the sequel:

Proposition 3.20. *Let $U \subseteq \mathbb{R}^n$ be open, and $g \in \mathcal{C}^\infty(U)$. Then there are $h, k \in \mathcal{C}^\infty(\mathbb{R}^n)$ with $U_k = U$ and $g \cdot k \restriction_U \equiv h \restriction_U$. where $U_k = \mathbb{R}^n \setminus Z(k)$ and $Z(k) = \{x \in \mathbb{R}^n | k(x) = 0\}$.*

Now we turn to prove a very useful result which describes, in detail, the elements of $A\{S^{-1}\}$.

Theorem 3.21. *Let A be a $\mathcal{C}^\infty$-ring and $S \subseteq A$. An element $\lambda = \eta_S(c)/\eta_S(b)$ (with $c \in A$ and $b \in S^{\infty-\mathrm{sat}}$) is invertible in $A\{S^{-1}\}$ if, and only if, there are elements $d \in S^{\infty-\mathrm{sat}}$ and $c' \in A$ such that $dc'c \in S^{\infty-\mathrm{sat}}$, that is,*

$$\frac{\eta_S(c)}{\eta_S(b)} \in (A\{S^{-1}\})^\times \iff (\exists d \in S^{\infty-\mathrm{sat}})(\exists c' \in A)(d \cdot c' \cdot c \in S^{\infty-\mathrm{sat}}).$$

Proof. Suppose $\eta_S(c)/\eta_S(b) \in (A\{S^{-1}\})^\times$, so there are $c' \in A$ and $b' \in S^{\infty-\mathrm{sat}}$ such that:

$$\frac{\eta_S(c)}{\eta_S(b)} \cdot \frac{\eta_S(c')}{\eta_S(b')} = 1_{A\{S^{-1}\}} = \eta_S(1_A).$$

Thus,

$$\eta_S(c \cdot c') = \eta_S(b \cdot b')$$

and

$$\eta_S(c \cdot c' - b \cdot b') = 0$$

By Theorem 3.17, there is some $d \in S^{\infty-\text{sat}}$ such that:

$$d \cdot (c \cdot c' - b \cdot b') = 0$$

that is, such that $d \cdot c \cdot c' = d \cdot b \cdot b' \in S^{\infty-\text{sat}}$, where $d \cdot b \cdot b' \in S^{\infty-\text{sat}}$ (since it is a product of elements of $S^{\infty-\text{sat}}$, which is a submonoid of A).

Conversely, let $\eta_S(c)/\eta_S(b) \in A\{S^{-1}\}$ with $b \in S^{\infty-\text{sat}}$ be an element for which there are $d \in S^{\infty-\text{sat}}$ and $c' \in A$ such that $d \cdot c \cdot c' \in S^{\infty-\text{sat}}$. We have $\eta_S(d \cdot c' \cdot c) \in (A\{S^{-1}\})^{\times}$ and $b \in S^{\infty-\text{sat}}$, so $\eta_S(b) \in (A\{S^{-1}\})^{\times}$, and

$$\frac{\eta_S(d \cdot c' \cdot c)}{\eta_S(b)} \in (A\{S^{-1}\})^{\times}.$$

Since

$$\frac{\eta_S(c)}{\eta_S(b)} \cdot \eta_S(d \cdot c') = \frac{\eta_S(d \cdot c' \cdot c)}{\eta_S(b)} \in (A\{S^{-1}\})^{\times}$$

it follows that $\eta_S(c)/\eta_S(b) \in (A\{S^{-1}\})^{\times}$, for the product of invertible elements is, again, invertible. □

Proposition 3.22. *Let $U \subseteq \mathbb{R}^n$ be any open subset and define $S_U = \{g \in \mathcal{C}^\infty(\mathbb{R}^n) | U \subseteq U_g\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n)$. The $\mathcal{C}^\infty$-ring of fractions of $\mathcal{C}^\infty(\mathbb{R}^n)$ with respect to the set S_U:*

$$\eta_{S_U} : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)\{{S_U}^{-1}\}$$

is isomorphic to the restriction map:

$$\begin{array}{cccc} \rho : & \mathcal{C}^\infty(\mathbb{R}^n) & \to & \mathcal{C}^\infty(U) \\ & h & \mapsto & h \restriction_U \end{array}$$

Proof. For a detailed proof, see Proposition 3.30 of [5].

□

Remark 3.23. Let A be a $\mathcal{C}^\infty$-ring and $a \in A$. In general, the $\mathcal{C}^\infty$-ring of fractions of A with respect to a **is not a local** $\mathcal{C}^\infty$**-ring**. Let us consider the case in which $A = \mathcal{C}^\infty(\mathbb{R}^n)$ and $a = f : \mathbb{R}^n \to \mathbb{R}$ is such that $f \not\equiv 0$. By Theorem 1.3 of [17], $A\{a^{-1}\} \cong \mathcal{C}^\infty(\mathbb{R}^n)\{f^{-1}\}$, and $\mathcal{C}^\infty(\mathbb{R}^n)\{f^{-1}\} \cong \mathcal{C}^\infty(U_f)$, where $U_f = \mathrm{Coz}\,(f) = \mathbb{R}^n \setminus Z(f)$. Thus, for every $x \in U_f$, we have a maximal ideal $\mathfrak{m}_x = \{g \in \mathcal{C}^\infty(U_f) | g(x) = 0\}$, hence a *continuum* of maximal ideals in $\mathcal{C}^\infty(\mathbb{R}^n)$.

Remark 3.24. In the context of Proposition 3.22, note that if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ is such that $U = U_f$, we have $S_{U_f} = \{f\}^{\infty-\text{sat}}$.

Now we are ready to prove Theorem 3.17. We do so first proving the result for quotients and then for colimits.

Proof of Theorem 3.17: The case where $A = \mathcal{C}^\infty(\mathbb{R}^n)$ was proved in Proposition 3.22. Suppose that $A = \mathcal{C}^\infty(\mathbb{R}^n)/I$ for some ideal I and $\Sigma = \{f\}$. By Corollary 3.46 of [5], we have $(\mathcal{C}^\infty(\mathbb{R}^n)/I)\{(f+I)^{-1}\} \cong \mathcal{C}^\infty(\mathbb{R}^n)\{f^{-1}\}/\langle \eta_f[I] \rangle$, so items (i) and (ii) of Theorem 3.18 hold for the quotient. The slightly more general case, in which Σ is finite, follows immediately from the fact that $A\{S^{-1}\} = A\{\prod_{i=1}^{\ell} {f_i}^{-1}\}$.

Now let A be a finitely generated $\mathcal{C}^\infty$-ring and $S \subseteq A$ be *any* set, so we can write $S = \bigcup_{S' \subseteq_{\text{fin}} S} S'$. Since $A\{S^{-1}\} \cong \varinjlim_{S' \subseteq S} A\{S'^{-1}\}$, items (i) and (ii) hold for $A\{S^{-1}\}$.

Finally, given *any* $\mathcal{C}^\infty$-ring B (not a necessarily finitely generated one) and *any* set $S \subseteq B$, we write B as the directed colimit of its finitely generated $\mathcal{C}^\infty$-subrings, $\varinjlim_{B_i \subseteq_{\text{f.g.}} B} B_i$ and define $S_i = \jmath_i^{\dashv}[S]$, where

$\jmath_i : B_i \rightarrowtail B$ is the injection in the colimit. Since by the cases already proven items (i) and (ii) hold for every $B_i\{S_i^{-1}\}$, the same is true for $B\{S^{-1}\} \cong \varinjlim_{B_i \subseteq_{\text{fin}} B} B_i\{S_i^{-1}\}$. □

Now we turn to the concept of a "$\mathcal{C}^\infty$-radical ideal" in the theory of $\mathcal{C}^\infty$-rings, which plays a similar role to the one played by radical ideals in Commutative Algebra. This key concept was first introduced by I. Moerdijk and G. Reyes in [17] in 1986, and explored in more details in [18]. Contrary to the concepts of $\mathcal{C}^\infty$-fields, $\mathcal{C}^\infty$-domains and local $\mathcal{C}^\infty$-rings, the concept of a $\mathcal{C}^\infty$-radical of an ideal can not be brought from Commutative Algebra via the forgetful functor. Recall that the radical of an ideal I of a commutative unital ring R is given by $\sqrt{I} = \{x \in R | (\exists n \in \mathbb{N})(x^n \in I)\}$. The most fitting characterization of this concept to Smooth Commutative Algebra is given below:

$$\sqrt{I} = \bigcap \{\mathfrak{p} \in \operatorname{Spec}(R) | I \subseteq \mathfrak{p}\} = \left\{ x \in R | \left(\frac{R}{I}\right)[(x+I)^{-1}] \cong 0 \right\}.$$

Moerdijk and Reyes use the latter equality in order to motivate their "ad hoc" definition:

Definition 3.25. Let A be a $\mathcal{C}^\infty$-ring and let $I \subseteq A$ be an ideal. The $\mathcal{C}^\infty$-**radical of** I is given by:

$$\sqrt[\infty]{I} := \left\{ a \in A | \left(\frac{A}{I}\right)\{(a+I)\}^{-1} \cong \{0\} \right\}$$

Unlike what happens in ordinary Commutative Algebra, it is not evident, up to this point, that whenever I is an ideal of a $\mathcal{C}^\infty$-ring, $\sqrt[\infty]{I}$ is also an ideal. This fact shall be addressed later on.

The $\mathcal{C}^\infty$-radical of an ideal may be characterized in terms of the smooth saturation, as we show in the following:

Proposition 3.26 (Proposition 3.48 of [5]). *Let A be a $\mathcal{C}^\infty$-ring and let $I \subseteq A$ be any ideal. We have the following equalities:*

$$\sqrt[\infty]{I} = \{a \in A | (\exists b \in I)\&(\eta_a(b) \in (A\{a^{-1}\})^\times)\} = \{a \in A | I \cap \{a\}^{\infty-\text{sat}} \neq \varnothing\}$$

where $\eta_a : A \to A\{a^{-1}\}$ is the morphism of fractions with respect to $\{a\}$.

Proposition 3.27. *Let A and B be two $\mathcal{C}^\infty$-rings and $S \subseteq A$ and $f : A \to B$ a $\mathcal{C}^\infty$-homomorphism. By the universal property of $\eta_S : A \to A\{S^{-1}\}$ we have a unique $\mathcal{C}^\infty$-homomorphism $f_S : A\{S^{-1}\} \to B\{f[S]^{-1}\}$ such that the following square commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\eta_S} & A\{S^{-1}\} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \exists! f_S} \\ B & \xrightarrow[\eta_{f[S]}]{} & B\{f[S]^{-1}\} \end{array} .$$

Moreover, if $f : A \to B$ is surjective, i.e., $B \cong A/ker(f)$, then $f_S : B \to B\{f[S]^{-1}\}$ is surjective and $B\{f[S]^{-1}\} \cong A\{S^{-1}\}/ker(f_S)$.

