A metric approach to representations of compact Lie groups

Claudio Gorodski*

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Abstract

Lecture 1 introduces polar representations from two different viewpoints and briefly explains their classification and importance. Lecture 2 presents a generalization of polar representations. Lecture 3 introduces the idea of reductions and describes some of the results of the papers [12, 11]. Lecture 4 is about isometric actions on spheres with an orbifold quotient space and corresponds to [10]. The Appendix has some standard background on Alexandrov spaces and some (perhaps not so standard) background on Riemannian orbifolds.

Introduction

These are the notes for a series of talks at Ohio State University on August 27-29, 2014. The aim has been to give a gentle introduction to a new approach to representations of compact Lie groups via metric considerations. We have striven to be as non-technical as possible. Proofs, if any, are either sketched or given in simpler cases. These ideas fit into the general program of understanding how much of a representation can be recovered from the metric structure of the orbit space. Such questions can also be considered for proper isometric actions or even singular Riemannian foliations, but we do not discuss them here. Most of the results herein described as well as the program outlined are the joint work of the author with Alexander Lytchak (Cologne). We wish to thank Michael Davis and Luis Casian for the invitation to give these talks.

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*Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, São Paulo, SP 05508-090, Brazil, *E-mail address*: gorodski@ime.usp.br

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1 Lecture 1: Polar representations

1.1 Basic example

Consider $G = SO(n)$-conjugation of real $n \times n$ symmetric matrices, that is for $g \in G$ and $A \in \tilde{V} = \mathrm{Sym}(n, \mathbb{R})$, put

$$g \cdot A = gAg^{-1} = gAg^t.$$

Then the orbits

$$G \cdot A = \{g \cdot A \mid g \in G\}$$

are the orthogonal conjugation classes of symmetric matrices, or orthogonal equivalence classes of real quadratic forms. We can also talk about normal forms, which are distinguished representatives of equivalence classes, namely, each orbit contains a diagonal matrix, uniquely defined up to permutation of the diagonal entries (eigenvalues). Thus we can view the orbit space

$$\mathrm{Sym}(n, \mathbb{R})/SO(n) = \mathbb{R}^n/\mathcal{S}_n \tag{1}$$

where $\mathbb{R}^n \cong \tilde{\Sigma}$ is the subspace of diagonal matrices of $\mathrm{Sym}(n,\mathbb{R})$ and $\mathcal{S}_n$ denotes the symmetric group on n letters. Note there is an splitting

$$\mathbb{R}^n/\mathcal{S}_n = \mathbb{R}^{n-1}/\mathcal{S}_n \times \mathbb{R}$$

since $\mathcal{S}_n$ acts trivially on the line $\mathbb{R}(1,\ldots,1)$ and leaves invariant the linear hyperplane of $\mathbb{R}^n$ whose coordinates add up to zero. In fact, if we restrict the action of G to the subspace $V = \mathrm{Sym}_0(n,\mathbb{R})$ of traceless matrices then

$$\mathrm{Sym}_0(n,\mathbb{R})/SO(n) = \mathbb{R}^{n-1}/\mathcal{S}_n \tag{2}$$

where $\mathbb{R}^{n-1} \cong \Sigma$ is the subspace of diagonal matrices of V.

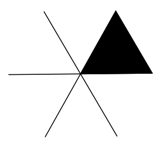

FIG. 1A: X for $n = 3$

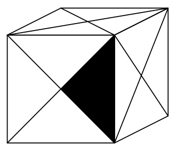

FIG. 1B: $S(X)$ for $n = 4$

Let us analyze a number of nice features of this example.

• $\mathcal{S}_n$ is a finite group generated by reflections on Σ. In fact, it has a special presentation so that it is a Coxeter group.

• The orbit space $X = V/G$ (as well as the orbit space $S(X)$ of the unit sphere $S(V)$) is the quotient of Σ (resp. $S(\Sigma)$) by a finite group of isometries and hence is a good Riemannian orbifold.

• $\dim \Sigma = n-1$ equals the *cohomogeneity* of the G-action on V, that is, the codimension of the principal orbits. In fact, the principal orbits are exactly those containing matrices with pairwise different eigenvalues. The isotropy group of a matrix is its centralizer in G. Therefore, if A has pairwise different eigenvalues, its isotropy group consists of diagonal matrices in $SO(n)$ with ± 1 along the diagonal, namely, it is isomorphic to $(\mathbb{Z}_2)^{n-1}$ and hence finite. Now $\dim G\cdot A = \dim G = \frac{n(n-1)}{2}$ and the cohomogeneity of the G-action on V is $\dim V - \dim G \cdot A = \left(\frac{n(n+1)}{2} - 1\right) - \frac{n(n-1)}{2} = n-1 = \dim \Sigma$.

• Σ is normal to every orbit it meets and thus it is the normal space to every principal orbit it meets. Indeed the natural inner product in V is given by $\langle A, B\rangle = \mathrm{trace}(AB)$ for A, $B \in V$. The tangent space to the orbit $G \cdot A$ at $A \in V$ is

$$T_A(G\cdot A) = \{XA - AX \mid X \in \mathfrak{so}(n)\}.$$

Therefore, for every A, $B \in \Sigma$:

$$\begin{aligned}\langle XA - AX, B\rangle &= \mathrm{trace}(XAB) - \mathrm{trace}(AXB)\\ &= \mathrm{trace}((XB)A) - \mathrm{trace}(A(XB))\\ &= 0\end{aligned}$$

where we have used that A and B commute as they are diagonal matrices.

• The equalities in (1) and (2) are isometries. In fact, there is a natural structure of metric space on the orbit space $X = V/G$, simply by declaring the distance between two points in X to be the distance in V between the corresponding G-orbits; note that this distance is always realized by the length of a minimizing geodesic, which is orthogonal to every G-orbit it meets. It follows that a minimizing geodesic between two principal G-orbits can always be chosen entirely contained in Σ. This essentially shows that the maps are isometries on the regular sets; the extension to include the singular sets follows, see Proposition 2.2.1 for details.

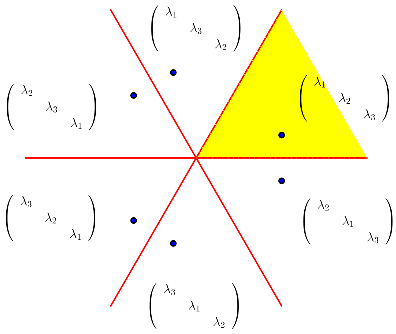

FIG. 2: A typical principal orbit meets Σ for $n = 3$

1.2 Generalization

We continue with the above example. Note that

$$\begin{aligned} \mathfrak{so}(n) \oplus V &= \{X \mid X^t = -X\} \oplus \{X \mid X^t = X, \operatorname{tr}(X) = 0\} \\ &= \mathfrak{sl}(n, \mathbb{R}) \end{aligned} \tag{3}$$

and we have the bracket relations

$$\begin{aligned} [\mathfrak{so}(n), \mathfrak{so}(n)] &\subset \mathfrak{so}(n) \\ [\mathfrak{so}(n), V] &\subset V \\ [V, V] &\subset \mathfrak{so}(n) \end{aligned}$$

Here the first relation expresses the fact that $\mathfrak{so}(n)$ is a Lie algebra, the second one expresses the action of $\mathfrak{so}(n)$ on V, and the third is a simple computation. Therefore the decomposition (3) is the ± 1-eigenspace decomposition of an involutive automorphism

$$\sigma : \mathfrak{sl}(n, \mathbb{R}) \to \mathfrak{sl}(n, \mathbb{R}).$$

In fact $\sigma(X) = -X^t$. Passing to the group level, we have an automorphism of $G = SL(n, \mathbb{R})$, given by $g \mapsto (g^{-1})^t$, with fixed point set $SO(n)$. The pair

$(SL(n,\mathbb{R}), SO(n))$ is a symmetric pair, and the homogeneous space $SL(n,\mathbb{R})/SO(n)$ is the corresponding symmetric space (of non-compact type). The decomposition (3) is the Cartan decomposition, and the isotropy representation of the symmetric space turns out to be the representation of $SO(n)$ on V. Finally, $\Sigma \subset V$ is a maximal Abelian subspace of V and hence a Cartan subspace.

Remark 1.2.1. There is also a Cartan decomposition on the group level. Namely, $SL(n,\mathbb{R}) = SO(n) \cdot \exp[\mathrm{Sym}_0(n,\mathbb{R})]$ is equivalent to the polar decomposition of a matrix into the product of an orthogonal matrix and a positive-definite symmetric matrix. The isotropy group $SO(n)$ acts on the left on $SL(n,\mathbb{R})/SO(n)$ and this action is called the *isotropy action*. The image of Σ under the Riemannian exponential map at the basepoint is a maximal flat totally geodesic submanifold of the symmetric space and meets all the orbits of the isotropy action orthogonally.

Remark 1.2.2. The *Cartan dual* of the involutive Lie algebra $\mathfrak{g} \oplus V$ is the Lie algebra obtained by taking the real form $\mathfrak{g} \oplus iV$ of the complexification $\mathfrak{g}\otimes\mathbb{C}\oplus V\otimes\mathbb{C}$. Namely, $\mathfrak{so}(n)\oplus i\mathrm{Sym}_0(n,\mathbb{R})$ yields $\mathfrak{su}(n)$ with involution $\sigma^*(X) = -X^t = \bar{X}$. We obtain a symmetric space $SU(n)/SO(n)$ of compact type. Note that a symmetric space and its Cartan dual have equivalent isotropy representations.

