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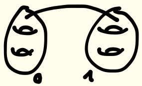
If f is a 3-manifold, that can be written as: $S \times [0,1] / \sim$,

S of finite type (cpct with finitely many pts removed), $f: S \to S$ orient. preserving homeo.,

$(x,0) \sim (f(x),1)$.

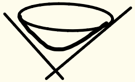


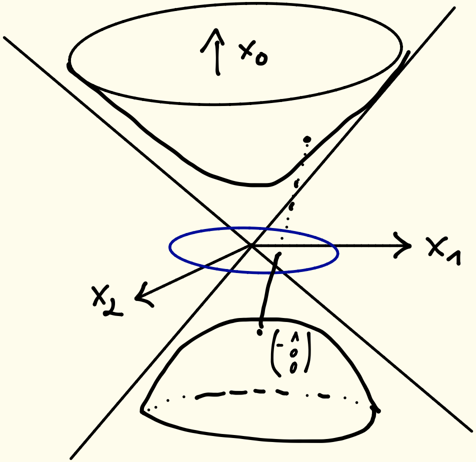
Then M_f admits a hyperbolic structure (complete, finite volume)

iff f homotopic to a pseudo-Anosov homeomorphism.

Some reminder:

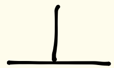
$\mathbb{H}^n \subset \mathbb{R}^{n+1} = \{-x_0^2 + (x_1^2 + \cdots + x_n^2) = 1, x_0 > 0\}$ with inner product $dx_1^2 + \cdots + dx_n^2 - dx_0^2$.

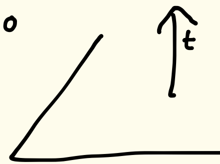
 This expression is a Riem.-metric on tangent space to upper hyperboloid.



We will need 2 other models:

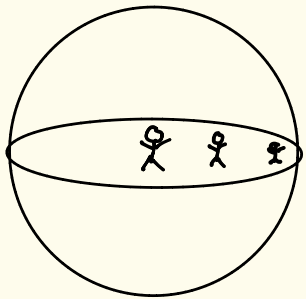
1) $\mathbb{R}^{n-1}\times(0,\infty)$ $(\underline{x}, t)$ $\qquad \dfrac{d\underline{x}^2+dt^2}{t^2}$

ex:  $x+iy,\ y>0$ $\qquad \dfrac{|dz|}{y}$



$\dfrac{|dz|^2+dt^2}{t^2}$

2) Ball model: $\{\underline{x}\in\mathbb{R}^n,\ |\underline{x}|^2<1\}$, $\dfrac{4|d\underline{x}|^2}{(1-|\underline{x}|^2)^2}$



Hyp. manifold: is a Riem. manifold locally isometric to $\mathbb{H}^n$, where universal cov. sp. is isometric to $\mathbb{H}^n$.

$\Gamma \subset SO^+(n,1)$. For $n=3$: $SO^+(3,1) \simeq PSL_2\mathbb{C}$.

Let $\mathcal{H}^2$ be space of hermitian matrices: $\begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$, $a,d \in \mathbb{R}$, $b \in \mathbb{C}$, det is $ad - |b|^2 = \frac{1}{4}\left((a+d)^2 - (a-d)^2 - (\mathrm{Re}\, b)^2 - (\mathrm{Im}\, b)^2\right)$

(due to Von Neumann).

Now: $A.H = A^{-1} H A$ is action of $SL_2\mathbb{C}$ on these matrices.

so gives a map $SL_2\mathbb{C} \longrightarrow SO(3,1)$.

$\boxed{\mathrm{Aut}\, \mathbb{H}^3 = PSL_2\mathbb{C}}$ (Rk: Poincaré knew: $\mathrm{Aut}\, \mathbb{H}^2 = PSL_2\mathbb{R}$.

Thm: an orientation preserving homeo. of a surface of finite type is homotopic to:

- a map of finite order,
- a reducible map,
- a pseudo-Anosov

The group of or. preserving homeos. of S up to the homotopy is mapping class group $MCG(S)$.

(a finitely gen. group). Most homeos are pseudo Anosov.

Def: $f : S \to S$ is pseudo Anosov if $\exists$ homeo $\varphi : S \to X$ a Riem. surface, and an integrable holomorphic quadratic differential $q \in Q^1(X)$ and a map (homeo) $g : X \to X$ s.t $g^*(|\mathrm{Re}\sqrt{q}|) = \lambda |\mathrm{Re}\sqrt{q}|$, $g^*(|\mathrm{Im}\sqrt{q}|) = \frac{1}{\lambda}(|\mathrm{Im}\sqrt{q}|)$, $\lambda > 1$ and:

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ S & \xrightarrow{\varphi} & X \end{array}$$

commutes up to homotopy.

Def: quadr. diff is locally $f(\xi)\,d\xi^2$, section of $\Omega_X \otimes \Omega_X$.

it gives the surface locally the structure of lined paper: $\exists$ coord. in which q is written as dz^2.

Globally: can't be done (Gauss-Bonnet): so these q have zeroes.

Structure near a zero.

gluing.

$|f(z)\,dz^2| = |f(z)|\,dx\,dy$ so speaking of integrable means: as a flat surface, it has finite area.

Rk: $\int \frac{dx\,dy}{|z|} < +\infty$: simple pole is integrable $= \int_0^{2\pi}\int_0^1 \frac{\cancel{r}\,dr\,d\theta}{\cancel{r}}$.

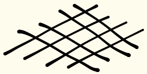
near simple pole: cone of angle π.

Thus: hor. vect. are stretched by λ
vert. vectors are contracted by λ. } $\leftarrow$ situation in pseudo Anosov case.