Proposition 3.28 (Theorem 3.41 of [5]). *Let B be the directed colimit of a system $\{A_\ell \stackrel{t_{\ell j}}{\to} A_j | \ell, j \in I\}$ of $\mathcal{C}^\infty$-rings, that is,*

$$\begin{array}{ccccc} & & B = \varinjlim_{\ell \in I} A_\ell & & \\ & \nearrow^{t_i} & & \nwarrow^{t_j} & \\ A_i & & \xrightarrow{t_{ij}} & & A_j \end{array}$$

is a limit co-cone, so given any $u \in B$, there are $j \in I$ and $u_j \in A_j$ such that $t_j(u_j) = u$. Under those circumstances, we have:

$$\varinjlim_{k \geq j} A_k\{u_k^{-1}\} \cong B\{u^{-1}\}$$

Using this proposition, we can prove that:

$$\left(\varinjlim_{i \in I} S_i\right)^{\infty-\mathrm{sat}} = \varinjlim_{i \in I} S_i^{\infty-\mathrm{sat}} \subseteq \varinjlim_{i \in I} A_i$$

(for a detailed proof, see Corollary 3.43 of [5]).

4. Some classes of $\mathcal{C}^\infty$-rings, the $\mathcal{C}^\infty$-spectrum and $\mathcal{C}^\infty$-saturation

In this section we present some distinguished classes of $\mathcal{C}^\infty$-rings, such as $\mathcal{C}^\infty$-fields, $\mathcal{C}^\infty$-domains, $\mathcal{C}^\infty$-local rings and reduced $\mathcal{C}^\infty$-rings. The concept of von Neumann regular $\mathcal{C}^\infty$-ring is explored in details in [6]).

In [17], we find definitions of $\mathcal{C}^\infty$-fields, $\mathcal{C}^\infty$-domains and $\mathcal{C}^\infty$-local rings: they are $\mathcal{C}^\infty$-rings such that their underlying $\mathbb{R}$-algebras are fields, domains and local rings in the ordinary sense, respectively.

Following I. Moerdijk and G. Reyes, we use the $\mathcal{C}^\infty$-version of the prime spectrum in Smooth Commutative Algebra by taking, among all prime ideals of a $\mathcal{C}^\infty$-ring, only the $\mathcal{C}^\infty$-radical ones (since "being prime" does not imply " being $\mathcal{C}^\infty$-radical", cf. [18]). We denote the set of all prime proper $\mathcal{C}^\infty$-radical ideals of a $\mathcal{C}^\infty$-ring A by $\mathrm{Spec}^\infty(A)$.

Proposition 4.1 (Proposition 4.3 of [5]). *Let A be a $\mathcal{C}^\infty$-ring and $\mathfrak{p} \in \mathrm{Spec}^\infty(A)$. Then $A_\mathfrak{p} := A\{A \setminus \mathfrak{p}^{-1}\}$ is a local $\mathcal{C}^\infty$-ring whose unique maximal ideal is given by $\mathfrak{m}_\mathfrak{p} = \left\{\frac{\eta_{A\setminus\mathfrak{p}}(x)}{\eta_{A\setminus\mathfrak{p}}(y)} | (x \in \mathfrak{p}) \& (y \in A \setminus \mathfrak{p})\right\}$.*

Proposition 4.2 (Proposition 4.4 of [5]). *Let A be a $\mathcal{C}^\infty$-ring. The following assertions are equivalent.*
i) A is a $\mathcal{C}^\infty$-field;
ii) For every subset $S \subseteq A \setminus \{0\}$, the canonical map $\mathrm{Can}_S : A \to A\{S^{-1}\}$ is a $\mathcal{C}^\infty$-ring isomorphism;
iii) For any $a \in A \setminus \{0\}$, we have that $\mathrm{Can}_a : A \to A\{a^{-1}\}$ is a $\mathcal{C}^\infty$-isomorphism.

In ordinary Commutative Algebra, given an element x of a ring R, we say that x is a nilpotent infinitesimal if and only if there is some $n \in \mathbb{N}$ such that $x^n = 0$. Let A be a $\mathcal{C}^\infty$-ring and $a \in A$. D. Borisov and K. Kremnizer in [8] call a an ∞-infinitesimal if, and only if $A\{a^{-1}\} \cong 0$. The next definition describes the notion of a $\mathcal{C}^\infty$-ring being free of ∞-infinitesimals - which is analogous to the notion of "reducedness", of a commutative ring.

Definition 4.3. A $\mathcal{C}^\infty$-ring A is $\mathcal{C}^\infty$-**reduced** if, and only if, $\sqrt[\infty]{(0)} = (0)$.

Among the $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-rings we can highlight the $\mathcal{C}^\infty$-fields.
The following result is crucial to the proof of the main theorems of this work.

Theorem 4.4. *Let A be a $\mathcal{C}^\infty$-ring, $S \subseteq A$, and $I \subseteq A$ any ideal. Then:*

$$\langle \eta_S[I] \rangle = \left\{ \frac{\eta_S(b)}{\eta_S(d)} | b \in I \& d \in S^{\infty-\mathrm{sat}} \right\}$$

Proof. Given $h \in \langle \eta_S[I] \rangle$, there are $n \in \mathbb{N}$, $b_1, \ldots, b_n \in I$ and $\alpha_1, \ldots, \alpha_n \in A\{S^{-1}\}$ such that $h = \sum_{i=1}^n \alpha_i \cdot \eta_S(b_i)$.

For each $i \in \{1, \dots, n\}$ there are $c_i \in A$ and $d_i \in S^{\infty-\text{sat}}$ such that $\alpha_i \cdot \eta_S(d_i) = \eta_S(c_i)$, so

$$h = \sum_{i=1}^{n} \alpha_i \cdot \eta_S(b_i) = \sum_{i=1}^{n} \frac{1}{\eta_S(d_i)} \cdot \eta_S(c_i) \cdot \eta_S(b_i),$$

and denoting $b'_i := c_i \cdot b_i \in I$, we get:

$$h = \sum_{i=1}^{n} \frac{\eta_S(b'_i)}{\eta_S(d_i)}$$

For each $i = 1, \dots, n$, let

$$b''_i = b'_i \prod_{j \neq i} d_j,$$

so

$$h = \frac{\eta_S\left(\prod_{j\neq 1} d_j\right)\eta_S(b'_1)}{\eta_S(d_1 \dots d_n)} + \frac{\eta_S\left(\prod_{j\neq 2} d_j\right)\eta_S(b'_2)}{\eta_S(d_1 \dots d_n)} + \cdots + \frac{\eta_S\left(\prod_{j\neq n} d_j\right)\eta_S(b'_n)}{\eta_S(d_1 \dots d_n)}$$

Hence:

$$h \cdot \eta_S(d_1 \dots d_n) = \eta_S\left(\prod_{j\neq 1} d_j\right)\eta_S(b'_1) + \eta_S\left(\prod_{j\neq 2} d_j\right)\eta_S(b'_2) + \cdots + \eta_S\left(\prod_{j\neq n} d_j\right)\eta_S(b'_n).$$

Let $b''_i := \left(\prod_{j\neq i} d_i\right) \cdot \overbrace{b'_i}^{\in I} \in I$, so we have $h \cdot \eta_S(d_1 \dots d_n) = \sum_{i=1}^{n} \eta_S(b''_i) = \eta_S\left(\sum_{i=1}^{n} b''_i\right)$. Since $\eta_S(d_1 \dots d_n) \in A\{S^{-1}\}^{\times}$, $d = d_1 \dots d_n \in S^{\infty-\text{sat}}$, so taking $b = \sum_{i=1}^{n} b''_i$, we have $b \in I$, since it is a sum of elements of I, we can write $h \cdot \eta_S(d) = \eta_S(b)$, and $h = \dfrac{\eta_S(b)}{\eta_S(d)}$, with $b \in I$ and $d \in S^{\infty-\text{sat}}$.

The other way round is immediate. □

As a consequence of Theorem 4.4, we characterize $\mathcal{C}^\infty$-reducedness in terms of the $\mathcal{C}^\infty$-saturation:

Corollary 4.5. *If A is a reduced $\mathcal{C}^\infty$-ring, then we have:*

$$(\forall a \in A)((0 \in \{a\}^{\infty-\text{sat}}) \leftrightarrow (a = 0))$$

4.1. The $\mathcal{C}^\infty$-radical of an ideal is an ideal

Given $I \subseteq \mathcal{C}^\infty(\mathbb{R}^n)$ any ideal, $\widehat{I} = \{A \subseteq \mathbb{R}^n | A \text{ is closed and } (\exists f \in I)(A = Z(f))\}$ is a filter on the set of all the closed subsets of $\mathbb{R}^n$. Moreover, I is a proper ideal if, and only if, $\widehat{I}$ is a proper filter (see Proposition 4.9 of [5]). Also, given any filter $\mathcal{F}$ on the set of all the closed subsets of $\mathbb{R}^n$, we define $\check{\mathcal{F}} = \{f \in \mathcal{C}^\infty(\mathbb{R}^n) | Z(f) \in \mathcal{F}\}$, which is an ideal of $\mathcal{C}^\infty(\mathbb{R}^n)$. Moreover, $\mathcal{F}$ is a proper filter if, and only if, $\check{\mathcal{F}}$ is a proper ideal. (see Proposition 4.10 of [5]).

Let $I \subseteq \mathcal{C}^\infty(\mathbb{R}^n)$ be any ideal. Then we have, by definition, $\widehat{I} = \{A \subseteq \mathbb{R}^n | (\exists h \in I)(A = Z(h))\}$, and therefore $\check{\widehat{I}} = \{g \in \mathcal{C}^\infty(\mathbb{R}^n) | (\exists h \in I)(Z(g) = Z(h))\}$.

Let $\mathcal{F}$ be a proper filter on the set of all closed subsets of $\mathbb{R}^n$. Since every closed subset $B \subseteq \mathbb{R}^n$ is a zero set of some smooth function, we have:

$$\widehat{\check{\mathcal{F}}} = \{A \subseteq \mathbb{R}^n | (\exists g \in \check{\mathcal{F}})(A = Z(g))\} = \{A \subseteq \mathbb{R}^n | (\exists g \in \check{\mathcal{F}})((Z(g) \in \mathcal{F}) \& (A = Z(g)))\} = \mathcal{F}$$

As a consequence of the discussion above, we have:

Proposition 4.6 (Proposition 4.12 of [5]). *Let $I \subseteq \mathcal{C}^{\infty}(\mathbb{R}^n)$ be an ideal. Then $\check{\widehat{I}} = \{g \in \mathcal{C}^{\infty}(\mathbb{R}^n) | (\exists h \in I)(Z(g) = Z(h))\} = \sqrt[\infty]{I}$. In particular, the $\mathcal{C}^{\infty}$-radical of an ideal I of the free $\mathcal{C}^{\infty}$-ring on finitely many generators, $\mathcal{C}^{\infty}(\mathbb{R}^n)$, is again an ideal.*

As a consequence, we have the following:

Corollary 4.7. *Let $A = \mathcal{C}^{\infty}(\mathbb{R}^n)$ be a finitely generated free $\mathcal{C}^{\infty}$-ring. $I \subseteq \mathcal{C}^{\infty}(\mathbb{R}^n)$ is a $\mathcal{C}^{\infty}$-radical ideal, that is, $\sqrt[\infty]{I} = I$, if, and only if:*

$$(\forall g \in \mathcal{C}^{\infty}(\mathbb{R}^n))((g \in I) \leftrightarrow (\exists f \in I)(Z(f) = Z(g))).$$

The following result gives us another characterization of "being $\mathcal{C}^{\infty}$-radical" as a consequence of a comment made by Moerdijk and Reyes in the p. 330 of [18]:

Corollary 4.8. *Let $\mathcal{C}^{\infty}(\mathbb{R}^n)$ be the free $\mathcal{C}^{\infty}$-ring and let $I \subseteq \mathcal{C}^{\infty}(\mathbb{R}^n)$ be a finitely generated ideal, that is, $I = \langle g_1, \ldots, g_k \rangle$, for some $g_1, \ldots, g_k \in \mathcal{C}^{\infty}(\mathbb{R}^n)$. I is a $\mathcal{C}^{\infty}$-radical ideal if, and only if:*

$$(\forall x \in Z(g_1, \ldots, g_k))(f(x) = 0) \rightarrow (f \in I)$$

Proof. Suppose $I = \sqrt[\infty]{I}$ and let $f \in \mathcal{C}^{\infty}(\mathbb{R}^n)$ be such that $(\forall x \in Z(g_1, \ldots, g_k))(f(x) = 0)$. We have $g = g_1^2 + \cdots + g_k^2 \in I$ such that $Z(g) \subseteq Z(f)$, so $Z(f) \in \widehat{I}$ (since $\widehat{I}$ is a filter) and by Proposition 4.6, $f \in \check{\widehat{I}} = \sqrt[\infty]{I} = I$.