In general, a *symmetric pair* is a pair (L, G) where L is a connected real semisimple Lie group (with finite center) and G is open in the fixed point set of an involutive automorphism of L [18, 41]. The symmetric pair is called of *compact type* (resp. *non-compact type*) if all simple factors of L are compact (resp. non-compact) Lie groups. There is a decomposition on the Lie algebra level $\mathfrak{l} = \mathfrak{g} \oplus V$ into the ± 1-eigenspaces of the involution (especially in the case of non-compact type, it is called the associated *Cartan decomposition*). The homogeneous space L/G is called a *symmetric space*; it is equipped with natural L-invariant Riemannian metrics (say, one whose value at the basepoint is induced by a multiple of the Killing form of L); the tangent space of L/G at the basepoint is identified with V. The isotropy group G acts on the tangent space of L/G at the basepoint; this action is called the *isotropy representation* of the symmetric space, and is equivalent to the adjoint action of G on V. Isotropy representations of symmetric spaces are sometimes called *s-representations*. Since $[V, V] \subset \mathfrak{g}$, any subalgebra of V must be Abelian. A maximal Abelian subalgebra Σ of V is a called a *Cartan subspace*. Any two Cartan subspaces are G-conjugate and their common dimension is called the *rank* of the symmetric space. The *Weyl group* $\mathcal{W}$ of the symmetric space is the (conjugation class) of the maximal effective subquotient of G that acts on a Cartan subspace Σ, namely, $\mathcal{W} = N_G(\Sigma)/Z_G(\Sigma)$ where $N_G(\Sigma) = \{g \in G \mid g\Sigma = \Sigma\}$ and $Z_G(\Sigma) = \{g \in G \mid g|_\Sigma = \mathrm{id}\}$. It turns out $\mathcal{W}$ is a Coxeter group acting by orthogonal transformations on Σ. A Cartan subspace Σ meets all G-orbits in V, always orthogonally, and thus there is an isometric identification $V/G = \Sigma/\mathcal{W}$. It follows that $X = V/G$ and $S(X) = S(V)/G$ inherit the structure of good Riemannian orbifolds. Since $\mathcal{W}$ contains reflections, X and $S(X)$ have boundary in the Alexandrov sense.

Let G be a compact Lie group. Representations of G will be assumed faithful.

We give two equivalent definitions of polar representations.

Definition 1.2.1. A representation $\rho : G \to O(V)$ is called *polar* if there exists a subspace Σ, called a *section*, that meets all G-orbits and meets them always orthogonally.

Definition 1.2.2. A representation $\rho : G \to O(V)$ is called *polar* if there exists a representation $\tau : H \to O(W)$ of a finite group H such that $V/G = W/H$.

In general, a representation τ like in Definition 1.2.2, where H is not necessarily finite but $\dim H < \dim G$, is called a *reduction* of ρ (cf. Definition 2.3.1).

Proof of equivalence between Definitions 1.2.1 and 1.2.2. We will see that the first definition implies the second one in a broader context later (cf. Proposition 2.2.1). To see the reverse implication, consider the projections

$$\begin{tikzcd} V \arrow[d, "\pi_G"'] & W \arrow[d, "\pi_H"] \\ V/G & = W/H \end{tikzcd}$$

Since H is a finite group, π_H is a local isometry on the regular set W_{reg} so $V_{reg}/G = W_{reg}/H$ is flat. Now we apply O'Neill's formula for Riemannian submersion $\pi_G : V_{reg} \to V_{reg}/G$ to relate the sectional curvatures of tangent 2-planes on V_{reg} and V_{reg}/G:

$$K(X,Y) = K(\tilde{X}, \tilde{Y}) + 3||\nabla^v_{\tilde{X}} \tilde{Y}||^2$$

where X, Y are (smooth) vector fields on V_{reg}/G, $\tilde{X}$, $\tilde{Y}$ are their horizontal lifts, and ∇^v denotes the vertical component of the Euclidean covariant derivative on V. Since both terms with sectional curvatures vanish, this formula shows that the horizontal distribution $\mathcal{H}$ on V_{reg}, consisting of normal spaces to the principal orbits, is integrable with totally geodesic leaves.

Let L be a leaf of $\mathcal{H}$. Then L is a non-empty connected open subset of an affine subspace Σ of V. We claim Σ is a section of ρ. In fact, $\Sigma = T_pL$ is the normal space $\nu_p(Gp)$ for $p \in L$ (since p is a regular point), so Σ meets all G-orbits (since there is a minimizing geodesic from p to any given G-orbit, which must be entirely contained in Σ). To see that it meets always orthogonally, let X be a Killing field on V induced by the G-action and let γ be any horizontal geodesic with $\gamma(0) = p \in L$. Then the image of γ is entirely contained in Σ and $J := X \circ \gamma$ is a Jacobi field along γ. Since Σ is totally geodesic, also the horizontal component J^h with respect to Σ is a Jacobi field. Now J^h vanishes on a neighborhood of $t = 0$, so it vanishes identically. This shows that $X_{\gamma(t)}$ is orthogonal to Σ for all t. Since γ is an arbitrary horizontal geodesic and X is an arbitrary G-Killing field, Σ is orthogonal to all orbits it meets. □

1.3 Classification

The standard examples of polar representations are the s-representations. What about the classification? A polar representation can fail to be an s-representation for an unexceptionable reason, namely, it has the same orbits as an s-representation but it is not one of them. For instance, the standard action of $SU(n)$ on $\mathbb{C}^n$ has concentric round spheres as orbits, thus it is polar. However, this is not an s-representation; namely, it has the same orbits as the standard action of $U(n)$ on $\mathbb{C}^n$, which is the isotropy representation of $\mathbb{C}P^n = SU(n+1)/U(n)$.

Definition 1.3.1. Two representations are called *orbit-equivalent* if they have the same orbits after isometric identification of their orbit spaces.

It turns out this is the worst that can happen, as the following celebrated result shows.

Theorem 1.3.1 (Dadok 1985 [6])**.** *Every polar representation of a connected compact Lie group is orbit-equivalent to an s-representation.*

Dadok proved this theorem by classification. So far there is no known complete, purely geometric proof.

1.4 More properties

Polar representations have a number of connections with other important properties. In the following, we name a few.

Theorem 1.4.1 (Conlon 1971 [5], Di Scala-Olmos 2000 [8])**.** *A representation of a compact Lie group is polar if and only if it is variationally complete in the sense of Bott and Samelson.*

Variational completeness means roughly that the multiplicities of focal points of the orbits are determined by the group action, or that all Jacobi fields giving rise to focal points are Killing fields induced by the group action. One can also talk about absence of conjugate points in the orbit space [23].

Theorem 1.4.2 (Bott-Samelson 1958 [2], Conlon 1971 [5])**.** *Polar representations are taut. More precisely, let $\rho : G \to O(V)$ be polar. Then all Morse distance functions to orbits $L_q : Gp \to \mathbb{R}$, $L_q(x) = \frac{1}{2}||x - q||^2$ ($q \in V$), are $\mathbb{Z}_2$-perfect.*

Recall that a Morse function on a compact smooth manifold is called $\mathbb{Z}_2$-*perfect* if the Morse inequalities are equalities with respect to $\mathbb{Z}_2$-coefficients. A submanifold of Euclidean space all of whose Morse distance functions L_q are $\mathbb{Z}_2$-perfect is called $\mathbb{Z}_2$-*taut* [4]. A representation all of whose orbits are $\mathbb{Z}_2$-*taut* is called $\mathbb{Z}_2$-*taut*. Polar representations are taut, but there are examples of taut representations which are not polar [14, 15], see also Theorem 2.2.2.

Theorem 1.4.3 (Palais-Terng 1987 [30])**.** *The orbits of a polar representation $\rho : G \to O(V)$ yield an isoparametric foliation of $S(V)$ (resp. V). Conversely, every homogeneous isoparametric foliation of $S(V)$ (resp. V, with compact leaves) arises in this way.*

Recall that a submanifold of a space form is called *isoparametric* if the principal curvatures along any locally defined parallel normal vector field are constant and the normal bundle is flat. In particular, a hypersurface of the unit sphere is isoparametric if and only if it has constant principal curvatures. Every isoparametric submanifold can be extended to a complete one, and the parallel submanifolds, obtained by exponentiating parallel normal vector fields, comprise a singular Riemannian foliation whose regular leaves are isoparametric and whose singular leaves are focal manifolds [36, 1].

Let $N = Gp$ be a principal orbit of a polar representation $\rho : G \to O(V)$, where $p \in V$. Any normal vector $\xi \in \nu_p(Gp) =: \Sigma$ can be G-equivariantly extended to a normal vector field $\hat{\xi}$ along N, due to the fact that the slice representation at p is trivial. The equivariance says that

$$gp + \hat{\xi}(gp) = g(p + \xi) \in Gq$$

for all $g \in G$, where $q = p+\xi$, so differentiation with respect to g in the direction of $v \in T_pN$ gives

$$(v - A_\xi v) + \nabla^\perp_v \hat{\xi} \in T_q Gq \perp \Sigma,$$

where we have used the fact that Σ is a section, and A_ξ denotes the shape operator in the direction of ξ; this implies $\nabla^\perp_v \hat{\xi} = 0$. In other words, we have shown that equivariant normal vector fields are parallel in the normal connection. Isoparametricity of N follows. The converse to Theorem 1.4.3 is not hard either (cf. [30]).

Remark 1.4.1. There exist examples of inhomogeneous isoparametric submanifolds in spheres (Ozeki-Takeuchi 1975 [29]; Ferus-Karcher-Münzner 1981 [9]), but necessarily only in codimension one (Thorbergsson 1991 [37]).

The following result follows from the corresponding one for isoparametric foliations proved by Terng in 1985 [36].

Theorem 1.4.4 (Chevalley restriction theorem for polar representations)**.** *Let $\rho : G \to O(V)$ be a polar representation where we assume G is connected and fix a section Σ with associated Weyl group $\mathcal{W}$. Then the restriction map $\mathbb{R}[V]^G \to \mathbb{R}[\Sigma]^{\mathcal{W}}$ induces a isomorphism between the rings of invariant polynomials.*

The isomorphism in this theorem can be extended to the rings of invariant smooth functions [33], and further to basic differential forms [27] and Riemannian metrics [26].

