Summary: $f : S \to S$ or. preserving homeo.

$$M_f = (S \times \mathbb{R})/\sim \,, \quad (x,t) \sim (f(x), t-1).$$

Thm: M_f has a hyp. structure $\iff$ f is (homotopic to) a pseudo-Anosov homeomorphism.

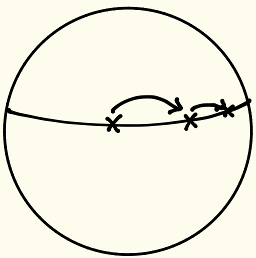
Recall: $\mathcal{T}_S = \{(X, \varphi) \,/\, X$ Riem. surface, $\varphi : S \to X$ is a $+$-homeo$\}/\sim$

where $(X_1, \varphi_1) \sim (X_2, \varphi_2)$ iff $\exists\, \alpha : X_1 \to X_2$ analytic is s.t $\alpha \circ \varphi_1$ homotopic to φ_2.

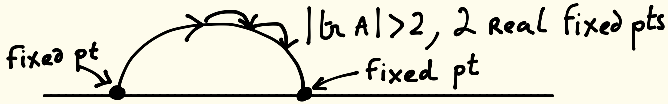
Then, $MCG(S)$ acts on $\mathcal{T}_S$ by: $f.(X, \varphi) = (X, \varphi \circ f)$.

Rk: homotopy $\simeq$ isotopy for <u>surfaces</u>.

f is p.Anosov $\iff$ the set of pts of $\mathcal{T}_S$ that are moved the least is a geodesic of $\mathcal{T}_S$, the <u>axis</u> of f.



if $S = \text{Torus} - \{x_0\}$, $3g-3+n = 1$, so picture is correct, f_A is the map. $A \in SL_2\mathbb{Z}$.
and map has 2 fixed pts.



<u>Hard part</u>: F pA $\Rightarrow M_F$ hyperbolic.

hyp. struct. given by conj. class of subgrps of $SL_2\mathbb{C}$ isomorphic to $\pi_1(M_F)$

$$\mathrm{Aut}^+\mathbb{H}^3 = SO^+(3,1)$$

$$\begin{array}{c} S \longrightarrow M_F \\ \downarrow \\ S^1 \end{array} \qquad \underset{\{1\}}{\overset{}{\pi_2(S^1)}} \to \pi_1(S) \to \pi_1(M_F) \to \underset{\mathbb{Z}}{\pi_1(S^1)} \to \{1\}$$

* We are looking for $\pi_1(S) \rtimes \mathbb{Z}$, semi-direct product.

So we need to find an injective discrete repr. $\rho : \pi_1(S) \longrightarrow \mathrm{Aut}\,\mathbb{H}^3$

that admits an enrichment, i.e an extra element g.

Giving M_F a hyp. structure means finding subgrp. G of $PSL_2\mathbb{C}$ s.t $\mathbb{H}^3/G \underset{\text{homeo.}}{\simeq} M_F$.

$$\begin{array}{c} \tilde{X} \\ \downarrow \Gamma \\ X \end{array} \quad \text{univ. cov.}, \ \Gamma \simeq \pi_1(X).$$

<u>Rk (by Marcel)</u>: Mostow's rigidity theorem.

We know lots of discrete injective representations of $\pi_1(S)$ into $\operatorname{Aut} \mathbb{H}^3$:
namely the quasi-fuchsian representations whose conjugacy classes are in 1:1 corresp.
with $\mathcal{T}_S \times \mathcal{T}_{S^*}$

Opti software: explore qf-gp with 2 generators, with commutator $z \mapsto z+2$.