Now, suppose $(\forall x \in Z(g_1, \ldots, g_k))(f(x) = 0) \rightarrow (f \in I)$. Given $h \in \sqrt[\infty]{I} = \check{\widehat{I}}$, we have $Z(h) \in \widehat{I}$, so there exists some $g \in I$ such that $Z(h) = Z(g)$, thus $(\forall x \in Z(g_1, \ldots, g_k))(h(x) = 0)$. By hypothesis, this means that $h \in I$, so $\sqrt[\infty]{I} \subseteq I$. Since $I \subseteq \sqrt[\infty]{I}$ always holds, it follows that I is a $\mathcal{C}^{\infty}$-radical ideal. □

By Proposition 4.5 of [19], it follows that any finitely generated $\mathcal{C}^{\infty}$-radical ideal I of $\mathcal{C}^{\infty}(\mathbb{R}^n)$ is such that:

$$(\forall x \in Z(I))(f \upharpoonright_x \in I \upharpoonright_x \rightarrow f \in I),$$

where $Z(I) := \bigcap_{f \in I} Z(f)$. This (important) condition an ideal of a $\mathcal{C}^{\infty}$-ring may satisfy is called "point-determinacy."

As a consequence, we have a particular version of the weak **Nullstellensatz** to finitely generated ideals:

Proposition 4.9. *For any finitely generated $\mathcal{C}^{\infty}$-radical ideal I of $\mathcal{C}^{\infty}(\mathbb{R}^n)$, we have $1 \in I \iff Z(I) = \varnothing$.*

Proof. See p. 45 of [19]. □

Let $\mathfrak{F}$ be the set of all the filters on the closed parts of $\mathbb{R}^n$ and $\mathfrak{I}$ be the set of all the ideals of $\mathcal{C}^{\infty}(\mathbb{R}^n)$. We have, so far, established that for every ideal $I \subseteq \mathcal{C}^{\infty}(\mathbb{R}^n)$ we have $\sqrt[\infty]{I} = \check{\widehat{I}}$ and for every filter $\mathcal{F} \in \mathfrak{F}$ we have $\widehat{\check{\mathcal{F}}}$.

Next we show that the following diagram:

$$\mathfrak{F} \underset{\wedge}{\overset{\vee}{\rightleftarrows}} \mathfrak{I}$$

where $\wedge(I) = \widehat{I}$ and $\vee(\mathcal{F}) = \check{\mathcal{F}}$, is a Galois connection with $\wedge \vdash \vee$.

Proposition 4.10 (Proposition 4.17 of [5]). *The adjunction $\wedge \vdash \vee$ is a covariant Galois connection between the posets $(\mathfrak{F}, \subseteq)$ and $(\mathfrak{I}, \subseteq)$, i.e.,*

(a)Given $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{F}$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ then $\check{\mathcal{F}}_1 \subseteq \check{\mathcal{F}}_2$;

(b) Given $I_1, I_2 \in \mathfrak{I}$ such that $I_1 \subseteq I_2$ then $\widehat{I_1} \subseteq \widehat{I_2}$;
(c) For every $\mathcal{F} \in \mathfrak{F}$ and every $I \in \mathfrak{I}$, we have $\widehat{I} \subseteq \mathcal{F} \iff I \subseteq \check{\mathcal{F}}$.

Proposition 4.11 (Proposition 4.18 of [5]). *Let $A = \mathcal{C}^\infty(\mathbb{R}^n)$ for some $n \in \mathbb{N}$. The Galois connection $\wedge \vdash \vee$ establishes a bijective correspondence between:*
(a) proper filters of $(\mathfrak{F}, \subseteq)$ and proper ideals of $(\mathfrak{I}, \subseteq)$;
(b) maximal filters of $(\mathfrak{F}, \subseteq)$ and maximal ideals of $(\mathfrak{I}, \subseteq)$;
(c) prime filters of $(\mathfrak{F}, \subseteq)$ and prime ideals of $(\mathfrak{I}, \subseteq)$.
(d) filters on the closed parts of $\mathbb{R}^n$, $\mathfrak{F}$, and the set of all $\mathcal{C}^\infty$-radical ideals of $\mathcal{C}^\infty(\mathbb{R}^n)$, $\mathfrak{I}^\infty = \{I \subseteq \mathcal{C}^\infty(\mathbb{R}^n) | \sqrt[\infty]{I} = I\}$.

By Proposition 4.11, whenever $\mathfrak{p}$ is a prime ideal of $\mathcal{C}^\infty(\mathbb{R}^n)$, the filter associated with $\mathfrak{p}$, $\widehat{\mathfrak{p}}$, is a prime filter. Again by Proposition 4.11, it follows that $\check{\widehat{\mathfrak{p}}} = \sqrt[\infty]{\mathfrak{p}}$ is a prime ideal. Thus, whenever $\mathfrak{p}$ is a prime ideal of $\mathcal{C}^\infty(\mathbb{R}^n)$, $\sqrt[\infty]{\mathfrak{p}}$ is also a prime ideal.

A consequence of Proposition 4.11 combined with Proposition 4.6 is that given the $\mathcal{C}^\infty$-ring $\mathcal{C}^\infty(\mathbb{R}^n)$, the operator $\vee \circ \wedge = \sqrt[\infty]{\cdot} : \mathfrak{I} \to \mathfrak{I}$ is a closure operator (that is, it is idempotent, inflationary and increasing). In virtue of these remarks, we have the following:

Theorem 4.12 (Theorem 4.21 of [5]). *Let $I, I_1, I_2 \subseteq \mathcal{C}^\infty(\mathbb{R}^n)$ be ideals. Then:*
(a) $\sqrt[\infty]{I}$ is an ideal of $\mathcal{C}^\infty(\mathbb{R}^n)$ and $I \subseteq \sqrt[\infty]{I}$;
(b) $I_1 \subseteq I_2 \Rightarrow \sqrt[\infty]{I_1} \subseteq \sqrt[\infty]{I_2}$
(c) $\sqrt[\infty]{\sqrt[\infty]{I}} = \sqrt[\infty]{I}$

The following definition will be helpful to prove that whenever I is an ideal of **any** $\mathcal{C}^\infty$-ring A, then $\sqrt[\infty]{I}$ is also an ideal.

Definition 4.13. Let A be a $\mathcal{C}^\infty$-ring. We say that A is **admissible** if for every ideal $I \subseteq A$, $\sqrt[\infty]{I}$ is an ideal in A.

We claim that *every* $\mathcal{C}^\infty$-ring is admissible. We reason as follows: first, given two $\mathcal{C}^\infty$-rings, A and A' such that $A \cong A'$, if A is admissible then so is A'. Then we use Proposition 3.27 to check that whenever a $\mathcal{C}^\infty$-ring, A, is admissible and J is one of its ideals, A/J is admissible. Finally, given a filtered diagram of admissible $\mathcal{C}^\infty$-rings, $\{A_i \stackrel{h_{ij}}{\to} A_j\}$, we prove that $\varinjlim A_i$ is an admissible $\mathcal{C}^\infty$-ring.

Summing all these results up yields:

Theorem 4.14. *Given any $\mathcal{C}^\infty$-ring A, whenever I is an ideal of A, $\sqrt[\infty]{I}$ is also an ideal of A.*

Now we present some properties of taking the $\mathcal{C}^\infty$-radical of an ideal of an *arbitrary* $\mathcal{C}^\infty$-ring.

Proposition 4.15. *Let A be a $\mathcal{C}^\infty$-ring, $I, J \subseteq A$ any of its ideals. Then:*
(i) $I \subseteq J \Rightarrow \sqrt[\infty]{I} \subseteq \sqrt[\infty]{J}$
(ii) $I \subseteq \sqrt[\infty]{I}$

Proof. Ad (*i*): Given $a \in \sqrt[\infty]{I}$, there is $b \in I$ such that $\eta_a(b) \in (A\{a^{-1}\})^\times$. Since $I \subseteq J$, the same b is a witness of the fact that $a \in \sqrt[\infty]{J}$, for $b \in J$ and $\eta_a(b) \in (A\{a^{-1}\})$. Ad (*ii*): We prove its contrapositive: $A \setminus \sqrt[\infty]{I} \subseteq A \setminus I$. Given $a \in A \setminus \sqrt[\infty]{I}$, we have that $(\forall b \in I)(\eta_a(b) \notin (A\{a^{-1}\})^\times)$, so $\eta_a[I] \cap (A\{a^{-1}\})^\times = \varnothing$. Since $\eta_a(a) \in (A\{a^{-1}\})^\times$, it follows that $\eta_a(a) \notin \eta_a[I]$, so $a \notin I$. □

Proposition 4.16. *Given any $\mathcal{C}^\infty$-ring B and $J \subseteq B$ any of its ideals, we have $\sqrt[\infty]{\{0_{B/J}\}} = \sqrt[\infty]{J}/J$.*

Proof. We have $a \in \sqrt[\infty]{J} \iff (\exists b \in J)(\eta(b) \in (B\{a^{-1}\})^{\times})$, so

$$a + J = \bar{a} \in \sqrt[\infty]{\{0_{B/J}\}} \iff (\exists \bar{b} \in \{0_{B/J}\})(\overline{\eta_a}(\bar{b}) \in \left((B/J)\{(a+J)^{-1}\}\right)^{\times})$$
$$\iff (B/J)\{(a+J)^{-1}\} \cong \{0\} \iff$$
$$\iff a \in \sqrt[\infty]{J} \iff a + J \in \frac{\sqrt[\infty]{J}}{J}$$

Now, since $J \subseteq \sqrt[\infty]{J}$, if $a' + J = a + J$, then $a \in \sqrt[\infty]{J} \iff a' \in \sqrt[\infty]{J}$. □

As an immediate consequence, we have the following:

Corollary 4.17. *Let A be a $\mathcal{C}^\infty$-ring. We have:*
(a) An ideal $J \subseteq A$ is a $\mathcal{C}^\infty$-radical ideal if, and only if, (A/J) is a $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-ring
(b) A proper prime ideal $\mathfrak{p} \subseteq A$ is $\mathcal{C}^\infty$-radical if, and only if, $(A/\mathfrak{p})$ is a $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-domain.