2 Lecture 2: Copolarity and reductions

2.1 More examples

(a) Consider the $U(n)$-action on the space $V = \mathrm{Sym}(n, \mathbb{C})$ of $n \times n$ complex symmetric matrices given by

$$g \cdot A = gAg^t$$

for $g \in U(n)$ and $A \in V$. The orbits are unitary equivalence classes of complex quadratic forms, and the normal forms are real diagonal matrices, uniquely defined up to permutation of diagonal entries. In fact we can reduce a complex symmetric matrix to a normal form in two stages, first transforming it into a complex diagonal matrix and then further transforming to a real diagonal matrix:

$$\mathrm{Sym}(n,\mathbb{C})/U(n) = \mathbb{C}^n/\mathcal{S}_n \cdot U(1)^n = \mathbb{R}^n/\mathcal{S}_n \cdot (\mathbb{Z}_2)^n. \tag{4}$$

Here $U(1)^n$ is the maximal torus of $U(n)$ consisting of diagonal matrices. This provides a reduction of the G-action to a finite group and shows that this representation is polar according to Definition 1.2.2. A section is given by the subspace Σ of real diagonal matrices. In fact this representation is the s-representation associated to the symmetric space $Sp(n)/U(n)$.

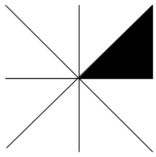

FIG. 3A: X for $n = 2$

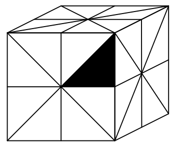

FIG. 3B: $S(X)$ for $n = 3$

(b) Next we restrict the above action on V to the subgroup $SU(n)$. Comparing with (4), now we can still find a complex diagonal matrix in each orbit but no further:

$$\mathrm{Sym}(n,\mathbb{C})/SU(n) = \mathbb{C}^n/\mathcal{S}_n \cdot S(U(1)^n).$$

Here $S(U(1)^n)$ is the maximal torus of $SU(n)$ consisting of diagonal matrices; the condition that the determinant is one makes it impossible to find a real diagonal matrix in all orbits.

This representation is not polar, as can be seen for instance from the classification. Let Σ be the subspace of V consisting of complex diagonal matrices. It is the fixed point set of the principal isotropy group $S(\mathbb{Z}_2^n)$, which consists of diagonal matrices with ± 1 entries and determinant 1. Since the principal isotropy group at a regular point always acts trivially on the normal space to the orbit (i.e. the slice representation is trivial; this in fact characterizes regular points), Σ in particular contains the normal space to every principal orbit it meets; it can be shown that Σ is the minimal subspace of V with this property (one can show that Σ equals the span of the normal spaces to principal orbits along some broken horizontal geodesic, starting at a regular point and broken only at regular points).

Alternatively, note that the group $\mathcal{S}_n \cdot S(U(1)^n)$ is not discrete; it can be shown, and indeed it will follow from our boundary analysis in subsection 3.1, that this is a minimal reduction of the $SU(n)$-action on V.

In the special case $n = 2$, $\mathrm{Sym}(2,\mathbb{C})$ has complex dimension 3, so it is the complexified adjoint representation. Recall that the adjoint representation of

$SU(2)$ is geometrically equivalent to the standard representation of $SO(3)$ on $\mathbb{R}^3$. It follows that

$$\begin{aligned}\mathrm{Sym}(2,\mathbb{C})/SU(2) &= \mathbb{R}^3 \oplus \mathbb{R}^3/SO(3) \\ &= \mathbb{R}^3 \oplus \mathbb{R}^3/O(3) \\ &= \mathbb{R}^2 \oplus \mathbb{R}^2/O(2),\end{aligned}$$

where we have used Luna-Richardson-Straume reduction in the last equality (cf. Example 2.2.1); in fact, in this presentation, the $\mathbb{R}^2 \oplus \mathbb{R}^2$ corresponds to the subspace Σ above. Now for the quotient of the unit sphere we have that

$$\begin{aligned}S(X) &= S^3/O(2) \\ &= \mathbb{C}P^1/\mathbb{Z}_2 \\ &= S^2_+(1/2)\end{aligned}$$

is a 2-hemisphere of constant curvature 4.

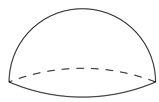

FIG. 4: $S(X)$ for $n = 2$

2.2 Copolarity

Example (b) in subsection 2.1 suggests the following definition.

Definition 2.2.1 (G-Olmos-Tojeiro 2004 [13])**.** Let $\rho : G \to O(V)$ be a representation. A *generalized section* of ρ is a subspace Σ of V that meets all G-orbits and always contains the normal spaces to the principal orbits it meets.

A polar representation will have a section as a *minimal* generalized section, thus of dimension equal to its cohomogeneity. For a general representation, the whole V trivially fits into the definition of a generalized section; more interestingly, the intersection of two generalized sections through a regular point is a generalized section, too. It follows that any representation admits a unique minimal generalized section through a given regular point. Clearly any two minimal generalized sections are conjugate.

Definition 2.2.2 (G-Olmos-Tojeiro 2004)**.** The *copolarity* of a representation is the excess of the dimension of a minimal generalized section over the cohomogeneity of ρ.

Thus the copolarity is a measure of non-polarity of a representation. One also says the copolarity is non-trivial in case there is a non-trivial (different from V) generalized section.

Example 2.2.1 (Luna-Richardson-Straume reduction [21, 34]). Let $\rho : G \to O(V)$ be a representation. Fix a principal isotropy group H. Then the fixed point set V^H is a generalized section, the subquotient $\bar{N} = N_G(H)/Z_G(H)$ of G acts on V^H and $V/G = V^H/\bar{N}$.

It is often more interesting to apply LRS reduction after passing to the maximal group with the same orbits. In fact this has the effect of enlarging the principal isotropy group and shrinking its fixed point set.

Question 2.2.1. Does every minimal generalized section is obtained via LRS reduction, after passing to the maximal group in the orbit-equivalence class?

Remark 2.2.1. For a generalized section Σ of an arbitrary representation of G, the largest subquotient of G that acts on Σ is $N_G(\Sigma)/Z_G(\Sigma)$; the latter has trivial principal isotropy group if Σ is minimal, for otherwise we could find a smaller generalized section by applying LRS reduction to the action on Σ. Recall that representations of connected groups with non-trivial principal isotropy representations have been classified by W.-C. Hsiang and W.-Y. Hsiang [19].

Example 2.2.2. In subsection 2.1(b), the doubling representation of $SO(3)$ on $\mathbb{R}^3 \oplus \mathbb{R}^3$ has $\mathbb{R}^2 \oplus \mathbb{R}^2$ as a generalized section. Since it has cohomogeneity 3, it has (non-trivial) copolarity $4 - 3 = 1$.

More generally, $(SU(n), \mathrm{Sym}(n, \mathbb{C}))$ has cohomogeneity

$$\begin{aligned}\dim \mathrm{Sym}(n, \mathbb{C}) - \dim SU(n)/S((\mathbb{Z}_2)^n) &= (n^2 + n) - (n^2 - 1)\\ &= n + 1\end{aligned}$$

and $\dim \Sigma = 2n$ so it has copolarity $2n - (n + 1) = n - 1$.

The paper that introduced copolarity contains the proof of the following theorem.

Theorem 2.2.1 (G-Olmos-Tojeiro 2004 [13])**.** *An irreducible representation of copolarity* 1 *has cohomogeneity* 3*, hence it is one of*

$$\begin{array}{rcl} SO(2) \times Spin(9) & \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^{16} & \\ U(2) \times Sp(n) & \mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^{2n} & (n \geq 2) \\ SU(2) \times Sp(n) & S^3(\mathbb{C}^2) \otimes_{\mathbb{H}} \mathbb{C}^{2n} & (n \geq 2) \end{array}$$

The motivation to introduce generalized sections came from the study of the geometry and topology of the orbits of the above representations. In fact:

Theorem 2.2.2 (G.-Thorbergsson 2000 [15])**.** *A taut non-polar irreducible representation has cohomogeneity* 3.

The following result shows that a minimal generalized section yields a representation of smaller dimension with an underlying group of smaller dimension and the same orbit space.

Proposition 2.2.1. *Let Σ be a minimal generalized section of $\rho : G \to O(V)$ and let $\bar{N} = N_G(\Sigma)/Z_G(\Sigma)$. Then the inclusion $\Sigma \hookrightarrow V$ induces an isometry $\Sigma/\bar{N} \to V/G$.*

We need:

Lemma 2.2.1. *$\Sigma \cap V_{reg}$ is dense in Σ.*

Proof. Given $p \in \Sigma$, let $q \in \Sigma \cap V_{reg}$ and let $\gamma : [0, 1] \to \Sigma$ be the geodesic segment from p to q. Then $\gamma(t) \in \Sigma \cap V_{reg}$ for sufficiently small $t > 0$. □

Proof of Proposition 2.2.1. We first observe that the map $\iota : \Sigma/N = \Sigma/\bar{N} \to V/G$ is well defined (since $N := N_G(\Sigma) \subset G$), continuous (since the orbit spaces have the quotient topologies) and non-expanding (or 1-Lipschitz), namely,

$$d(\iota(x), \iota(y)) \leq d(x, y)$$

for all x, $y \in \Sigma/N$ (since every geodesic in Σ is a geodesic in V).

Next we show that the restriction of ι to $\Sigma \cap V_{reg}$ is injective. Assume $\iota(x) = \iota(y)$ for some x, $y \in \Sigma \cap V_{reg}/N$. Then $x = Np$, $y = Nq$ for some p, $q \in \Sigma \cap V_{reg}$ and $q = gp$ for some $g \in G$. Now Σ, $g^{-1}\Sigma$ are two minimal generalized sections through the regular point p and thus they coincide. We deduce that $g \in N$ and hence $x = y$.