What is dim of sp. of rep. of F_2, with commutator

2 gen. $\dim_{\mathbb{C}} 6$, comm. is 3 cplex equ. $\Rightarrow \dim_{\mathbb{C}} = 3$
one can conj. by similarities (arbitrary): $\dim_{\mathbb{C}} 2$

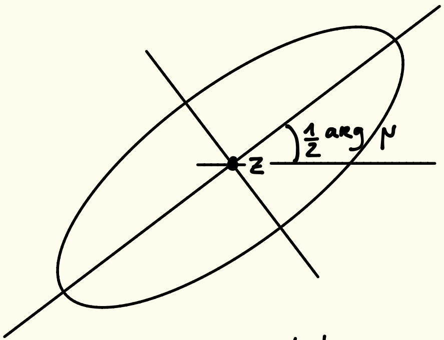
colored locus: discrete rep. / black.
pic: curve is limit set. In blue: 0 and 1, in red fixed pts of generators.

Ahlfors-Bers theory:
For any $\mu \in L^\infty(\mathbb{C})$, $\|\mu\|_\infty < 1$, $\exists!\ f : \mathbb{C} \to \mathbb{C}$ homeo. st $\underbrace{\frac{\partial f}{\partial \bar z} = \mu \frac{\partial f}{\partial z}}_{\text{in } L^2_{loc}}$, $\frac{\partial f}{\partial \bar z}, \frac{\partial f}{\partial z} \in L^2_{loc}(\mathbb{C})$.
and $f(0)=0$, $f(1)=1$.

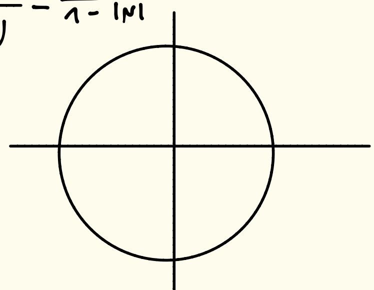
Rk: Gauss proved it in real analytic case.

Geometric meaning:

$$\frac{\text{big axis }(z)}{\text{Small axis }(z)} = \frac{1+|\mu|}{1-|\mu|}$$

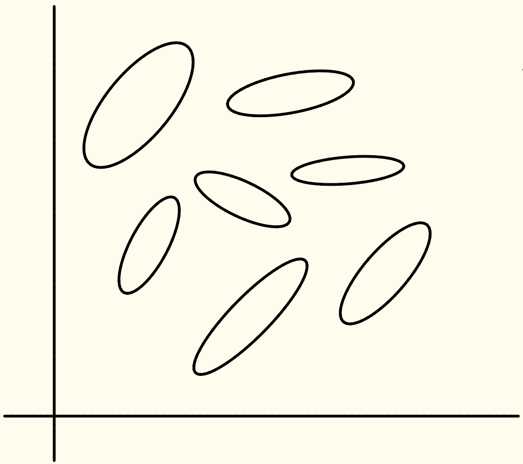


$\xrightarrow{Df(z)}$

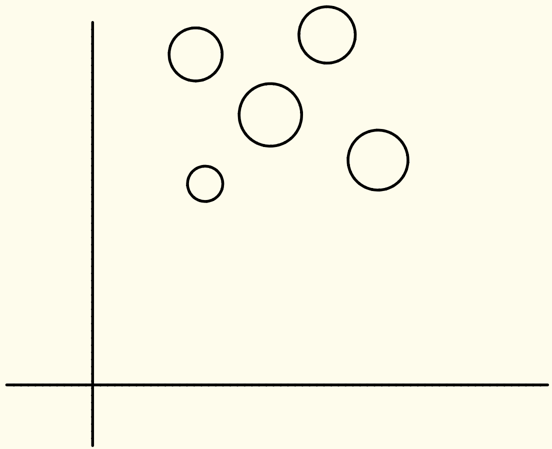


Field of ellipses with bounded eccentricities $\longrightarrow$ family of <u>round</u> circles.

Cauchy-Riemann: round circles $\rightarrow$ round circles.