Proposition 4.18 (Proposition 4.33 of [5]). *Let A', B' be two $\mathcal{C}^\infty$-rings and $\jmath : A' \to B'$ be a monomorphism. If B' is $\mathcal{C}^\infty$-reduced, then A' is also $\mathcal{C}^\infty$-reduced.*

In ordinary Commutative Algebra, we say a commutative unital ring, R is reduced if, and only if, $\sqrt{(0)} = (0)$. Within the Commutative Algebra of $\mathcal{C}^\infty$-rings, we say that a $\mathcal{C}^\infty$-ring, A is $\mathcal{C}^\infty$-**reduced** if, and only if $\sqrt[\infty]{(0)} = (0)$.

As a consequence of Propositions 4.18 and 3.26, we have the following:

Corollary 4.19. *Every $\mathcal{C}^\infty$-subring of a $\mathcal{C}^\infty$-field is a $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-domain.*

Proposition 4.20 (Proposition 4.37 of [5]). *Let $(I, \leq)$ be a directed set and suppose that $\{A_i\}_{i \in I}$ is a directed family of $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-rings. Then*

$$B = \varinjlim_{i \in I} A_i$$

is a $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-ring.

Theorem 4.21 (Theorem 4.38 of [5]). *Let A and B be two $\mathcal{C}^\infty$-rings, $J \subseteq B$ a $\mathcal{C}^\infty$-radical ideal in B and $f : A \to B$ any $\mathcal{C}^\infty$-homomorphism. Then $f^{\dashv}[J]$ is a $\mathcal{C}^\infty$-radical ideal in A.*

We register that, in general, holds:

Proposition 4.22 (Proposition 4.39 of [5]). *Let A, B be $\mathcal{C}^\infty$-rings, $f : A \to B$ a $\mathcal{C}^\infty$-homomorphism and $J \subseteq B$ any ideal. Then:*

$$\sqrt[\infty]{f^{\dashv}[J]} \subseteq f^{\dashv}[\sqrt[\infty]{J}].$$

Remark 4.23. With exactly the same method used in the proof of Theorem 4.14, one proves that whenever $\mathfrak{p} \subseteq A$ is a prime ideal of an arbitrary $\mathcal{C}^\infty$-ring, A, $\sqrt[\infty]{\mathfrak{p}}$ is also a prime ideal.

At this point it is natural to look for a $\mathcal{C}^\infty$-analog of the Zariski spectrum of a commutative unital ring. With this motivation, we give the following:

Definition 4.24 ($\mathcal{C}^\infty$-**spectrum**). For a $\mathcal{C}^\infty$-ring A, we define $\mathrm{Spec}^\infty(A) = \{\mathfrak{p} \in \mathrm{Spec}\,(A) | \mathfrak{p} \text{ is } \mathcal{C}^\infty - \text{radical}\}$, equipped with the smooth Zariski topology generated by the basic open sets $D^\infty(a) = \{\mathfrak{p} \in \mathrm{Spec}^\infty(A) | a \notin \mathfrak{p}\}$.

A detailed study of the smooth Zariski spectrum is made in Section 5.1 of [5].

Now we present some properties of $\mathcal{C}^\infty$-radical ideals of an arbitrary $\mathcal{C}^\infty$-ring A.

Proposition 4.25 (Proposition 4.42 of [5]). *Given a $\mathcal{C}^\infty$-ring, A, let $\mathfrak{I}_A^\infty$ denote the set of all its $\mathcal{C}^\infty$-radical ideals. The following results hold:*

(a) Suppose that $(\forall \alpha \in \Lambda)(I_\alpha \in \mathfrak{I}_A^\infty)$. Then $\bigcap_{\alpha\in\Lambda} I_\alpha \in \mathfrak{I}_A^\infty$, that is, if $(\forall \alpha \in \Lambda)(I_\alpha \in \mathfrak{I}_A^\infty)$, then:

$$\sqrt[\infty]{\bigcap_{\alpha\in\Lambda} I_\alpha} = \bigcap_{\alpha\in\Lambda} I_\alpha = \bigcap_{\alpha\in\Lambda} \sqrt[\infty]{I_\alpha}$$

(b) Let $\{I_\alpha | \alpha \in \Sigma\}$ an upward directed family of elements of $\mathfrak{I}_A^\infty$. Then $\bigcup_{\alpha\in\Sigma} I_\alpha \in \mathfrak{I}_A^\infty$.

From the above result, it follows that $\mathfrak{I}_A^\infty$ is a complete Heyting algebra.

Lemma 4.26 (Lemma 1.12 of [18]). *Let A be a $\mathcal{C}^\infty$-ring and $S \subseteq A$ any multiplicative subset. Consider the following posets:*

$$(\mathrm{Spec}^\infty(A\{S^{-1}\}), \subseteq)$$

and

$$(\{\mathfrak{p} \in \mathrm{Spec}^\infty(A) | \mathfrak{p} \cap S = \varnothing\}, \subseteq)$$

The following poset maps:

$$\begin{array}{rccc} \mathrm{Can}_S^* : & (\mathrm{Spec}^\infty(A\{S^{-1}\}), \subseteq) & \to & (\{\mathfrak{p} \in \mathrm{Spec}^\infty(A) | \mathfrak{p} \cap S = \varnothing\}, \subseteq) \\ & Q & \mapsto & \mathrm{Can}_S^{\dashv}[Q] \end{array}$$

and

$$\begin{array}{rccc} \mathrm{Can}_{S*} : & (\{\mathfrak{p} \in \mathrm{Spec}^\infty(A) | \mathfrak{p} \cap S = \varnothing\}, \subseteq) & \to & (\mathrm{Spec}^\infty(A\{S^{-1}\}), \subseteq) \\ & P & \mapsto & \langle \mathrm{Can}_S[P] \rangle \end{array}$$

are poset isomorphisms, each one inverse of the other.

Proposition 4.27. *Let A be a $\mathcal{C}^\infty$-ring. For every $\mathfrak{p} \in \mathrm{Spec}^\infty(A)$ let $\hat{\mathfrak{p}}$ denote the maximal ideal of $A_{\{\mathfrak{p}\}} = A\{A \setminus \mathfrak{p}^{-1}\}$ and consider:*

$$\mathrm{Can}_\mathfrak{p} : A \to A_{\{\mathfrak{p}\}}.$$

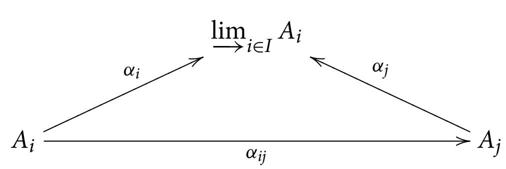
We have the following equalities: $\mathrm{Can}_\mathfrak{p}^{\dashv}[\hat{\mathfrak{p}}] = \mathfrak{p}$ *and* $\mathrm{Can}_\mathfrak{p}[\mathfrak{p}] = \hat{\mathfrak{p}}$.

Proof. Taking $S = A \setminus \mathfrak{p}$, since $\hat{\mathfrak{p}}$ is a maximal ideal, it is the largest element of $\mathrm{Spec}^\infty(A\{(A \setminus \mathfrak{p})^{-1}\})$. Hence $\mathrm{Can}_\mathfrak{p}^{\dashv}[\hat{\mathfrak{p}}]$ is the largest element of $\{\mathfrak{p}' \in \mathrm{Spec}^\infty(A) | \mathfrak{p}' \cap (A \setminus \mathfrak{p}) = \varnothing\}$. Thus, by Lemma 4.26, $\mathrm{Can}_\mathfrak{p}^{\dashv}[\hat{\mathfrak{p}}] = \mathfrak{p}$. □

Proposition 4.28. *If D is a reduced $\mathcal{C}^\infty$-domain, then $D\{a^{-1}\} \cong \{0\}$ implies $a = 0$.*

Proposition 4.29 (Proposition 4.47 of [5]). *Any free $\mathcal{C}^\infty$-ring is a reduced $\mathcal{C}^\infty$-ring.*

Proposition 4.30. *Let $\{A_i\}_{i\in I}$ be a directed family of $\mathcal{C}^\infty$-rings, so we have the diagram:*



and let $(\mathfrak{p}_i)_{i\in I}$ be a compatible family of prime $\mathcal{C}^\infty$-radical ideals, that is:

$$(\mathfrak{p}_i)_{i\in I} \in \varprojlim_{i\in I} \mathrm{Spec}^\infty(A_i).$$

Under those circumstances,

$$\varinjlim_{i\in I} \mathfrak{p}_i = \bigcup_{i\in I} \alpha_i[\mathfrak{p}_i]$$

is a $\mathcal{C}^\infty$-radical prime ideal of $\varinjlim_{i\in I} A_i$.

Proof. It suffices to prove that

$$\frac{\varinjlim_{i\in I} A_i}{\varinjlim_{i\in I} \mathfrak{p}_i}$$

is a $\mathcal{C}^\infty$-reduced domain.

It is a fact that colimits commute with quotients, so:

$$\frac{\varinjlim_{i\in I} A_i}{\varinjlim_{i\in I} \mathfrak{p}_i} \cong \varinjlim_{i\in I} \frac{A_i}{\mathfrak{p}_i}$$

Now, since every $\mathfrak{p}_i$ is a $\mathcal{C}^\infty$-radical prime ideal of A_i, we have, for every $i \in I$, that $\dfrac{A_i}{\mathfrak{p}_i}$ is a $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-domain.

The colimit of $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-domains is again a $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-domain, so $\dfrac{\varinjlim_{i\in I} A_i}{\varinjlim_{i\in I} \mathfrak{p}_i}$ is a domain and:

$$\varinjlim_{i\in I} \mathfrak{p}_i$$

is a $\mathcal{C}^\infty$-radical prime ideal of $\varinjlim_{i\in I} A_i$. □

5. Separation theorems for smooth commutative algebra

From the notions and results previously established, we are ready to state and prove the main result of this work:

Theorem 5.1 (Separation Theorems). *Let A be a $\mathcal{C}^\infty$-ring, $S \subseteq A$ be a subset of A and I be an ideal of A. Denote by $\langle S \rangle$ the multiplicative submonoid of A generated by S. We have:*
(a) If I is a $\mathcal{C}^\infty$-radical ideal, then:

$$I \cap \langle S \rangle = \varnothing \iff I \cap S^{\infty-\mathrm{sat}} = \varnothing$$

(b) If $S \subseteq A$ is a $\mathcal{C}^\infty$-saturated subset, then:

$$I \cap S = \varnothing \iff \sqrt[\infty]{I} \cap S = \varnothing$$

(c) If $\mathfrak{p} \in \mathrm{Spec}^\infty(A)$ then $A \setminus \mathfrak{p} = (A \setminus \mathfrak{p})^{\infty-\mathrm{sat}}$
(d) If $S \subseteq A$ is a $\mathcal{C}^\infty$-saturated subset, then:

$$I \cap S = \varnothing \iff (\exists \mathfrak{p} \in \mathrm{Spec}^\infty(A))((I \subseteq \mathfrak{p}) \& (\mathfrak{p} \cap S = \varnothing)).$$

(e) $\sqrt[\infty]{I} = \bigcap \{\mathfrak{p} \in \mathrm{Spec}^\infty(A) | I \subseteq \mathfrak{p}\}$

Proof. Ad (a): Since $\langle S\rangle \subseteq S^{\infty-\text{sat}}$, it is clear that $(ii) \to (i)$. We show that $(i) \to (ii)$ by contraposition. Suppose there exists some $b \in I \cap S^{\infty-\text{sat}}$, so $\eta_S(b) \in (B\{S^{-1}\})^{\times}$. We have:

$$B\{S^{-1}\} \cong_{\psi} \varinjlim_{S' \subseteq_{\text{fin}} S} B\{S'^{-1}\} \cong_{\varphi} \varinjlim_{S' \subseteq_{\text{fin}} S} B\left\{\prod S'^{-1}\right\},$$

so $(\varphi \circ \psi)(\eta_S(b)) \in \varinjlim_{S' \subseteq_{\text{fin}} S} B\{\prod S'^{-1}\}$ implies that there is some finite $S'' \subseteq S$ such that $\eta_{S''}(b) \in B\{S''^{-1}\} \cong B\{\prod S''^{-1}\}$. Let $a = \prod S''$, so $a \in \langle S\rangle$. We have that $\eta_a(b) \in B\{S''^{-1}\}$ implies $\eta_a(b) \in (B\{a^{-1}\})^{\times}$, and by hypothesis, $b \in I$ so $a \in \sqrt[\infty]{I} = I$. Hence $a \in I \cap \langle S\rangle \neq \varnothing$, and the result is proved.