In view of the continuity of ι and Lemma 2.2.1, to finish the proof we need only show that ι is an isometry on $\Sigma \cap V_{reg}$. In fact let $x = Np$, $y = Nq$ where p, $q \in \Sigma \cap V_{reg}$. The minimal geodesic γ in V from p to Gq is entirely contained in Σ. Let $r \in \Sigma \cap Gq$ be the endpoint of γ. Clearly γ minimizes the distance from Np to Nr. Since $Gr = Gq$, by the argument in the previous paragraph we have $Nr = Nq$. Hence $d(\iota(x), \iota(y)) = \text{length}(\gamma) = d(x, y)$ as desired. □

Corollary 2.2.1. *With the above notation:*

(i) *$Np = Gp \cap \Sigma$ for all $p \in \Sigma$.*

(ii) *The copolarity of ρ equals the dimension of $\bar{N}$.*

(iii) *G_p is transitive on the set of minimal generalized sections through p for all $p \in V$.*

(iv) *$\Sigma \cap V_{reg}$ coincides with the set of $\bar{N}$-regular points Σ_{reg}.*

Proof. (i) is equivalent to the injectivity of ι. To prove (ii), we can argue that the cohomogeneity of a representation is the topological dimension of its orbit space, so the proposition yields that the copolarity of ρ is $\dim \Sigma - \dim \Sigma/\bar{N} = \dim \bar{N}$, due to Remark 2.2.1. For (iii), take two minimal generalized sections Σ, Σ' through p. Since they meet a common principal orbit, $\Sigma' = g\Sigma$ for some $g \in G$. Now p, $g^{-1}p \in \Sigma$ so the proposition gives $n \in N$ such that $np = g^{-1}p$. Hence $gn\Sigma = \Sigma'$ with $gnp = p$, as we wished. Finally (iv) follows from the fact that ι maps Σ_{reg}/N onto V_{reg}/G as those are exactly the manifold points of the quotients Σ/N, V/G, respectively. □

2.3 Quotient-equivalence

Proposition 2.2.1 fosters a further generalization.

Definition 2.3.1 ([12])**.** Two representations are called *quotient-equivalent* if they have isometric orbit spaces; in addition, if one representation is such that its underlying group has dimension smaller than the dimension of the group underlying the other, then the former representation is said to be a *reduction* of the latter.

Example 2.3.1. Any representation admits the LRS reduction to the effectivized normalizer of the principal isotropy group (cf. Example 2.2.1).

It is interesting to find representations $\rho : G \to O(V)$ such that $\dim G$ is minimal in the quotient-equivalence class of ρ. In view of LRS reduction, such a representation must have trivial principal isotropy groups, thus it satisfies $\dim G + \dim V/G = \dim V$. We see it is an equivalent problem to find representations $\rho : G \to O(V)$ such that $\dim V$ is minimal in the quotient equivalence class of ρ.

Definition 2.3.2. A representation of minimal dimension in a quotient-equivalence class is called *reduced.* A reduced representation which is a reduction of another representation is also called a *minimal reduction* of the latter.

Example 2.3.2. In subsection 1.2 we explained that a representation reduces to a finite group action if and only if it is polar. More generally, Proposition 2.2.1 shows that if Σ is a minimal generalized section of a representation (G, V) then $(\bar{N}, \Sigma)$ is a reduction of (G, V).

Question 2.3.1. Does a minimal reduction of a representation always come from a minimal generalized section? (Compare with Question 2.2.1.)

The answer to Questions 2.2.1 and 2.3.1 is yes if: G is polar, by Dadok's Theorem 1.3.1; or G is connected, V is irreducible and (G, V) is quotient-equivalent to a representation of a group whose identity component is a torus [G-Lytchak 2014], see Theorem 3.3.2.

The idea behind quotient-equivalence is that the orbit space somehow determines the transverse geometry of the action. Since linear orthogonal actions are so rigid (as opposed for instance to general isometric actions), we can go one step further and ask:

Question 2.3.2. What kind of algebraic invariants of a representation $\rho : G \to O(V)$ can be recovered from the metric space structure of the orbit space $X = V/G$?

In the remaining of these lectures we aim to give (very) partial answers to this questions.

Example 2.3.3 (*invariance of irreducibility*). The cohomogeneity of a representation $\rho : G \to O(V)$ is the topological dimension of its orbit space X. The invariant subspaces can also be detected from the metric distance on the orbit space. In fact it is easy to see that a representation has a non-zero fixed vector if and only if $\operatorname{diam} S(X) = \pi$; otherwise $\operatorname{diam} S(X) \leq \frac{\pi}{2}$, and it is not hard to see that the projections of invariant subspaces of V are exactly the closed subsets of $S(X)$ for which there exists another closed subset at a distance $\frac{\pi}{2}$; hence $\operatorname{diam} S(X) < \frac{\pi}{2}$ if and only if the representation is irreducible. On the other hand, $\dim V$ cannot be read off X as non-trivial reductions exist.

Example 2.3.4. If $S(X)$ is isometric to a finite quotient of a sphere of constant curvature then ρ is taut (Wiesendorf 2014 [40]).

It is apparent that most representations do not admit reductions. The existence of a reduction entails the presence of interesting geometric properties and bounds the complexity of the representations and its orbit space.

Question 2.3.3. (Existence) Which representations admit reductions? Which representations can be minimal reductions of some representation?

Question 2.3.4. (Uniqueness) If $\rho_i : G_i \to O(V_i)$ for $i = 1, 2$ are minimal reductions of the same representation (resp. quotient-equivalent and reduced) is it true that $\rho_1(G_1)$ and $\rho_2(G_2)$ must be conjugate by an isometry $V_1 \cong V_2$?

Question 2.3.4 has a positive answer if the G_i are finite groups, cf. Lemma 5.2.1.

3 Lecture 3: Basic theory of reductions

3.1 Boundary in the orbit space

The orbit space of a s-/polar representation of a connected group is obtained as the fundamental domain of a Coxeter group acting on the Cartan subspace/section by linear orthogonal transformations and generated by reflections, and hence that domain is a simplicial cone (Weyl chamber). In particular, it has a non-empty boundary in the Alexandrov sense. More generally:

Proposition 3.1.1 ([12])**.** *Let $\rho_i : G_i \to O(V_i)$ for $i = 1, 2$ be quotient-equivalent representations. If V_1/G_1^0 has empty boundary, then $\dim V_1 \leq \dim V_2$. In particular, if $V_1/G_1 = V_2/G_2$ has empty boundary then $\dim V_1 = \dim V_2$.*

Proof. Suppose $\partial(V_1/G_1^0) = \varnothing$. By transversality, we can find a horizontal geodesic $\tilde{\gamma}$ in $S(V_1)$ of length π entirely contained in the regular set which thus projects to a geodesic γ in $S(V_1)/G_1^0$. Let η be the projection of γ to $S(V_1)/G_1 = S(V_2)/G_2$. Since $S(V_1)/G_1^0 \to S(V_1)/G_1$ is a finite covering, η is contained in the orbifold part of $S(V_1)/G_1$ and is an orbifold-geodesic, by definition. Consider its lift to a G_2-horizontal geodesic $\tilde{\eta}$ in $S(V_2)$. We may assume $\tilde{\gamma}$ was chosen so that all of the above curves start at regular points.

$$\begin{array}{ccc}
\tilde{\gamma} \subset S(V_1)_{reg} & & S(V_2) \supset \tilde{\eta} \\
\downarrow & & \\
\gamma \subset (S(V_1)/G_1^0)_{reg} & & \downarrow \\
\downarrow & & \\
\eta \subset (S(V_1)/G_1))_{orb} & = & (S(V_2)/G_2)_{orb}
\end{array}$$

Let m be the index of γ (viz. the number of conjugate points along γ, counted with multiplicities). Then m is also the L_1-index of $\tilde{\gamma}$ (viz. the number of L_1-focal points along γ, counted with multiplicities), where L_1 is the G_1^0-orbit through the initial point $\tilde{\gamma}(0)$ and $\tilde{\gamma}$ is considered as an L_1-geodesic. But in the unit sphere $S(V_1)$, the L_1-index of of any L_1-geodesic of length π is $\dim L_1$. Thus $m = \dim L_1$.

On the other hand, for L_2 equal to the G_2-orbit through $\tilde{\eta}(0)$, it has been shown in [23] that the L_2-index of the geodesic $\tilde{\eta}$ is equal to the sum of the index of the orbifold-geodesic η and a "vertical index", a non-negative number that counts the number of intersections of $\tilde{\eta}$ with G_2-singular orbits. In particular, it is not smaller than m, the index of the orbifold-geodesic η. Using again that the L_2-index of $\tilde{\eta}$ is given by $\dim L_2$, we get $\dim L_2 \geq \dim L_1$ and hence $\dim V_2 \geq \dim V_1$.

Any boundary point of V_1/G_1^0 projects to a boundary point of V_1/G_1 under the finite covering $V_1/G_1^0 \to V_1/G_1$. The last assertion follows. $\square$

It follows from the proposition that ρ_1 can admit a non-trivial reduction only if $\partial(V_1/G_1^0) \neq \varnothing$ (only if $\partial(V_1/G_1) \neq \varnothing$). In fact $\partial(V_1/G_1^0) = \varnothing$ implies that G_1 has finite principal isotropy groups so $\dim G_1 + \dim V_1/G_1 = \dim V_1 \leq \dim V_2 \leq \dim G_2 + \dim V_2/G_2$ and hence $\dim G_1 \leq \dim G_2$.

3.2 Coxeter groups induced by reductions

The Weyl group of a symmetric space is a finite Coxeter group acting by linear transformations on the Cartan subspace. More generally:

Proposition 3.2.1 ([12]). *Let $\rho_i : G_i \to O(V_i)$ for $i = 1, 2$ be quotient-equivalent representations. If G_1 is connected and G_2 is not, then G_2/G_2^0 acts by reflections on V_2/G_2^0; in fact, its image in $\mathrm{Iso}(V_2/G_2^0)$ is a Coxeter group.*

Here a *reflection* on a Riemannian manifold is an isometry whose fixed point set has a connected component of codimension one, and a *reflection* on an orbit space X is an isometry whose restriction to the regular part X_{reg} is a reflection.