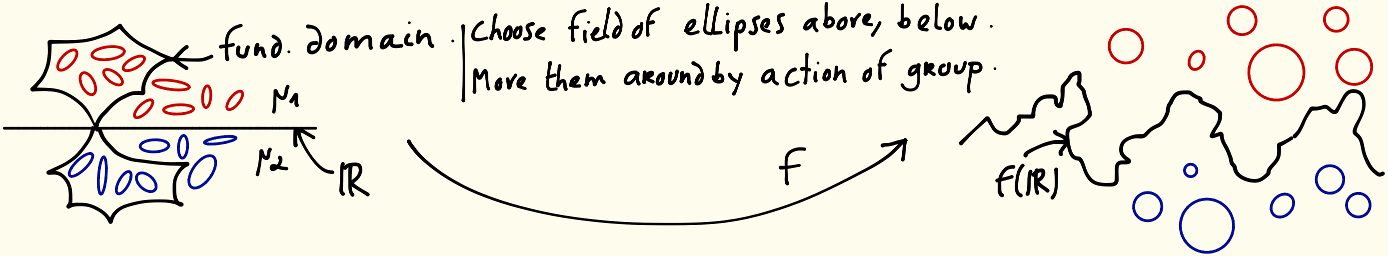


$\longrightarrow$



Why is it relevant?

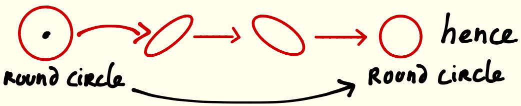
Find a fuchsian group G s.t $\mathbb{H}^2/G \simeq S$.



Choose field of ellipses above, below.
Move them around by action of group.

$\underbrace{f \circ G \circ f^{-1}}_{\text{analytic!}}$ is in $PSL_2\mathbb{C}$ (f very irregular!)

↳ take some $g \in G$, some pt



hence analytic!

[pic: quasi-circle, more complicated]

We are going to take limits in $\mathcal{C}_S \times \mathcal{C}_{S^*}$. The extra element will be in the boundary, with a fixed pt.

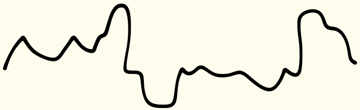
f acts on $\mathcal{C}_S \times \mathcal{C}_{S^*}$, acting separately on each factor. The fixed pt will be at ∞.

The group we want should corresp. to a fixed pt of the action. In the bdry of space of qf-rep,

we will find it. Prove that the bdry rep. is the one we want.

In the limit: limit set becomes a Peano curve.

show ~~it~~ the limit exists as a discrete grouprepresentation, is the hard part.

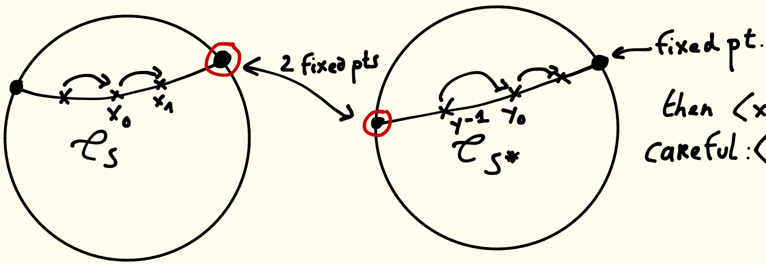


upper half$/\varphi G \varphi^{-1}$ is Riem. surface, homeo. to S, so an elt of $\mathcal{T}_S$

lower half$/\varphi G \varphi^{-1}$ " " ", homeo. to S, so gives an elt of $\mathcal{T}_{S^*}$.

Thus: space of G-inv. Belt. forms of qf-grps $\longrightarrow \mathcal{T}_S \times \mathcal{T}_{S^*}$

We know how f acts on $\mathcal{T}_S \times \mathcal{T}_{S^*}$: it acts by transl. along the axis:



then $\langle x_n, y_{-n} \rangle$ is a conj. class of qf. groups Γ_n

careful: $\langle x_n, y_n \rangle$ would not work!

<u>Question</u>: does $n \to \underbrace{\Gamma_n}_{\text{fin. generated grps}}$ converge? (in the sense of limits of generators.) Yes! (Thurston)