Ad (b): Given $b \in \sqrt[\infty]{I} \cap S$, there must be some $x \in I$ such that $\eta_b(x) \in A\{b^{-1}\}^{\times}$, so $x \in \{b\}^{\infty-\text{sat}} \subseteq S^{\infty-\text{sat}} = S$. Thus $x \in I \cap S$ and $I \cap S = \varnothing$. The other way round is immediate since $I \subseteq \sqrt[\infty]{I}$.

Ad (c): Since $A \setminus \mathfrak{p} = \langle A \setminus \mathfrak{p}\rangle$, by item (a) we have $\mathfrak{p} \cap (A \setminus \mathfrak{p}) = \varnothing \iff \mathfrak{p} \cap (A \setminus \mathfrak{p})^{\infty-\text{sat}} = \varnothing$, so $(A \setminus \mathfrak{p})^{\infty-\text{sat}} \subseteq A \setminus \mathfrak{p}$. The other inclusion always holds, so $A \setminus \mathfrak{p} = (A \setminus \mathfrak{p})^{\infty-\text{sat}}$.

Ad (d): Consider the set $\Gamma_S := \{J \in \mathfrak{I}(A) | (I \subseteq J)\&(S \cap J = \varnothing)\}$, ordered by inclusion. It is straightforward to check that $(\Gamma_S, \subseteq)$ satisfies the hypotheses of **Zorn's Lemma**. Let M be a maximal member of Γ_S. We show that $M \in \text{Spec}^{\infty}(A)$.

Note that M is a proper prime ideal of A, since $1 \in S$ and $S \cap M = \varnothing$.

The proof that M is prime is made by contradiction. If $a, a' \notin M$, then by maximality there are $\alpha, \alpha' \in A$ and $m, m' \in M$ such that $m + \alpha \cdot a \in S$ and $m' + \alpha' \cdot a' \in S$. Since $\mathcal{C}^{\infty}$-saturated sets are submonoids, it follows that $(m + \alpha \cdot a) \cdot (m' + \alpha' \cdot a') \in S$. Thus,

$$\underbrace{(m \cdot m' + m \cdot \alpha' \cdot a' + m' \cdot \alpha \cdot a)}_{\in M} + (\alpha \cdot \alpha') \cdot (a \cdot a') \in S$$

If $a \cdot a' \in M$, we get $M \cap S \neq \varnothing$, a contradiction. Thus, $a, a' \notin M$, and M is a prime ideal.

We claim that $M = \sqrt[\infty]{M}$. In fact, since $M \cap S = \varnothing$ and S is $\mathcal{C}^{\infty}$-saturated, by item (b) it follows that $\sqrt[\infty]{M} \cap S = \varnothing$, so $\sqrt[\infty]{M} \in \Gamma_S$. Since $M \subseteq \sqrt[\infty]{M}$, $\sqrt[\infty]{M} \in \Gamma_S$ and M is a maximal element of Γ_S, it follows that $M = \sqrt[\infty]{M}$. Thus, $M \in \text{Spec}^{\infty}(A)$.

Ad (e): Clearly, $\sqrt[\infty]{I} \subseteq \bigcap\{\mathfrak{p} \in \text{Spec}^{\infty}(A) | I \subseteq \mathfrak{p}\}$, so we need only to prove the reverse inclusion. Let $a \notin \sqrt[\infty]{I}$, so $S_a = \{1, a, a^2, \ldots\} \cap \sqrt[\infty]{I} = \varnothing$. Since $\dfrac{A}{I}\{(a+I)^{-1}\} \cong \dfrac{A}{I}\{(a^k + I)^{-1}\}$ for any $k \in \mathbb{N}$ such that $k \geq 1$ and since $\sqrt[\infty]{I}$ is a $\mathcal{C}^{\infty}$-radical ideal, by item (a), we have $(S_a)^{\infty-\text{sat}} \cap \sqrt[\infty]{I} = \varnothing$, and by item (d), there is some $\mathfrak{p} \in \text{Spec}^{\infty}(A)$ such that $\mathfrak{p} \supseteq \sqrt[\infty]{I} \supseteq I$ such that $a \notin \mathfrak{p}$. □

Proposition 5.2. *Let A be a C^{∞}-ring and let $\{a_i : i \in \mathcal{I}\} \subseteq A$. Denote $I := \langle\{a_i : i \in \mathcal{I}\}\rangle$. Then the following are equivalent:*
(a) $\text{Spec}^{\infty}(A) = \bigcup_{i \in \mathcal{I}} D^{\infty}(a_i)$
(b) $A = I$

Proof. (b) ⇒ (a): Since there exists $\{i_1, \ldots, i_n\} \subseteq \mathcal{I}$ such that $1_A = \sum_{j=1}^{n} \lambda_j.a_{i_j}$ for some $\lambda_1, \ldots, \lambda_n \in A$, then there is no $\mathfrak{p} \in \text{Spec}^{\infty}(A)$ such that $\{a_{i_1}, \ldots, a_{i_n}\} \subseteq \mathfrak{p}$, i.e., $\text{Spec}^{\infty}(A) \subseteq \bigcup_{i \in \mathcal{I}} D^{\infty}(a_i)$.

(a) ⇒ (b): Suppose that $A \neq I$. Then $A^{\times} \cap I = \varnothing$. Since $A^{\times} \subseteq A$ is a $\mathcal{C}^{\infty}$-saturated subset of A $(A^{\times} = \eta_1^{-1}[(A\{1^{-1}\})^{\times}])$, then by the **Separation** Theorem 5.1.(b), it follows that $A^{\times} \cap \sqrt[\infty]{I} = \varnothing$.

By the **Separation** Theorem 5.1.(d), there is $\mathfrak{p} \in \text{Spec}^{\infty}(A)$ such that $\sqrt[\infty]{I} \subseteq \mathfrak{p}$, thus $\mathfrak{p} \in \text{Spec}^{\infty}(A) \setminus \bigcup_{i \in \mathcal{I}} D^{\infty}(a_i)$. □

6. Order theory of $\mathcal{C}^\infty$-rings

The class of $\mathcal{C}^\infty$-rings carries good notions of order theory for rings. As pointed out by Moerdijk and Reyes in [18], every $\mathcal{C}^\infty$-ring A has a canonical (strict) preorder - this and other aspects of the order theory of $\mathcal{C}^\infty$-rings are developed in [7]. The key point of this section is to introduce the notion of the smooth real spectrum of a $\mathcal{C}^\infty$-rings (made in [2], denoted by Sper^∞) and to describe, as a consequence of the separation theorems presented in the previous section, a spectral bijection from the smooth Zariski spectrum to the real smooth spectrum of a $\mathcal{C}^\infty$-ring: comparing this smooth algebraic scenario with the usual commutative algebraic setting, this is a surprising result. We begin by describing the notion of "order":

Definition 6.1. Given a $\mathcal{C}^\infty$-ring (A, Φ), we write:

$$(\forall a \in A)(\forall b \in A)(a \prec b \iff (\exists c \in A^\times)(b - a = c^2))$$

Note that if $A \neq \{0\}$, $\prec$ is an irreflexive relation, i.e., $(\forall a \in A)(\neg(a \prec a))$.

According to [17], we have the following facts about the order $\prec$ defined above.

Fact 6.2. *Let $A = \dfrac{\mathcal{C}^\infty(\mathbb{R}^E)}{I}$ for some set E. Then, given any $f + I, g + I \in A$, with respect to the relation $\prec$, given in Definition 6.1, we have:*

$$f \prec g \iff (\exists \varphi \in I)((\forall x \in Z(\varphi))(f(x) < g(x)))$$

so $\prec$ is compatible with the ring structure which underlies A, i.e.:
(i) $0 + I \prec f + I, g + I \Rightarrow 0 \prec (f + I) \cdot (g + I)$;
(ii) $0 + I \prec f + I, g + I \Rightarrow 0 \prec f + g + I$

Fact 6.3. *Let $A = \dfrac{\mathcal{C}^\infty(\mathbb{R}^E)}{I}$ for some set E be a $\mathcal{C}^\infty$-field, so $I = \sqrt[\infty]{I}$. The relation $\prec$, given in Definition 6.1, is such that:*

$$(\forall f + I \in A)(f + I \neq 0 + I \rightarrow (f + I \prec 0 + I) \vee (0 + I \prec f + I))$$

We have the following:

Proposition 6.4 (Corollary 3.15 of [7]). *For any $\mathcal{C}^\infty$-ring A, we have:*

$$1 + \sum A^2 \subseteq A^\times,$$

where $\sum A^2 = \{\sum_{i=1}^n a_i^2 | n \in \mathbb{N}, a_i \in A\}$. In particular, every $\mathcal{C}^\infty$ ring A is such that its underlying commutative unital ring, $\mathcal{U}(A)$ is a semi-real ring.

Fact 6.5. *Let $n \in \mathbb{N}$. We have:*

$$h + I \in \left(\frac{\mathcal{C}^\infty(\mathbb{R}^n)}{I}\right)^\times \iff (\exists \varphi \in I)(\forall x \in Z(\varphi))(h(x) \neq 0)$$

Recall that a totally ordered field $(F, \leq)$ is **real closed** if it satisfies:
(a) $(\forall x \in F)(0 < x \rightarrow (\exists y \in F)(x = y^2))$;
(b) every polynomial of odd degree has, at least, one root;

Remark 6.6. A $\mathcal{C}^\infty$**-polynomial in one variable** is an element of $\mathcal{C}^\infty(\mathbb{R}^n)\{t\} = A \otimes_\infty \mathcal{C}^\infty(\mathbb{R})$ (cf. Section 4.3 of [4]). More generally, a $\mathcal{C}^\infty$**-polynomial in set S of variables** is an element of $A\{S\}$.

As pointed out in Theorem 2.10 of [17], we have the following:

Fact 6.7. *Every $\mathcal{C}^\infty$-field, F, together with its canonical preorder $\prec$ given in Definition 6.1, is such that $\mathcal{U}(F)$ is a real closed field.*

As a consequence of the above fact, given any polynomial $p(x) \in F[x]$ and any $f, g \in F$ such that $f \prec g$, if $p(f) < p(g)$ then there is some $h \in]f, g[$ such that $p(h) = 0$.