Before discussing the proof of the proposition, we state a lemma. In general, the quotient of a simply-connected manifold by a connected compact Lie group is simply-connected, but the orbifold part of the quotient space need not be simply-connected as an orbifold. For instance, the standard action of $SO(n)$ on $\mathbb{R}^n$ yields a simply-connected orbit space $[0, \infty)$ which admits a double orbi-covering $\mathbb{R} \to [0, \infty)$. On the other hand, after removal of the boundary point, $(0, \infty)$ becomes simply-connected as an orbifold. This simple example conveys the general situation:

Lemma 3.2.1 ([22]). *Let the connected compact Lie group G act by isometries on the simply connected complete Riemannian manifold M. Denote by X the orbit space M/G, let X_{orb} be the set of orbifold points in X and set $X_0 = X_{orb} \setminus \partial X_{orb}$. Then X_0 is exactly the set of non-singular G-orbits. Moreover, X_0 has trivial orbifold fundamental group.*

Proof of Proposition 3.2.1 (Sketch). Let

$$X := V_1/G_1 = V_2/G_2 \qquad\qquad B := X_{orb}$$
$$X_0 := V_2/G_2^0 \qquad\qquad B_0 := (X_0)_{orb}$$

Since G_1 is connected and V_1 is simply-connected, the orbifold fundamental group $\pi_1^{orb}(B \setminus \partial B)$ is trivial due to Lemma 3.2.1.

It follows that $\Gamma := \pi_1^{orb}(B)$ is a group generated by reflections. In fact denote by Γ_{refl} the (normal) subgroup generated by all reflections in Γ. Then $\Gamma' := \Gamma/\Gamma_{refl}$ acts on $B' := \tilde{B}/\Gamma_{refl}$, where $\tilde{B}$ denotes the universal orbi-covering of B. No element γ of Γ' can act as a reflection on B', for otherwise we could find a manifold point p of $\tilde{B}$ that is projected to a manifold point of B' but whose projection to B lies on a stratum of codimension one in B. Then p would have to be fixed by a reflection in $\Gamma \setminus \Gamma_{refl}$, a contradiction. For the projection $B' \to B$, now the preimage of the ∂B is contained in $\partial B'$, so the preimage of $B \setminus \partial B$ is equal to the connected orbifold $B' \setminus \partial B'$ and thus $B \setminus \partial B = (B' \setminus \partial B')/\Gamma'$. Since $\pi_1^{orb}(B \setminus \partial B)$ is trivial, the group Γ' acts trivially on B' and hence $\Gamma = \Gamma_{refl}$.

Now Γ acts by reflections on the universal orbi-covering $\tilde{B}$ of B, so the quotient Γ/Γ_0, where $\Gamma_0 = \pi_1^{orb}(B_0)$, acts by reflections on B_0 and also on $C := B_0 \setminus \partial B_0$, and this group is the image of G_2/G_2^0 in $\mathrm{Iso}(V_2/G_2^0)$. By Lemma 3.2.1 $\pi_1^{orb}(B_0 \setminus \partial B_0)$ is trivial, but a reflection group acting on a Riemannian orbifold with trivial orbifold fundamental group has a presentation as a Coxeter group. Indeed Γ/Γ_0 can be viewed as the orbifold fundamental group of $C := (B_0 \setminus \partial B_0)/(\Gamma/\Gamma_0)$; looking at the canonical presentation of $\pi_1^{orb}(C)$ in terms of strata of codimension 1 and 2 [7], we deduce that Γ/Γ_0 has a presentation as a Coxeter group. □

3.3 Applications

Based on Propositions 3.1.1 and 3.2.1, there is the following method to find which representations $\tau : H \to O(W)$ can be minimal reductions of some irreducible representation of a connected compact Lie group (cf. Question 2.3.3). First of all, H must act with trivial principal isotropy groups, for otherwise LRS reduction would yield a non-trivial reduction. Second, H must act irreducibly, by invariance of irreducibility (cf. Example 2.3.3). Now there is a dichotomy: either H is connected and W/H has non-empty boundary by Proposition 3.1.1; or H^0 is normalized by an involution in $O(V)$ that acts as a reflection on W/H^0 by Proposition 3.2.1. Following this program and using some classifications of representations of low cohomogeneity, we can prove the following results.

Let $\rho : G \to O(V)$ be non-polar and irreducible with G connected, and let $\tau : H \to O(W)$ be a non-trivial minimal reduction of ρ.

Theorem 3.3.1 ([12])**.** *If* $\dim H \leq 6$*, then* H^0 *is a torus* T^k*.*

Theorem 3.3.2 ([12, 11])**.** *If* H^0 *acts reducibly on* W*, then* $H^0 = T^k$ *and its action on* W *can be identified with the action of the maximal torus of* $SU(k+1)$

on $\mathbb{C}^{k+1}$; in particular, the cohomogeneity of ρ is $k+2$. Moreover, such ρ can be classified and fall into three classes:

(I) Representations of cohomogeneity 3 *(listed in Theorem 2.2.1).*

(II) Semisimple factors of an s-representation of Hermitian type:

$$\begin{array}{ccc} SU(n) & S^2\mathbb{C}^n & (n \geq 3) \\ SU(n) & \Lambda^2\mathbb{C}^n & (n = 2p \geq 6) \\ SU(n) \times SU(n) & \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^n & (n \geq 3) \\ E_6 & \mathbb{C}^{27} & \end{array}$$

(III) One of two exceptions: $SO(3) \otimes G_2$ and $SO(4) \otimes Spin(7)$.

Corollary 3.3.1. *If ρ has non-trivial copolarity $k \leq 6$, then it has cohomogeneity $k+2$.*

Remark 3.3.1. $SO(3)\otimes U(2)$ is a minimal reduction of $U(3)\otimes Sp(2)$, where $SO(3)\times U(2)$ has dimension 7, so Theorem 3.3.1 is sharp.

3.4 Toy example

We illustrate the arguments used in the proofs of the above results by considering the case $\dim H = 1$, i.e. ρ has non-trivial copolarity 1. We will prove that ρ has cohomogeneity 3.

In fact, $H^0 = S^1$, which we identify with the unit complex numbers, so $\tau|_{H^0}$ decomposes into a sum $\mathbb{C}_{r_1} \oplus \cdots \oplus \mathbb{C}_{r_n}$ of weight spaces, where r_i is a positive integer (the order of the kernel of the action on $\mathbb{C}_{r_i}$), plus a fixed subspace (in other words, $\mathbb{C}_{r_i}$ is the representation $H^0 = S^1 \to U(1) = S^1$ given by $z \mapsto z^{r_i}$). By invariance of irreducibility, τ is irreducible so the fixed subspace is trivial and H/H^0 acts transitively on the set of H^0-isotypical components.

Indeed for $z \in H^0$ and $h \in H$ and $i = 1, \ldots, n$, the image $\tau(h)\mathbb{C}_{r_i}$ is an H^0-irreducible subspace and

$$\begin{aligned} \tau(z)\tau(h) &= \tau(h)\tau(h^{-1}zh) \\ &= \tau(h)(\tau \circ \varphi)(z) \end{aligned}$$

where φ is an automorphism of $H^0 = S^1$, so we may assume $\varphi(z) = z$ or $\varphi(z) = \bar{z}$. Therefore

$$\tau(h)\mathbb{C}_{r_i} = \mathbb{C}_{\pm r_i}$$

which is equivalent to $\mathbb{C}_{r_i}$ as a real representation (complex conjugation of $\mathbb{C}$ yields an equivalence between $\mathbb{C}_{-r_i}$ and $\mathbb{C}_{r_i}$!). We deduce that there is only one H^0-isotypical component, and then $r_1 = 1$ by effectiveness:

$$\tau|_{H^0} = \mathbb{C}_1 \oplus \cdots \oplus \mathbb{C}_1 \qquad (\ell \text{ summands}).$$

This is the Hopf action. The case $\ell = 1$ is polar, so $\ell > 1$. Note that W/H^0 is the cone over $S(W)/H^0 = \mathbb{C}P^{\ell-1}$ and thus has no boundary. Due to Proposition 3.1.1 or invariance of irreducibility, H is disconnected and, owing to Proposition 3.2.1, we can find an element of H/H^0 that acts as a reflection on W/H^0;

its fixed point set yields a totally geodesic hypersurface of $\mathbb{C}P^{\ell-1}$, but such hypersurfaces can exist only if $\ell = 2$. Hence the cohomogeneity of ρ is the dimension of the cone over $\mathbb{C}P^1$, namely, 3.

4 Lecture 4: Orbit-spaces and orbifolds

4.1 Characterization of orbifold points

Consider a proper and isometric action of a Lie group on a Riemannian manifold M, and let $X = M/G$ be its orbit space. The set X_{reg} of regular points of X is exactly the set of points that have neighborhoods isometric to Riemannian manifolds, whereas the slightly larger set X_{orb} of orbifold points of X consists of the set of points that have neighborhoods isometric to quotients of Riemannian manifolds by finite groups of isometries.

Theorem 4.1.1 (Lytchak-Thorbergsson 2010 [23]). *A point $x = Gp \in X$ is an orbifold point if and only if the slice representation at $p \in M$ is polar.*

Representations all of whose slice representations are polar are called *infinitesimally polar*.