· finite order : $f \sim g$, $g^N = Id$.

· reducible : $\exists$ family of disj. simple closed curves, preserved.

Rk: Anosov maps $A \in SL_2\mathbb{Z}$, $\bar{A} : \mathbb{R}^2/\mathbb{Z}^2 \circlearrowleft$ det $= 1$, trace : $0, 1, 2$ or something else : $\exists$ 2 real eigenval.

So plane is foliated  $\rightsquigarrow$ irr. foliations on torus, one stretched, one expanded.

Such maps are structurally stable.

Sketch of Bers's proof of Thurston's thm :

Teichmüller space $\mathcal{T}_S = \{(X, \varphi) : \varphi : S \longrightarrow X$ homeo, or. preserving$\}/\sim$

$$\begin{tikzcd} & X_1 \arrow[d, dotted, "\alpha"] \\ S \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2"'] & X_2 \end{tikzcd}$$

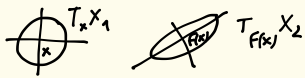
$(X_1, \varphi_1) \sim (X_2, \varphi_2)$ iff $\exists\ \alpha : X_1 \to X_2$ analytic isom. s. that φ_2 homotopic to $\alpha \circ \varphi_1$.

$\mathcal{T}_S$ is :

· a complex man. of dim $3g-3$

· a complete metric space $d((X_1, \varphi_1), (X_2, \varphi_2)) = \inf\limits_{f : X_1 \to X_2,\ f \circ \varphi_1 \text{ hom. to } \varphi_2,\ f\ C^1} \log K(f)$ / is contractible.

where $K(f) = \sup\limits_{x \in X_1} \dfrac{\sup_{\xi \in T_x X_1} [Df(x)](\xi)/|\xi|}{\text{Inf (same quantity)}}$



Action of MCG: $f.(X,\varphi) = (X, \varphi \circ f^{-1})$.

Proof of Th. thm:

understanding Möbius maps: $\mathbb{H}^2 \longrightarrow \mathbb{H}^2$ classified as elliptic, parab., hyp.:

$\xi \mapsto \dfrac{a\xi+b}{c\xi+d}$

Look at $D(A) = \inf_{z \in \mathbb{H}^2} D(z, Az)$

$D(A)$ — 0 : realized, A elliptic conj. to $\xi \mapsto \lambda\xi$, $|\lambda|=1$ in $\mathbb{D}$

not realized, parab., conj. to $z \mapsto z \pm 1$ in $\mathbb{H}$.

>0 — Realized: hyp.: 2 fixed pts at ∞, conj. to $z \mapsto \lambda z$ $|\lambda|>1$

not Realized: does not happen.

Action of MCG:

$f \in MCG(S)$, $D(f) = \inf_{\tau \in \mathcal{T}_S} d_{Teich}(\tau, f.\tau)$

0 — realized finite order: easy part, using Hurwitz theorem. 84(g-1) thm.

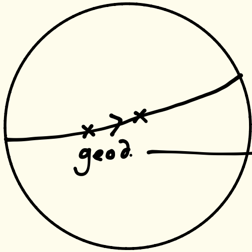
not realized reducible: use Mumford compactness thm

>0 — realized pAnosov.: OK.

not Realized red.

Axis of hyp. map:

analogy in teich. sp:



$\exists!$ geodesic joigning 2 pts in $\mathcal{T}_S$.

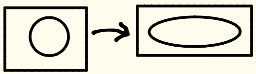
surfaces obtained by stretching along foliation: they form a geodesic.

The pts in $\mathcal{T}_S$ minimizing $D(\tau, f.\tau)$ when f is pAnosov are of following form:

(X, φ)

$S \xrightarrow{\varphi} X$

$\lambda|\mathrm{Re}\sqrt{q}| + i\frac{1}{\lambda}|\mathrm{Im}\sqrt{q}|$: a quad. diff on a different R. surface.

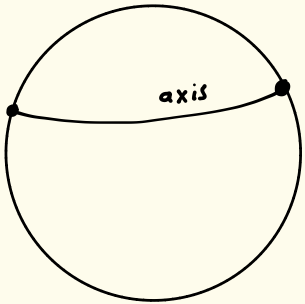


get $\log|\lambda|$.

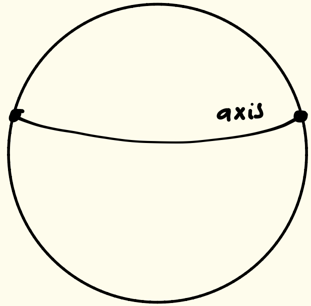
Summary: we have M_f, also $\tilde{M}_f \to M_f$, where $\tilde{M}_f \simeq S \times \mathbb{R}$, a repr. of $\pi_1(S) \longrightarrow PSL_2\mathbb{C}$.

$$\begin{array}{ccc} M_f & & \tilde{M}_f & \to & M_f \\ \downarrow & & \downarrow & & \downarrow \\ S^1 & & \mathbb{R} & \longrightarrow & S^1 \end{array}$$

it turns out: space of repres., discrete is $\underset{\text{(Bers)}}{\simeq} \mathcal{T}_S \times \mathcal{T}_{S^*}$, by analytic isomorphism.



$\mathcal{C}_S$



$\mathcal{C}_{S^*}$

We need to add 1 generator: of S^1, that will act on both spaces.

Unfortunately: no fixed point inside, but only on the boundary: such pts do correspond to ~~re~~ groups.

Goal: find $\pi_1(M_F)$ in $PSL_2\mathbb{C}$, using limits of Kleinian groups.