We have the $\mathcal{C}^\infty$-analog of the notion of "real closedness":

Definition 6.8. Let $(F, \prec)$ be a $\mathcal{C}^\infty$-field. We say that $(F, \prec)$ is $\mathcal{C}^\infty$-**real closed** if, and only if:

$$(\forall f \in F\{x\})((f(0) \cdot f(1) < 0)\&(1 \in \langle\{f, f'\}\rangle \subseteq F\{x\}) \to (\exists \alpha \in]0,1[\subseteq F)(f(\alpha) = 0))$$

Fact 6.9. *As proved in* Theorem 2.10' of [17], *every $\mathcal{C}^\infty$-field is $\mathcal{C}^\infty$-real closed.*

From the preceding proposition, we conclude that $f + I \prec g + I$ occurs if, and only if, there is a "witness" $\varphi \in I$ such that $(\forall x \in Z(\varphi))(f(x) < g(x))$.

Given any $\mathcal{C}^\infty$-ring A and any $\mathfrak{p} \in \mathrm{Spec}^\infty(A)$, let:

$$k_{\mathfrak{p}} := \left(\frac{A}{\mathfrak{p}}\right)\left\{\frac{A}{\mathfrak{p}} \setminus \{0+\mathfrak{p}\}^{-1}\right\},$$

that is, $k_{\mathfrak{p}}(A)$ is the $\mathcal{C}^\infty$-field obtained by taking the quotient $\frac{A}{\mathfrak{p}}$:

$$q_{\mathfrak{p}} : A \to \frac{A}{\mathfrak{p}}$$

and then taking its $\mathcal{C}^\infty$-ring of fractions with respect to $\frac{A}{\mathfrak{p}} \setminus \{0+\mathfrak{p}\}$,

$$\eta_{\frac{A}{\mathfrak{p}}\setminus\{0+\mathfrak{p}\}} : \frac{A}{\mathfrak{p}} \to \left(\frac{A}{\mathfrak{p}}\right)\left\{\frac{A}{\mathfrak{p}} \setminus \{0+\mathfrak{p}\}^{-1}\right\}.$$

The family of $\mathcal{C}^\infty$-fields $\{k_{\mathfrak{p}}(A) | \mathfrak{p} \in \mathrm{Spec}^\infty(A)\}$ has the following multi-universal property:

"Given any $\mathcal{C}^\infty$-homomorphism $f : A \to \mathbb{K}$, where $\mathbb{K}$ is a $\mathcal{C}^\infty$-field, there is a unique $\mathcal{C}^\infty$-radical prime ideal $\mathfrak{p}$ and a unique $\mathcal{C}^\infty$-homomorphism $\widetilde{f} : k_{\mathfrak{p}}(A) \to \mathbb{K}$ such that the following diagram commutes:

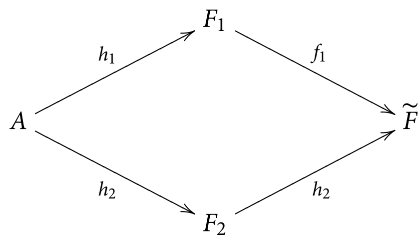
$$\begin{array}{ccc} A & \xrightarrow{\alpha_{\mathfrak{p}}} & k_{\mathfrak{p}}(A) \\ & \searrow^{f} & \vdots\, \widetilde{f} \\ & & K \end{array},$$

where $\alpha_{\mathfrak{p}} = \eta_{\frac{A}{\mathfrak{p}}\setminus\{0+\mathfrak{p}\}} \circ q_{\mathfrak{p}} : A \to k_{\mathfrak{p}}(A)$."

Thus, given $f : A \to K$, take $\mathfrak{p} = \ker(f)$, so $\overline{f}$ is injective, $\frac{A}{\mathfrak{p}}$ is a $\mathcal{C}^\infty$-reduced $\mathcal{C}^\infty$-ring. By the universal property of the smooth fraction field $k_{\mathfrak{p}}(A)$, there is a unique arrow $\widetilde{f} : k_{\mathfrak{p}}(A) \to K$ such that the following diagram commutes:

$$\begin{array}{ccc} \frac{A}{\mathfrak{p}} & \xrightarrow{\alpha_{\mathfrak{p}}} & k_{\mathfrak{p}}(A) \\ & \searrow^{f} & \downarrow \widetilde{f} \\ & & K \end{array}$$

Definition 6.10. Let $\mathcal{F}$ be the (proper) class of all the $\mathcal{C}^\infty$-homomorphisms of A to some $\mathcal{C}^\infty$-field. We define the following relation $\mathcal{R}$: given $h_1 : A \to F_1$ and $h_2 : A \to F_2$, we say that h_1 is related with h_2 if, and only if, there is some $\mathcal{C}^\infty$-field $\widetilde{F}$ and some $\mathcal{C}^\infty$-fields homomorphisms $\mathcal{C}^\infty$ $f_1 : F_1 \to \widetilde{F}$ and $f_2 : F_2 \to \widetilde{F}$ such that the following diagram commutes:



The relation $\mathcal{R}$ defined above is symmetric and reflexive.

Let $h_1 : A \to F_1$ and $h_2 : A \to F_2$ be two $\mathcal{C}^\infty$-homomorphisms of A to the $\mathcal{C}^\infty$-fields F_1, F_2 such that $(h_1, h_2) \in \mathcal{R}$, and let $f_1 : F_1 \to \widetilde{F}$ and $f_2 : F_2 \to \widetilde{F}$ be two $\mathcal{C}^\infty$-homomorphisms to the $\mathcal{C}^\infty$-field $\widetilde{F}$ such that $f_1 \circ h_1 = f_2 \circ h_2$, so:

$$(f_1 \circ h_1)^{\dashv}[\{0\}] = (f_2 \circ h_2)^{\dashv}[\{0\}]$$

Then

$$\ker(h_1) = h_1^{\dashv}[\{0\}] = h_1^{\dashv}[f_1^{\dashv}[\{0\}]] = h_2^{\dashv}[f_2^{\dashv}[\{0\}]] = h_2^{\dashv}[\{0\}] = \ker(h_2).$$

The above considerations prove the following:

Proposition 6.11. *If $h_1 : A \to F_1$ and $h_2 : A \to F_2$ be two $\mathcal{C}^\infty$-homomorphisms from the $\mathcal{C}^\infty$-ring A to the $\mathcal{C}^\infty$-fields F_1, F_2 such that $(h_1, h_2) \in \mathcal{R}$, then* $\ker(h_1) = \ker(h_2)$.

The above proposition has the following immediate consequence:

Corollary 6.12. *Keeping the same notations of the above result, let $\mathcal{R}^t$ be the transitive closure of $\mathcal{R}$. Then $\mathcal{R}^t$ is an equivalence relation on $\mathcal{F}$. We are going to denote the quotient set, $\dfrac{\mathcal{F}}{\mathcal{R}^t}$, by $\widetilde{\mathcal{F}}$.*

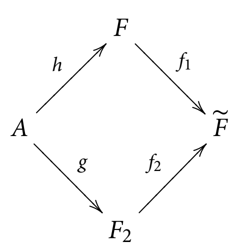
Let $F_1, F_2, \widetilde{F}$ be $\mathcal{C}^\infty$-fields and $f_1 : F_1 \to \widetilde{F}$ and $f_2 : F_2 \to \widetilde{F}$ be two following $\mathcal{C}^\infty$-fields homomorphisms. Since $\mathcal{C}^\infty$-fields homomorphisms must be injective maps, we have the following:

Proposition 6.13. *The following relation*

$$\beta = \{([h : A \to F], \ker(h)) | F \textit{ is a } \mathcal{C}^\infty - \textit{field}\} \subseteq \widetilde{\mathcal{F}} \times \mathrm{Spec}^\infty(A)$$

is a functional relation whose domain is $\widetilde{\mathcal{F}}$.

Proof. Suppose $[h : A \to F_1] = [g : A \to F_2]$, so there are maps $f_1 : F_1 \to \widetilde{F}$ and $f_2 : F_2 \to \widetilde{F}$ for some $\mathcal{C}^\infty$-field $\widetilde{F}$, such that the following diagram commutes:



Now, if $([h : A \to F_1], \ker(h)), ([g : A \to F_2], \ker(g)) \in \beta$ are such that $[h : A \to F_1] = [g : A \to F_2]$, then:

$$f_1 \circ h = f_2 \circ g$$

so

$$\ker(h) = h^{\dashv}[\{0\}] = h^{\dashv}[f_1^{\dashv}[\{0\}]] = \ker(f_1 \circ h) = \ker(f_2 \circ g) = g^{\dashv}[f_2^{\dashv}[\{0\}]] = g^{\dashv}[\{0\}] = \ker(g)$$

□

Definition 6.14. Let A be a $\mathcal{C}^\infty$-ring. A $\mathcal{C}^\infty$-**ordering** in A is a subset $P \subseteq A$ such that:
(O1) $P + P \subseteq P$;
(O2) $P \cdot P \subseteq P$;
(O3) $P \cup (-P) = A$
(O4) $P \cap (-P) = \mathfrak{p} \in \mathrm{Spec}^\infty(A)$

Definition 6.15. Let A be a $\mathcal{C}^\infty$-ring. Given a $\mathcal{C}^\infty$-ordering P in A, the $\mathcal{C}^\infty$-**support** of A is given by:

$$\mathrm{supp}^\infty(P) := P \cap (-P)$$

Definition 6.16. Let A be a $\mathcal{C}^\infty$-ring. The $\mathcal{C}^\infty$-**real spectrum of** A is given by:

$$\mathrm{Sper}^\infty(A) = \{P \subseteq A | P \text{ is an ordering of the elements of } A\}$$

together with the (spectral) topology generated by the sets:

$$H^\infty(a) = \{P \in \mathrm{Sper}^\infty(A) | a \in P \setminus \mathrm{supp}^\infty(P)\}$$

for every $a \in A$. The topology generated by these sets will be called "smooth Harrison topology," and will be denoted by Har^∞.

Proposition 6.17. *Given a $\mathcal{C}^\infty$-ring A, we have a function given by:*

$$\begin{array}{cccc} \mathrm{supp}^\infty : & (\mathrm{Sper}^\infty(A), \mathrm{Har}^\infty) & \to & (\mathrm{Spec}^\infty(A), \mathrm{Zar}^\infty) \\ & P & \mapsto & P \cap (-P) \end{array}$$

which is spectral, and thus continuous, since given any $a \in A$, ${\mathrm{supp}^\infty}^{\dashv}[D^\infty(a)] = H^\infty(a) \cup H^\infty(-a)$.

Proof. Given $P \in \mathrm{Sper}^\infty(A)$, since $P \cup -P = A$, we have $P \in H^\infty(a) \cup H^\infty(-a)$ if, and only if $a \in (P \setminus (P \cap -P)) \cup (-P \setminus (P \cap -P))$ if, and only if $a \in A \setminus \mathrm{supp}^\infty(P)$ if, and only if $\mathrm{supp}^\infty(P) \in D^\infty(a)$. □

Unlike what happens to a general commutative unital ring R, for which the mapping:

$$\begin{array}{cccc} \mathrm{supp} : & \mathrm{Sper}(R) & \to & \mathrm{Spec}(R) \\ & P & \mapsto & P \cap (-P) \end{array}$$

is seldom surjective or injective, within the category of $\mathcal{C}^\infty$-rings, supp^∞ is, as a matter of fact, a bijection. In order to prove this fact, we are going to need some preliminary results, given below.