The crux of the proof of one direction in Theorem 4.1.1 is a curvature estimate which we now explain in a particular case. Let $\rho : G \to O(V)$ be a representation and denote by g the Euclidean metric in V. For $v \in V_{reg}$, denote by $\bar\kappa(g)(v)$ the supremum over all of the sectional curvatures of V_{reg}/G at the projection of v with respect to the quotient Riemannian metric induced by g. The map

$$h_\lambda : (V, g) \to (V, \frac{1}{\lambda^2}\, g), \qquad h_\lambda(v) = \lambda v$$

for $\lambda > 0$ is an isometry which induces an isometry between the regular parts of the respective quotients. Therefore

$$\bar\kappa(g)(v) = \bar\kappa\left(\frac{1}{\lambda^2}\, g\right)(\lambda v) = \lambda^2 \bar\kappa(g)(\lambda v)$$

or

$$\bar\kappa(g)(\lambda v) = \frac{\bar\kappa(g)(v)}{\lambda^2}$$

for $v \in V_{reg}$ and $\lambda > 0$. Since a Riemannian orbifold has locally bounded curvature, the projection of the origin can be an orbifold point of X only if $\bar\kappa(g)(v) = 0$. Moreover if X_{reg} is flat then the horizontal distribution of $V_{reg} \to X_{reg}$ is integrable and hence ρ is polar as in the argument at the end of subsection 1.2.

4.2 Quotients of spheres which are orbifolds

Let $\rho : G \to O(V)$ be a representation. If ρ is polar and Σ is a section with associated Weyl group $\mathcal{W}$, then the orbit space V/G is isometric to $\Sigma/\mathcal{W}$ which is a good Riemannian orbifold. Conversely, we have seen in the previous subsection that V/G can be a Riemannian orbifold only if ρ is polar.

Things become more interesting if we restrict an arbitrary representation $\rho : G \to O(V)$ to an isometric action on the unit sphere $S(V)$ (note that the slice representations of ρ along any given line through the origin are all equivalent, except at the origin itself). There are interesting examples of non-polar representations for which $S(V)/G$ is isometric to a Riemannian orbifold; the Hopf action of S^1 on $\mathbb{C} \oplus \mathbb{C}$, with $S^3/S^1 = \mathbb{C}P^1$, immediately comes to mind.

Let $\rho : G \to O(V)$ be a representation of a compact connected Lie group G and assume that the quotient space $S(X) = S(V)/G$ of the unit sphere is isometric to a Riemannian orbifold. In the sequel we shall describe all such representations [10].

By Theorem 4.1.1, all slice representations at non-zero points are polar; equivalently, the action on $S(V)$ is infinitesimally polar.

The case ρ is polar. We have seen that, owing to O'Neill's formula, this case is characterized by $S(X)$ being an orbifold of constant curvature 1.

The case $S(X)$ has empty boundary. A first consequence of infinitesimal polarity is that all singular points of $S(V)$ project to the boundary of $S(X)$. In fact for a singular point $p \in S(V)$, the slice representation at p is polar and the corresponding Weyl chamber $\mathcal{C}$ has non-empty boundary; the isotropy subgroup K of G_p, corresponding to a boundary face of $\mathcal{C}$ containing 0 in its closure, is also an isotropy subgroup of G; and since $K \subset G_p$, p lies in the closure of the stratum (of codimension one) corresponding to the orbit type (K) and hence is a boundary point of $S(X)$.

It follows that if $S(X)$ has empty boundary then all isotropy groups of G have the same dimension. Moreover they have to be all discrete — i.e. the action must be *almost free* — because a non-trivial principal isotropy group yields a non-trivial reduction (via LRS) but this is forbidden by Proposition 3.1.1.

On the other hand, a representation of a k-torus splits into the direct sum of 2-dimensional representations, so we can always find an isotropy group of rank $k-1$.

We have thus proved that if $S(X)$ has empty boundary (and G is not transitive on $S(V)$) then G has rank one and its action is almost free. If $G = U(1)$ then ρ is a sum of complex 1-dimensional representations, each parametrized by a positive integer (the order of the kernel) and $S(X)$ is called a *weighted complex projective space*. If G is covered by $SU(2)$ then all almost free irreducible representations are of quaternionic type, so $G = SU(2)$ and ρ is an arbitrary sum of irreducible representations of even complex dimension; in this case we call $S(X)$ a *weighted quaternionic projective space*.

It turns out weighted projective spaces have trivial orbifold fundamental group, so they can be good orbifolds only if they are classical projective spaces (in which case the action of G is free).

The corresponding homogeneous regular foliations of spheres have been studied by Gromoll and Grove [17].

The case $S(X)$ has dimension 2. If $\dim S(X) = 2$ then the representation ρ has cohomogeneity 3 on V. The non-polar representations of cohomogeneity 3 have been extensively studied by Straume [34]. He showed that any such representation has a reduction to a representation of a one-dimensional group on $\mathbb{R}^4$. In fact, if $G \neq U(1)$ then the reduction is either a finite extension of the Hopf action or a two-fold extension of the action of $U(1)$ on $\mathbb{C} \oplus \mathbb{C} = \mathbb{R}^4$ with parameters $(1, 2)$. In the latter case, $S(X)$ is the bad orbifold given by a disc with one singularity with angle $\pi/2$ (half a teardrop); there are three strata: the open disk, the boundary circle minus the singular point and the singular point.

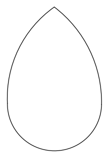

Fig. 5: A bad 2-orbifold

An example of a representation falling in this case is $U(2)$ acting on $\mathbb{C}^2 \oplus \mathbb{R}^3$ where the first summand is the standard representation and the second one is the adjoint representation on $\mathfrak{su}(2)$. Here the singularity in the boundary is the orbit through a point $p_1 \in S^2 \subset \mathbb{R}^3$; the corresponding isotropy groups are maximal tori in $U(2)$. The boundary of the disk is the image of the arc of great circle in $S(V)$ from p_1 to a point $p_2 \in S^3 \subset \mathbb{C}^2$ and then to $-p_1$; the isotropy groups along the boundary minus p_1 are circles in $U(2)$. The principal stratum corresponds to the open disk and the principal isotropy group is trivial.

In all cases above in which $S(X)$ is a good orbifold, it has constant curvature 4.

The remaining cases. Suppose now ρ is not polar, $S(X)$ has dimension $k \geq 3$ and non-empty boundary.

A Riemannian orbifold is called a *Coxeter orbifold* if all local groups are Coxeter groups acting as reflection groups on the corresponding tangent spaces (Davis [7] calls them *reflectofolds*).

We refer to [10] for details about the following arguments. We have $S(X)$ is a compact positively curved orbifold. A Soul Theorem like argument based on strict concavity of the distance function to the boundary shows that the non-manifold points of the orbifold $S(X)$ all lie in the boundary of $S(X)$. Next, using the local structure of Coxeter chambers, one sees that this property implies that $S(X)$ is a Coxeter orbifold. An argument based on the Petrunin-Frankel theorem involving intersections of closures of strata shows that a compact positively curved Coxeter orbifold of dimension at least 3 is a good orbifold. Then $S(X)$ is a good orbifold, and its universal orbi-covering Y is a compact positively curved

Riemannian manifold with an isometric reflection at a hypersurface, since $S(X)$ has non-empty boundary; thus Y is diffeomorphic to a sphere.

The next step is the following construction. Let K be any isotropy group of G and let F be its fixed point set in $S(V)$. Then F is a totally geodesic subsphere of $S(V)$, the normalizer N of K in G acts on F, the inclusion $F \subset S(V)$ induces a totally geodesic isometric immersion of orbifolds $F/N \to S(V)/G$, and F/N is a good Riemannian orbifold of dimension $k-1$ whose universal orbi-covering is diffeomorphic to a sphere. Moreover, the action of N on F is non-polar because the action of G on $S(V)$ is assumed non-polar, owing to a special case of a Theorem by Hang and Wang. Thus one can proceed by induction on k. At this juncture, we have used the interesting fact that a polar representation of a compact connected Lie group with trivial principal isotropy groups is orbit equivalent to the product of a number of standard representations of $U(1)$ and $Sp(1)$ on $\mathbb{C}$ and $\mathbb{H}$, respectively.

The above arguments yield that $k \leq 5$, $S(X)$ has constant curvature 4 and ρ must be one of the following sums of two representations of cohomogeneity one (the orbit space is also indicated; here $S^k(r)$, $S^k_+(r)$, $S^k_{++}(r)$, $S^k_{+++}(r)$ denote the round sphere of constant curvature $\frac{1}{r^2}$ quotiented by the group Γ which is respectively generated by 0, 1, 2, 3 commuting reflections).

$Spin(9)$	$\mathbb{R}^{16} \oplus \mathbb{R}^{16}$	$S^3_{++}(\frac{1}{2})$	
$SU(n)$	$\mathbb{C}^n \oplus \mathbb{C}^n$	$S^3_+(\frac{1}{2})$	$(n \geq 3)$
$U(n)$	$\mathbb{C}^n \oplus \mathbb{C}^n$	$S^3_+(\frac{1}{2})$	$(n \geq 2)$
$Sp(n)$	$\mathbb{H}^n \oplus \mathbb{H}^n$	$S^5_+(\frac{1}{2})$	$(n \geq 2)$
$Sp(n)U(1)$	$\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$	$S^4_{++}(\frac{1}{2})$	$(n \geq 2)$
$Sp(n)Sp(1)$	$\mathbb{R}^{4n} \oplus \mathbb{R}^{4n}$	$S^3_{++}(\frac{1}{2})$	$(n \geq 2)$
$T^2 \times Sp(n)$	$\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$	$S^3_{+++}(\frac{1}{2})$	$(n \geq 2)$

All quotient orbifolds described in this section have curvature ≤ 4 at some tangent plane. Owing to Theorem 4.1.1, a non-orbifold quotient $S(V)/G$ always has manifold points with arbitrary large curvatures at some tangent plane.

Question 4.2.1. How large can the infimum of the sectional curvatures be in a general quotient space $S(V)/G$?

Question 4.2.1 is related to the estimation of diameter of orbit spaces of .isometric actions on unit spheres [16, 25].

4.3 Concluding remarks

It is interesting to compare the results explained in this section with the world of inhomogeneous singular Riemannian foliations on spheres.

We have seen above that the orbit space of a non-polar isometric action of a compact Lie group of rank at least two on a sphere can be a Riemannian orbifold only if it has dimension at most 5. In a recent breakthrough, M. Radeschi [32] has constructed examples of non-isoparametric inhomogeneous singular Riemannian foliations on spheres, including examples whose leaf space is isometric to a hemisphere $S^k_+(\frac{1}{2})$, where k can be arbitrarily large!

On the other hand, for the other classes of infinitesimally polar actions on spheres, the differences are not that big. For polar foliations, it is known that the only quotients that arise, arise as quotients of group representations [38]. Moreover, in higher codimensions essentially all polar foliations are homogeneous [37]. The case of G having rank one and acting almost freely corresponds to the case of regular Riemannian foliations. All such foliations but the 7-dimensional Hopf fibration are homogeneous ([24] and the literature therein). We also refer to [31] for a related result.

5 Appendix

5.1 Alexandrov geometry of orbit spaces

Consider a proper and isometric action of a Lie group G on a connected complete Riemannian manifold M. Let $X = M/G$ be the orbit space equipped with the quotient topology. Then there is a natural stratification on M by G-invariant manifolds consisting of points with conjugate isotropy groups; a conjugacy class of isotropy groups is called an *orbit type*. The *slice representation* at a point $p \in M$ is the action of the isotropy group G_p on the normal space $\nu_p(Gp)$ to the orbit Gp. A point $p \in M$ is called *regular* if the slice representation is trivial. The set M_{reg} of regular points is open and dense, the corresponding orbits are called *principal orbits*, and the associated isotropy groups are called *principal isotropy groups*, and consist of a unique minimal conjugacy class of isotropy groups. The complement $M \setminus M_{reg}$ consists of *exceptional orbits*, corresponding to points whose slice representations have finite orbits, and the other orbits which are called *singular orbits*. The *cohomogeneity* of the G-action on M is by definition the codimension of the principal orbits, and equals the topological dimension of X.

From the geometric point of view, it is more natural to consider the more refined stratification of M by *normal isotropy types*, namely, conjugacy classes of slice representations. It turns out that connected components of isotropy strata and normal isotropy strata coincide; we will call such components simply the *strata* of M, and their projections to X, the *strata* of X. For a point $p \in M$, the set of fixed vectors of G_p in the slice representation is tangent to the stratum through p; G_p acts on its orthogonal complement in $\nu_p M$ with cohomogeneity equal to the codimension in X of the stratum through $x = Gp$. The *principal stratum* $X_{reg} = M_{reg}/G$ is full-dimensional, open, dense and connected (in fact, convex). The *boundary* ∂X of X is the closure of the union of the strata of codimension one.

A very basic result in the theory of isometric actions, which in fact can be used to prove many facts herein stated, is the *normal slice theorem*. Let $p \in M$, let S be the exponential image of an open ball in $\nu_p(Gp)$, centered at the origin, with sufficiently small radius, and put $U = G \cdot S$. Then *U is a G-invariant tubular neighborhood of the orbit Gp, which is equivariantly diffeomorphic to the homogeneous vector bundle $G \times_{G_p} \nu_p(Gp)$.* Here the *homogeneous vector* *bundle* $G \times_H V$ over G/H, for H a subgroup of G and V a representation of H, has total space given by the set of equivalence classes $[g, v]$ ($g \in G$, $v \in V$), where $(g, v) \sim (gh^{-1}, hv)$ for all $h \in H$.

We next discuss the natural metric structure of X. The *quotient metric* in X is defined by declaring the distance between two points of X to be the distance between the corresponding G-orbits in M (the latter is well defined since the orbits are properly embedded submanifolds). Owing to the Hopf-Rinow theorem and the first variation formula for the length, the distance between two orbits in M is realized by the length of a minimizing geodesic, orthogonal to all orbits it meets (that is, *horizontal*). It follows from the completeness of M that this metric in X is also complete: in fact every closed ball in X, being the projection of a closed ball of the same radius, is compact.

Recall that the *length* $L(\gamma)$ of a continuous path $\gamma : [a, b] \to X$ in a metric space (X, d) is the supremum over $\sum_i d(\gamma(t_i), \gamma(t_{i+1}))$ for all partitions $a = t_0 < t_1 < \ldots < t_n = b$, and γ is called *rectifiable* if its length is finite.

Coming back to our isometric action of G on M and induced quotient metric on $X = M/G$, for a minimizing geodesic between two G-orbits in M, the length of its projection to X cannot increase. It follows that the distance between two points in X is realized by the length of a rectifiable curve,

$$d(x, y) = \inf\{L(\gamma) \mid \gamma : [a, b] \to X \text{ is a rectifiable curve from } x \text{ to } y\}.$$

Metric spaces with this property are called *inner metric spaces* or *length spaces*.

A *geodesic* in a length space is a continuous curve all of whose sufficiently small arcs minimize the distance between their endpoints. A geodesic which minimizes the distance between its endpoints is called a *minimizing curve* or *shortest path*. We have seen that any two points in X can be joined by a minimizing geodesic. The projections of horizontal geodesics in M are suitable concatenations of geodesics in X: in fact, it follows from the slice theorem that such a projection γ has the property that for each t, $\gamma|_{[t-\epsilon,t]}$ and $\gamma|_{[t,t+\epsilon]}$ are shortest paths for sufficiently small $\epsilon > 0$.

Let $\gamma_i : [0, \epsilon) \to X$ for $i = 1, 2$ be two minimizing geodesics emanating from a point $x \in X$, which are parametrized by arc-length, that is, $L(\gamma_i|_{[0,t]}) = t$ for all t. One uses the (Euclidean) cosine law to define the *angle* between γ_1 and γ_2 to be

$$\angle(\gamma_1, \gamma_2) = \lim_{t \to 0^+} \arccos\left(1 - \frac{d(\gamma_1(t), \gamma_2(t))^2}{2t^2}\right)$$

(in our case, the limit can be shown to exist.) From the above description of geodesics, one sees that two minimizing geodesics emanating from x, parametrized by arc-length, with angle zero, coincide. Therefore such a geodesic defines an *initial direction* at x, and the angle defines a metric on the space $\Sigma_x X$ of directions at x. The cone over $\Sigma_x X$ is called the *tangent cone* of X at x and is denoted by $C_x X$ or $T_x X$. It follows from the slice theorem that the tangent cone at a point $x = Gp \in X$ is isometric to the orbit space of the slice representation at $p \in M$. We deduce that strata in X can also be characterized as the connected components of the sets of points in X with isometric tangent cones.

In most of our practical considerations, M is either Euclidean space or the unit sphere. Suppose the sectional curvatures of M are bounded below by a constant κ. Since the projection from M to X cannot increase lengths, one sees that the curvature of X is *bounded below by κ in the Alexandrov sense*, that is, for any sufficiently small triangle in X formed by three geodesics, there exists a comparison triangle in the simply-connected complete space of constant curvature κ with congruent sides and whose angles bound the corresponding angles of the original triangle from below; in other words, the given triangle is "fatter" than the comparison triangle.

A complete locally compact length space with curvature bounded below by κ is called an *Alexandrov space*. We have seen that if M has curvature bounded from below, then X is a finite dimensional Alexandrov space.

5.2 Riemannian orbifolds

For more complete treatments of orbifolds, [39, 7, 3, 20, 28] are excellent sources.

Let V be an Euclidean space and denote by $S(V)$ its unit sphere. If Γ, Γ' are two finite subgroups of $O(V)$ which are conjugate, then the orbit spaces V/Γ, V/Γ' are isometric. In fact, if $\Gamma' = f\Gamma f^{-1}$ for some $f \in O(V)$, then there is an induced isometry

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\pi"'] & V \arrow[d, "\pi'"] \\ V/\Gamma \arrow[r, dotted, "\bar{f}"'] & V/\Gamma' \end{tikzcd}$$

given by $\bar{f}(\Gamma v) = \Gamma' f(v)$.

Conversely:

Lemma 5.2.1 ([35]). *If V/Γ, V/Γ' are isometric then Γ, Γ' are conjugate in $O(V)$*

Proof. We proceed by induction on $n = \dim V$. In the initial case of $n = 1$, $V \cong \mathbb{R}$ and the only possibilities for Γ are $\{1\}$ and $\{\pm 1\}$, which yield $\mathbb{R}$ and $[0, +\infty)$, resp., non-isometric orbit spaces, so we are done.

Assume now $n \geq 2$. It is enough to work with $X = S(V)/\Gamma$, $X' = S(V)/\Gamma'$. Suppose $F : X \to X'$ is an isometry. Let $x \in X$ be such that $\Gamma \cdot x$ and $\Gamma' \cdot F(x)$ are principal orbits. Choose points $p \in \pi^{-1}(x)$, $p' \in \pi'^{-1}(x')$ and open neighborhoods U_p, $U_{p'}$, U_x, $U_{x'}$ such that $\pi|_{U_p} : U_p \to U_x$, $\pi'|_{U_p'} : U_{p'} \to U_{x'}$ are isometries and $F(U_x) = U_{x'}$. Then $(\pi'|_{U_{p'}})^{-1} F \pi : U_p \to U_{p'}$ is an isometry, where $\pi : S(V) \to X$, $\pi' : S(V) \to X'$ are the projections; since $S(V)$ is a sphere of constant curvature, we can (uniquely) extend it to a global isometry $\tilde{F} : S(V) \to S(V)$. Let $\bar{\tilde{F}} : X'' \to X'$ be the isometry induced on the level of quotients, where $\Gamma'' := \tilde{F}^{-1}\Gamma'\tilde{F}$ and $X'' = S(V)/\Gamma''$. Then $\pi'' = \bar{\tilde{F}}^{-1}\pi'\tilde{F}$:

$S(V) \to X''$. Therefore, identifying $X \cong X''$ using the isometry $\bar{\tilde{\bar{F}}}^{-1}F$, we get $\pi''|_{U_p} = \pi|_{U_p}$. We will show that $\Gamma'' = \Gamma$ as subgroups of $O(V)$.