Lemma 6.18. *Let A be a $\mathcal{C}^\infty$-ring and $\mathfrak{p}$ any $\mathcal{C}^\infty$-radical prime ideal, and let $\hat{\mathfrak{p}}$ be the maximal ideal of $A_{\{\mathfrak{p}\}}$. Then:*

$$\mathrm{Can}_{\mathfrak{p}}^{\dashv}[\hat{\mathfrak{p}}] = \mathfrak{p}.$$

Proof. Let $a \in \mathfrak{p}$, then $\mathrm{Can}_{\mathfrak{p}}(a) \in \hat{\mathfrak{p}}$, and $\mathfrak{p} \subseteq \mathrm{Can}_{\mathfrak{p}}^{\dashv}[\hat{\mathfrak{p}}]$. Now, if $a \in A \setminus \mathfrak{p}$ then $\mathrm{Can}_{\mathfrak{p}}(a) \in \mathcal{U}(A\{(A\setminus\mathfrak{p})^{-1}\})$. Since $A_{\{\mathfrak{p}\}}$ is a local ring, $A_{\{\mathfrak{p}\}} = \hat{\mathfrak{p}} \dot{\cup} \mathcal{U}(A\{(A \setminus \mathfrak{p})^{-1}\})$, so $\mathrm{Can}_{\mathfrak{p}}(a) \in A\{(A \setminus \mathfrak{p})^{-1}\} \setminus \hat{\mathfrak{p}}$, and therefore $\mathrm{Can}_{\mathfrak{p}}^{\dashv}[\hat{\mathfrak{p}}] \subseteq \mathfrak{p}$. □

Theorem 6.19. *Let A be a $\mathcal{C}^\infty$-ring and $\mathcal{R}^t$ be the relation defined above. The function:*

$$\begin{array}{rccc} \alpha' : & \mathrm{Sper}^\infty(A) & \to & \frac{\mathcal{F}}{\mathcal{R}^t} \\ & P & \mapsto & [\eta_{P\cap(-P)}] \end{array}$$

is the inverse function of:

$$\begin{array}{rccc} \beta' : & \frac{\mathcal{F}}{\mathcal{R}^t} & \to & \mathrm{Sper}^\infty(A) \\ & [h : A \to K] & \mapsto & h^{\dashv}[K^2] \end{array}$$

Proof. Note that:

$$(\alpha' \circ \beta')([h : A \to F]) = \alpha'(h^{\dashv}[F^2]) = [\eta_{\mathrm{supp}\,(h^{\dashv}[F^2])}],$$

where $\mathrm{supp}^\infty(h^{\dashv}[F^2]) = h^{\dashv}[F^2] \cap (-h^{\dashv}[F]^2) = h^{\dashv}[\{0\}] = \ker(h)$.

Thus we have:

$$(\alpha' \circ \beta')([h : A \to F]) = [\eta_{\ker(h)} : A \to k_{\ker(h)}(A)].$$

We claim that β' is the left inverse function for α', that is:

$$(\forall P \in \mathrm{Sper}^\infty(A))((\beta' \circ \alpha')(P) = P).$$

Thus, it will follow that α' is injective and β' is surjective.

We have $\beta'(\alpha'(P)) = \eta^{\dashv}_{\mathrm{supp}^\infty(P)}[k_{\mathfrak{p}}(A)^2]$, so we need to show that:

$$\eta^{\dashv}_{\mathrm{supp}(P)}[k_{\mathfrak{p}}(A)^2] = P$$

Let $\mathfrak{p} = \mathrm{supp}^\infty(P)$.

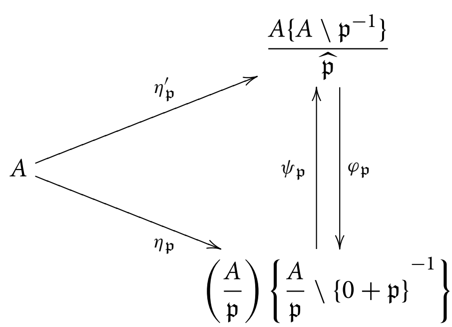
Ab absurdo, suppose

$$\eta^{\dashv}_{\mathfrak{p}}[k_{\mathfrak{p}}(A)^2] \nsubseteq P$$

There must exist some $x \in A$ such that $x \in \eta^{\dashv}_{\mathfrak{p}}[k_{\mathfrak{p}}(A)^2]$ and $x \notin P$. We have:

$$\eta_{\mathfrak{p}}(x) \in \left(\frac{A\{A \setminus \mathfrak{p}^{-1}\}}{\widehat{\mathfrak{p}}}\right)^2 \quad \text{and} \quad x \notin P.$$

Now, since by Theorem 24, p. 97 of [5] (denoting $\mathfrak{m}_{\mathfrak{p}}$ by $\widehat{\mathfrak{p}}$ instead) the following diagram commutes:



where $\eta_{\mathfrak{p}} = \eta_{\frac{A}{\mathfrak{p}}\setminus\{0+\mathfrak{p}\}} \circ q_{\mathfrak{p}}$ and $\eta'_{\mathfrak{p}} = q_{\widehat{\mathfrak{p}}} \circ \eta_{A\setminus\mathfrak{p}}$ and $\varphi_{\mathfrak{p}}$ and $\psi_{\mathfrak{p}}$ are the isomorphisms described in that theorem. Thus, we have:

$$\eta_{\mathfrak{p}}(x) \in \left(k_{\mathfrak{p}}(A)\right)^2 \Rightarrow \psi_{\mathfrak{p}}(\eta_{\mathfrak{p}}(x)) = \eta'_{\mathfrak{p}}(x) \in \left(\frac{A\{A \setminus \mathfrak{p}^{-1}\}}{\widehat{\mathfrak{p}}}\right)^2$$

and

$$\eta_{\mathfrak{p}}(x) \in \left(k_{\mathfrak{p}}(A)\right)^2 \Rightarrow (\exists (g+\widehat{\mathfrak{p}}) \in \frac{A\{A\setminus \mathfrak{p}^{-1}\}}{\widehat{\mathfrak{p}}})(\eta'_{\mathfrak{p}}(x) = g^2 + \widehat{\mathfrak{p}})$$

Since $q_{\widehat{\mathfrak{p}}}$ is surjective, given this $g + \widehat{\mathfrak{p}} \in \left(\frac{A\{A\setminus\mathfrak{p}^{-1}\}}{\widehat{\mathfrak{p}}}\right)$, there is some $\theta \in A\{A\setminus \mathfrak{p}^{-1}\}$ such that $q_{\widehat{\mathfrak{p}}}(\theta) = g + \widehat{\mathfrak{p}}$.

By Theorem 1.4, item (i) of [17], given this $\theta \in A\{A\setminus\mathfrak{p}^{-1}\}$, there are $a \in A$ and $b \in A \setminus \mathfrak{p}^{\infty-\mathrm{sat}}$, that is,

$$\mathrm{Can}_{\mathfrak{p}}(b) \in (A\{A\setminus\mathfrak{p}^{-1}\})^{\times}$$

such that:

$$\theta = \frac{\mathrm{Can}_{\mathfrak{p}}(a)}{\mathrm{Can}_{\mathfrak{p}}(b)}$$

or equivalently, since $(A\setminus\mathfrak{p})^{\infty-\mathrm{sat}} = A\setminus\mathfrak{p}$:

$$b \notin \mathfrak{p} \tag{1}$$

Hence,

$$\begin{gathered}
\eta'_{\mathfrak{p}}(x) = g^2 + \widehat{\mathfrak{p}} = q_{\widehat{\mathfrak{p}}}\left(\frac{\mathrm{Can}_{\mathfrak{p}}(a)}{\mathrm{Can}_{\mathfrak{p}}(b)}\right)^2 = \left(\frac{\mathrm{Can}_{\mathfrak{p}}(a)}{\mathrm{Can}_{\mathfrak{p}}(b)}\right)^2 + \widehat{\mathfrak{p}}\\
\eta'_{\mathfrak{p}}(x)\cdot(\mathrm{Can}^2_{\mathfrak{p}}(b) + \widehat{\mathfrak{p}}) = \mathrm{Can}^2_{\mathfrak{p}}(a) + \widehat{\mathfrak{p}}\\
\mathrm{Can}_{\mathfrak{p}}(x\cdot b^2 - a^2) \in \widehat{\mathfrak{p}}.\\
(x\cdot b^2 - a^2) \in \mathrm{Can}^{\dashv}_{\mathfrak{p}}[\widehat{\mathfrak{p}}].
\end{gathered}$$

By Lemma 6.18, $\mathfrak{p} = \mathrm{Can}^{\dashv}_{\mathfrak{p}}[\widehat{\mathfrak{p}}]$, so

$$x\cdot b^2 - a^2 \in \mathfrak{p} \subseteq P$$

Let $y = x\cdot b^2 + (-a^2) \in \mathfrak{p} \subseteq P$. Note that since $x \notin P$, $x \in (-P)\setminus\mathfrak{p}$ and

$$x\cdot b^2 \in (-P) \tag{2}$$

Since $y \in P$,

$$x\cdot b^2 = \underbrace{y}_{\in P} + \overbrace{a^2}^{\in P} \in P,$$

$$x\cdot b^2 \in P \tag{3}$$

By (2) and (3), it follows that $x\cdot b^2 \in P\cap(-P) = \mathfrak{p}$. Since $\mathfrak{p}$ is prime, either $x \in \mathfrak{p}$ or $b^2 \in \mathfrak{p}$. However, since $x \notin P$, *a fortiori*, $x \notin \mathfrak{p}$, so we must have $b^2 \in \mathfrak{p}$. Once again, since $\mathfrak{p}$ is prime, it follows that $b \in \mathfrak{p}$ which contradicts (1). Hence,

$$\eta'_{\mathrm{supp}^{\infty}(P)}{}^{\dashv}\left[\left(\frac{A\{A\setminus\mathrm{supp}^{\infty}(P)^{-1}\}}{\widehat{\mathfrak{p}}}\right)^2\right] \subseteq P$$

.

Now we claim that:

$$P \subseteq \eta'_{\mathrm{supp}(P)}{}^{\dashv}\left[\left(\frac{A\{A\setminus\mathrm{supp}^{\infty}(P)^{-1}\}}{\widehat{\mathfrak{p}}}\right)^2\right]$$

Conversely, suppose, *ab absurdo* that

$$P \nsubseteq {\eta'_{\mathrm{supp}(P)}}^{\dashv}\left[\left(\frac{A\{A \setminus \mathrm{supp}^{\infty}(P)^{-1}\}}{\widehat{\mathfrak{p}}}\right)^2\right] \tag{4}$$

so there must exist some $x \in P$ such that

$$(\forall (g + \widehat{\mathfrak{p}}) \in \frac{A\{A \setminus \mathfrak{p}^{-1}\}}{\widehat{\mathfrak{p}}})(\eta'_{\mathfrak{p}}(x) \neq g^2 + \widehat{\mathfrak{p}}) \tag{5}$$

Equivalently, there must exist some $(h + \widehat{\mathfrak{p}}) \in \frac{A\{A\setminus\mathfrak{p}^{-1}\}}{\widehat{\mathfrak{p}}}$ such that $\eta_{\mathfrak{p}}(x) = -h^2 + \widehat{\mathfrak{p}}$.