It suffices to prove that:

(a) π is completely determined by its restriction to an open neighborhood U_p of a Γ-regular point p.

(b) Γ is completely determined by π.

Since π is a local isometry on the principal stratum, it is determined along any unit speed geodesic γ in $S(V)$ emanating from p, until γ reaches a non-regular point, say $q = \gamma(t_0)$ for some $t_0 > 0$. Now $\dot{\gamma}(t_0)$ belongs to the unit sphere S_q of $T_qS(V)$. The space of directions $\Sigma_y X$ for $y = \pi(q) \in X$ is isometric to S_q/Γ_q. Since $\dim T_qS(V) = n-1$, the action of Γ_q on S_q is known by the induction hypothesis. It follows that the exit direction of $\pi \circ \gamma$ from y is known and thus π is determined along γ beyond t_0; this proves (a). Finally, the elements of Γ are in bijective correspondence with the points in $\pi^{-1}(x)$ via the map $\gamma \mapsto \gamma(p)$. For each $\gamma \in \Gamma$, we have a commutative diagram:

$$\begin{tikzcd} U_p \arrow[d, "\pi"'] \arrow[dr, "\gamma"] & \\ U_x & \gamma(U_p) = U_{\gamma(p)} \arrow[l, "\pi"] \end{tikzcd}$$

Since γ is an isometry of $S(V)$, using (a) this completely determines it as an element of $O(V)$. Hence (b) is proved. $\square$

A metric space X is called a *Riemannian orbifold* if every point $x \in X$ admits a neighborhood U isometric to a quotient M/Γ, where M is a Riemannian manifold and Γ is a finite group of isometries. (Such an approach is non-standard but equivalent to the usual one. It has been suggested by Lytchak [23].) The next lemma shows that X is locally represented as a quotient in a unique way.

Lemma 5.2.2. *Every isometry $F : M/\Gamma \to M'/\Gamma'$ is* locally *induced by a locally defined isometry $f : M \to M'$. Namely, for every $x \in M/\Gamma$, there exist connected open neighborhoods U, U' of x, $x' = F(x)$ of the form V/Γ_p, $V'/\Gamma'_{p'}$, where V, V' are normal neighborhoods of $p \in \pi^{-1}(x)$, $p' \in \pi^{-1}(x')$, resp., and $U' = F(U)$ ($\pi : V \to V/\Gamma_p$, $\pi' : V' \to V'/\Gamma'_{p'}$ denote the canonical projections). Moreover $F \circ \pi = \pi' \circ f$ for some isometry $f : V \to V'$ with $f(p) = p'$ and $\Gamma'_{p'} = f\Gamma_p f^{-1}$.*

Proof. Let $G = \Gamma_p$, $G' = \Gamma'_{p'}$. By restriction we have an isometry $F : V/G \to V'/G'$ with $F(x) = x'$. Here V, V' can be taken to be metric balls of the same, small radius, around p, p', resp. Consider the actions of G, G' on T_pV, $T_{p'}V'$, resp. Then there is an isometry

$$T_pV/G \cong T_x(V/G) \to T_{x'}(V'/G') \cong T_{p'}V'/G',$$

which we denote by dF_x. By Lemma 5.2.1, there is an isometry $\varphi : T_pV \to T_{p'}V'$ such that

$$\begin{array}{ccc} T_pV & \xrightarrow{\varphi} & T_{p'}V' \\ {\scriptstyle d\pi_p}\downarrow & & \downarrow{\scriptstyle d\pi'_{p'}} \\ T_pV/G & \xrightarrow[dF_x]{} & T_{p'}V'/G' \end{array}$$

is commutative. Since the Riemannian exponential maps $\exp_p : T_pV \to V$, $\exp_{p'} : T_{p'}V' \to V'$ are G-, G'-equivariant diffeomorphisms, resp., we can define an equivariant diffeomorphism

$$\begin{array}{ccc} T_pV & \xrightarrow{\varphi} & T_{p'}V' \\ {\scriptstyle \exp_p}\downarrow & & \downarrow{\scriptstyle \exp_{p'}} \\ V & \xrightarrow[f]{} & V' \end{array}$$

Finally, there is an induced map

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ V/G & \xrightarrow[\bar{f}]{} & V'/G' \end{array}$$

We claim that $\bar{f} = F$. In fact, for a geodesic $\gamma(t) = \exp_p t\dot{\gamma}(0)$ in V:

$$\begin{aligned} \bar{f}\pi\gamma(t) &= \bar{f}\pi \exp_p t\dot{\gamma}(0) \\ &= \pi' \exp_{p'} t\varphi(\dot{\gamma}(0)) \end{aligned} \tag{5}$$

by commutativity of the last two diagrams. Now $\pi \circ \gamma$ is an orbifold-geodesic of V/G. The main point here is that orbifold-geodesics in general are (correctly chosen) concatenations of metric geodesics (locally minimizing curves). Since x is a fixed point of G, $\pi \circ \gamma$ is also a metric geodesic and thus it is mapped under F to a metric geodesic emanating from x', hence, the orbifold geodesic $\pi \circ \gamma'$, where $\dot{\gamma}'(0) = \varphi(\dot{\gamma}(0))$:

$$F\pi\gamma(t) = F\pi \exp_p t\dot{\gamma}(0) = \pi' \exp_{p'} t\varphi(\dot{\gamma}(0)). \tag{6}$$

Comparison of (5) and (6) proves the claim.

It follows $f : V \to V'$ is a local isometry on the regular set and thus, by continuity, an isometry everywhere. Now the groups G', fGf^{-1} acting on V' are orbit-equivalent. If not coincident, they generate a strictly larger group with the same (finite) orbits and thus non-trivial principal isotropy groups, a contradiction (since the slice representation at regular points must be trivial). Hence $G' = fGf^{-1}$. $\square$

Let X be a Riemannian orbifold and let $x \in X$. Locally represent X around x as a quotient M/Γ and write $x = \Gamma p$ for some $p \in M$. Since the isotropy group Γ_p acts by isometries on M, it can be viewed as a subgroup of the orthogonal group $O(T_pM)$. Moreover, it follows from Lemma 5.2.2 that the congruence class of Γ_p is independent of the local representation of X as a quotient. After identification $T_pM \cong \mathbb{R}^n$, we get a congruence class of subgroups of $O(n)$, called the *local group* of X at x and denoted by $\mathrm{Iso}_x(X)$. A point $x \in X$ is called a *manifold point* of X if $\mathrm{Iso}_x(X) = \{1\}$.

Example 5.2.1. Let a Lie group G act by isometries on a Riemannian manifold. Then the orbit space has a canonical structure of Riemannian orbifold in the following two cases:

(a) G is discrete and the action is proper (such orbifolds are called *good* or *developable*; non-good orbifolds are also called *bad*);

(b) G is compact and connected and all orbits have the same dimension.

Let X be a Riemannian orbifold and locally represent $X \supset U \cong M/\Gamma$. Let $O(M)$ denote the orthonormal frame bundle of M. This is a principal $O(n)$-bundle. The action of Γ lifts to a free action on $O(M)$, commuting with the action of $O(n)$, and thus $O(M)/\Gamma$ is a smooth manifold with a right $O(n)$-action and quotient U. It follows again from Lemma 5.2.2 that the *orthonormal frame bundle* $O(X)$ of the orbifold X is well defined as a smooth manifold such that $O(M)/\Gamma$ canonically embeds into $O(X)$ for each local representation, and there is an induced projection $O(X) \to X$. Furthermore $O(n)$ acts on $O(X)$ and $O(X)/O(n) = X$, where for any frame $f \in O(X)$ at x, there is a canonical isomorphism $O(n)_f \cong \mathrm{Iso}_x(X)$. This construction shows that every Riemannian orbifold can be written as the quotient of a Riemannian manifold by an almost free isometric action of a compact Lie group (namely, $O(n)$). The stratification of X by local isotropy groups coincide with the stratification by $O(n)$-orbit type.

An *orbi-covering* is a map $\pi : X \to Y$ between Riemannian orbifolds which is locally represented as the natural projection $M/\Gamma \to M/\tilde{\Gamma}$ for groups of isometries $\Gamma \subset \tilde{\Gamma}$ of a Riemannian manifold M. It is a fact that every connected orbifold X admits a *universal orbi-covering* $\tilde{X}$, unique up to equivalence, with the property that it orbi-covers any other orbi-covering space of X. The *orbifold fundamental group* of X is the group $\pi_1^{orb}(X)$ of deck transformations of the universal orbi-covering; it acts simply transitively on the fibers of this orbi-covering. The orbifold fundamental group is a refinement of the usual fundamental group in the sense that an orbifold can be simply-connected in the topological sense without being simply-connected in the orbifold sense.

Example 5.2.2. Let the cyclic group $\mathbb{Z}_m$ act by rotations around a fixed axis on the sphere S^2. The orbit space X is a Riemannian orbifold and topologically a 2-sphere but $\pi_1^{orb}(X) \cong \mathbb{Z}_m$.

A Riemannian orbifold with a non-empty boundary in the Alexandrov sense can be doubled. It follows that a Riemannian orbifold X has $\partial X \neq \varnothing$ if and only if $\pi_1^{orb}(X)$ contains a reflection.

Example 5.2.3. Let X be the quotient of S^2 by the reflection across the equator. Then $\pi_1^{orb}(X) \cong \mathbb{Z}_2$.

Example 5.2.4. There is a Riemannian orbifold structure $X_{m,n}$ on S^2 with exactly two non-manifold (conical) points whose local groups are respectively $\mathbb{Z}_m$ and $\mathbb{Z}_n$. Then $\pi_1^{orb}(X_{m,n}) = \mathbb{Z}_d$ where d is the greatest common divisor of m and n. The orbifold $X_{m,n}$ is good if and only if $m = n$. In particular, $X_{m,1}$ for $m > 1$ is called a *teardrop*, and the bad 2-orbifold depicted in Figure 5 is the quotient of $X_{2,1}$ by a reflection.

6 References

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