Thus, since $q_{\widehat{\mathfrak{p}}}$ is surjective, given such an $h + \widehat{\mathfrak{p}} \in \dfrac{A\{A \setminus \mathfrak{p}^{-1}\}}{\widehat{\mathfrak{p}}}$ there must be some $\zeta \in A\{A \setminus \mathfrak{p}^{-1}\}$ such that $q_{\widetilde{\mathfrak{p}}}(\zeta) = h + \widehat{\mathfrak{p}}$. By item (i) of Theorem 3.18, there are $c \in A$ and $d \in A$ with:

$$\mathrm{Can}_{\mathfrak{p}}(d) \in A\{A \setminus \mathfrak{p}^{-1}\}^{\times},$$

equivalently

$$d \in (A \setminus \mathfrak{p})^{\infty-\mathrm{sat}},$$

and since $(A \setminus \mathfrak{p})^{\infty-\mathrm{sat}} = A \setminus \mathfrak{p}$,

$$d \notin \mathfrak{p} \tag{6}$$

such that:

$$\zeta = \frac{\mathrm{Can}_{\mathfrak{p}}(c)}{\mathrm{Can}_{\mathfrak{p}}(d)}.$$

Hence,

$$\eta_{\mathfrak{p}}(x) = -\frac{\mathrm{Can}^2_{\mathfrak{p}}(c)}{\mathrm{Can}^2_{\mathfrak{p}}(d)} + \widehat{\mathfrak{p}}$$
$$\mathrm{Can}^2_{\mathfrak{p}}(c) + \mathrm{Can}_{\mathfrak{p}}(x) \cdot \mathrm{Can}^2_{\mathfrak{p}}(d) \in \widehat{\mathfrak{p}}$$
$$\mathrm{Can}_{\mathfrak{p}}(c^2 + x \cdot d^2) \in \widehat{\mathfrak{p}}$$

so

$$\underbrace{c^2}_{\in P} + \overbrace{x \cdot d^2}^{\in P} = z = \mathrm{Can}_{\mathfrak{p}}^{\dashv}[\widehat{\mathfrak{p}}] = \mathfrak{p} \subseteq (-P)$$
$$x \cdot d^2 = z - c^2 \in (-P)$$

Since $x \in P$, we also have $x \cdot d^2 \in P$, hence $x \cdot d^2 \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, either $x \in \mathfrak{p}$ or $d^2 \in \mathfrak{p}$. Now, if $x \in \mathfrak{p}$ then $\mathrm{Can}_{\mathfrak{p}}(x) = 0^2 + \mathfrak{p}$, which contradicts our hypothesis (5). On the other hand, if $d^2 \in \mathfrak{p}$, then $d \in \mathfrak{p}$, and this contradicts (6). Hence $x \notin \mathfrak{p}$ and $d^2 \notin \mathfrak{p}$, so $x \cdot d^2 \notin \mathfrak{p}$. Thus we achieved an absurdity: $(x \cdot d^2 \in \mathfrak{p})\&(x \cdot d^2 \notin \mathfrak{p})$. It follows that our premise (4) must be false, so:

$$P \subseteq {\eta'_{\mathrm{supp}(P)}}^{\dashv}\left[\left(\frac{A\{A \setminus \mathrm{supp}^{\infty}(P)^{-1}\}}{\widehat{\mathfrak{p}}}\right)^2\right].$$

Hence $P = {\eta'_{\mathrm{supp}(P)}}^{\dashv}\left[\left(\frac{A\{A\setminus\mathrm{supp}^{\infty}(P)^{-1}\}}{\widehat{\mathfrak{p}}}\right)^2\right]$.

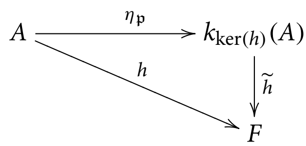
Now we need only to show that $\alpha' \circ \beta' = \mathrm{id}_{\widetilde{\mathcal{F}}}$.

Let $[h : A \to F] \in \widetilde{\mathcal{F}}$. We have:

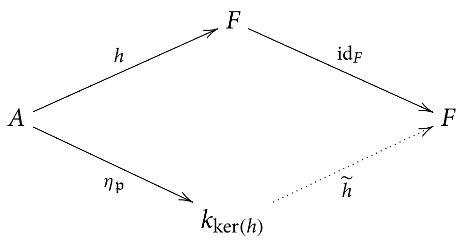
$$(\alpha' \circ \beta')([h : A \to F]) = \alpha'(h^{\dashv}[F^2]) = [\eta_{\mathrm{supp}^\infty(h^{\dashv}[F^2])} : A \to k_{\mathrm{supp}^\infty(h^{\dashv}[F^2])}(A)].$$

It suffices to show that $[h] = [\eta_{\mathrm{supp}^\infty(h^{\dashv}[F^2])}]$. Note that $\mathrm{supp}^\infty(h^{\dashv}[F^2]) = h^{\dashv}[F^2 \cap (-F^2)] = \ker(h)$.

By the universal property of the $\mathcal{C}^\infty$-field of fractions of $\left(\frac{A}{\ker(h)}\right)$, $k_{\ker(h)}(A)$, since $h\left[A^\times\right] \subseteq F^\times$ (for $\mathcal{C}^\infty$-homomorphisms preserve invertible elements), there is a unique $\mathcal{C}^\infty$-homomorphism $\widetilde{h} : k_{\ker(h)}(A) \to F$ such that the following diagram commutes:



We have, then, the following commutative diagram:



so $[h : A \to F] = [\eta_{\ker(h)} : A \to k_{\ker(h)}]$ and

$$(\alpha' \circ \beta')([h : A \to F]) = [h : A \to F].$$

Hence it follows that α' and β' are inverse bijections of each other. □

Theorem 6.20. *The map:*

$$\begin{array}{cccc} \alpha : & \mathrm{Spec}^\infty(A) & \to & \widetilde{\mathcal{F}} \\ & \mathfrak{p} & \mapsto & [\eta_{\mathfrak{p}}] = [q \circ \mathrm{Can}_{\mathfrak{p}}] \end{array}$$

is a bijection whose inverse is given by:

$$\begin{array}{cccc} \beta : & \widetilde{\mathcal{F}} & \to & \mathrm{Spec}^\infty(A) \\ & [h : A \to F] & \mapsto & \ker(h) \end{array}$$

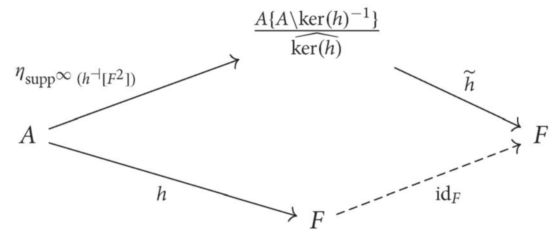
Proof. First we are going to show that $\alpha \circ \beta = \mathrm{id}_{\widetilde{\mathcal{F}}}$, so β is an injective map and α is its left inverse, hence it is surjective.

Let $[h : A \to F] \in \widetilde{F}$. We have:

$$(\alpha \circ \beta)([h : A \to F]) = \alpha(\ker(h)) = \left[\eta'_{\ker(h)} : A \to \left(A\{A \setminus \ker(h)^{-1}\}/\widehat{\ker(h)}\right)\right]$$

It suffices to show that $[h : A \to F] = \left[\eta'_{\ker(h)} : A \to \left(A\{A \setminus \ker(h)^{-1}\}/\widehat{\ker(h)}\right)\right]$.

Since $\left(A\{A \setminus \ker(h)^{-1}\}/\widehat{\ker(h)}\right)$ is (up to $\mathcal{C}^\infty$-isomorphism) the $\mathcal{C}^\infty$-field of fractions of $(A/\ker(h))$, $k_{\ker(h)}(A)$, there is a unique $\mathcal{C}^\infty$-homomorphism $\widetilde{h} : \left(A\{A \setminus \ker(h)^{-1}\}/\widehat{\ker(h)}\right) \to F$ such that the following diagram commutes:



and the equality holds, i.e.,

$$[h : A \to F] = \left[\eta_{\ker(h)} : A \to \left(A\{A \setminus \ker(h)^{-1}\}/\widehat{\ker(h)}\right)\right]$$

It follows that $\alpha \circ \beta = \mathrm{id}_{\widetilde{\mathcal{F}}}$, α is a surjective map and β is an injective map.

On the other hand, given $\mathfrak{p} \in \mathrm{Spec}^{\infty}(A)$, we have:

$$(\beta \circ \alpha)(P) = \beta([\eta_{\mathfrak{p}} : A \to k_{\mathfrak{p}}]) = \ker(\eta_{\mathfrak{p}}) = \mathrm{Can}_{\mathfrak{p}}^{\dashv}[\widehat{\mathfrak{p}}] = \mathfrak{p}$$

so:

$$(\beta \circ \alpha) = \mathrm{id}_{\mathrm{Spec}^{\infty}(A)}$$

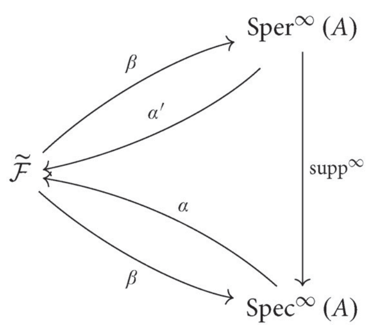
□

as a Corollary of the theorems 6.19 and 6.20, we have:

Lemma 6.21. *Let A be a $\mathcal{C}^{\infty}$-ring, and define:*

$$\begin{array}{rccc} \mathrm{supp}^{\infty}: & \mathrm{Sper}^{\infty}(A) & \to & \mathrm{Spec}^{\infty}(A) \\ & P & \mapsto & P \cap (-P) \end{array}$$

The following diagram commutes:



that is to say that:

$$\alpha \circ \mathrm{supp}^{\infty} = \alpha'$$

and

$$\mathrm{supp}^{\infty} \circ \beta' = \beta$$

Proof. Note that if we prove that $\alpha \circ \mathrm{supp}^{\infty} = \alpha'$, then composing both sides with β' yields:

$$(\alpha \circ \mathrm{supp}^{\infty}) \circ \beta' = \alpha' \circ \beta' = \mathrm{id}_{\widetilde{\mathcal{F}}}$$

so

$$\alpha \circ (\mathrm{supp}^{\infty} \circ \beta') = \mathrm{id}_{\widetilde{\mathcal{F}}}$$

and by the uniqueness of the inverse of α, it follows that:

$$\text{supp}^\infty \circ \beta' = \beta.$$

Now we are going to prove that $\alpha \circ \text{supp}^\infty = \alpha'$.

Given $P \in \text{Sper}^\infty(A)$ we have:

$$(\alpha \circ \text{supp}^\infty)(P) = \alpha(\text{supp}^\infty(P)) = [\eta_{\text{supp}^\infty(P)} : A \to k_{\text{supp}^\infty(P)}(A)] =: \alpha'(P),$$

so the result holds. □

About the map $P \mapsto P \cap (-P)$, from the real spectrum of a ring to the prime one, in p. 84 of [16], M. Marshall points out that "This is neither surjective nor injective in general." As an important result of the theory of $\mathcal{C}^\infty$-rings which distinguishes it from the theory of the rings, we have the following:

Theorem 6.22. *Let A be a $\mathcal{C}^\infty$-ring. The following map:*

$$\begin{array}{rccc} \text{supp}^\infty : & \text{Sper}^\infty(A) & \to & \text{Spec}^\infty(A) \\ & P & \mapsto & P \cap (-P) \end{array}$$

is a spectral bijection.

Proof. In Remark 6.17, we have already seen that supp^∞ is a spectral function, so we need only to show that is a bijection.

Just note that $\text{supp}^\infty = \beta \circ \alpha' = \alpha \circ \beta'$, and since supp^∞ is a composition of bijections, it is a bijection. □

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