An Introduction to Riemannian Symmetric Spaces

Claudio Gorodski*

September, 2021

Abstract

These are the notes for a series of lectures in the 7th School and Workshop on Lie Theory, at Universidade Federal de Juiz de Fora, Minas Gerais, Brazil (online event), in September 8-15, 2021. We wish to thank the organizers for the invitation to give these lectures. Lecture 1 introduces Riemannian symmetric spaces in terms of geodesic reflections and explain their basic structure. Lecture 2 presents a sketch of the classification of symmetric spaces, based on a good amount of Lie group theory. Lecture 3 discusses the intrinsic geometry of symmetric spaces, in terms of flats and restricted roots, and includes a brief survey on isometric actions on those spaces. The appendices contain a complete proof of the Cartan-Ambrose theorem (based on Cheeger-Ebin's book), and a review of semisimple Lie theory.

Introduction

Symmetric spaces were introduced and extensively studied by Élie Cartan around 1925-1930, over ten years after completion of his impressive work on real and complex Lie algebras, and constitute undoubtedly his most important work on Riemannian geometry, with ramifications in classical geometries, the theory of analytical functions of several complex variables, number theory, harmonic analysis and topology. It is often said that (in addition to Einstein's Relativity Theory) his work on symmetric spaces provoked further development of Riemannian geometry. Roughly speaking, a Riemannian manifold is called a symmetric space if it is reflectionally symmetric around any point. From the modern point of view, such manifolds form a class of Riemannian manifolds that simultaneously extends space forms and Lie groups with bi-invariant metrics, and include projective spaces, Grassmann manifolds and their classical generalizations, as well as some exceptional spaces and non-compact counterparts.

*Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, São Paulo, SP 05508-090, Brazil, *E-mail address*: gorodski@ime.usp.br

The classification of symmetric spaces is also the work of Cartan. In modern terminology, every symmetric space is a homogeneous space. The universal covering of a symmetric space is also a symmetric space. Every simply-connected symmetric space is a Riemannian product of irreducible symmetric spaces. The nonflat irreducible symmetric spaces are separated into compact type and non-compact type, which are interchanged by Cartan duality. Each of those two types, in turn, fall into two classes. The symmetric spaces of compact type can be class 2 (compact Lie groups with bi-invariant metrics), or class 1 (homogeneous spaces of a compact Lie group defined by an involution), and the symmetric spaces of non-compact type can be class 4 (homogeneous spaces of a complex Lie group by a real form, dual to class 2), or class 3 (homogeneous spaces of a non-compact, non-complex Lie group, by a maximal compact subgroup, dual to class 1).

In fact, originally Cartan considered a tensorial condition, namely, the parallelism of the curvature tensor, $\nabla R \equiv 0$. This condition defines the class of locally symmetric spaces (which can be equivalently defined as those Riemannian manifolds which are locally reflectionally symmetric around any point), and only later he brought the property of existence of symmetries to the forefront. In turns out that in the category of complete and simply connected Riemannian manifolds, the symmetric spaces can also be characterized by the property that $\nabla R \equiv 0$.

It is often said that Lie groups and homogeneous spaces equipped with invariant metrics are good testing spaces in Riemannian geometry. However these classes can be too general in some contexts, and then symmetric spaces appear as a more manageable class in which much more refined calculations can take place. For instance, a good amount of the geometry of symmetric spaces is described in terms of (non-reduced) root systems. The Jacobi equation along a geodesic has constant coefficients, and conjugate and cut loci can be accurately described. Totally geodesic submanifolds are abundant in symmetric spaces, and their complete classification is a formidable, possibly unreachable algebraic problem; nonetheless the flat ones are completely understood and indeed play a major role in the structural theory of symmetric spaces. The topology of symmetric spaces is also computable, and serves as a nice application of Morse Theory.

Herein our point of view is to avoid excessive technicality and use Riemannian geometric arguments whenever possible. This somehow contradicts some opinions that results of an algebraic character should have an algebraic proof, but, on the other hand, is in line with the feeling that Élie Cartan expressed in the final lines of [8]:

> *"J'espère vous avoir montré toute la variété des problèmes que la Théorie des groupes et la Géométrie, en s'appuyant mutuellement l'une sur l'autre, permettent d'aborder et de résoudre. Il y a encore là un champ de recherches à peine exploré et qui promet des résultats très intéressants."*

Contents

1 Lecture 1: Basic structure

The origins of the theory can be traced back to a joint paper with Schouten [11], in which Cartan studied some classes of bi-invariant connections on Lie groups, hence with curvature tensor invariant under parallel transport.[1] Cartan published a number of papers on the then called "espaces $\mathcal{E}$", at first building the theory mostly from the local point of view, and starting 1927, he combined his techniques with results and points of view of H. Weyl and embarked on a global theory, mixing differential geometry and semisimple Lie groups in a striking manner. This led to the monograph [10], in which Cartan outlined the foundations of a global theory of Lie groups and homogeneous spaces. From 1929 on, Cartan changed the name to "espace symétrique" and went on further developing the theory. Some highlights are the extension of the Peter-Weyl Theorem to compact symmetric spaces, the computation of the Betti numbers of compact Lie groups and their homogeneous spaces and, finally, the paper on bounded symmetric domains, which was influenced by the work of his son Henri Cartan.

The final chapters of Pontryagin's book [37] contains an exposition of some global aspects of Lie theory. However, Chevalley's Princeton book on Lie groups [15], born out of his reports on the subject for Bourbaki before the war, quickly became the standard reference in textbook form, for many years.

In the early fifties there was no exposition of the theory of symmetric spaces beyond Cartan's papers. Between 1953 and 1961, A. Borel gave three series of lectures on the subject (at IAS-Princeton, MIT and Tata Institute in Mumbai), hoping that the notes would lead to a publication. Early in January 1958, the available material was communicated to S.-S. Chern, who organized a seminar on symmetric spaces at the University of Chicago in the first quarter, which generated informal seminar notes. At that time, R. Palais was an instructor at Chicago, S. Helgason was assistant professor and J. Wolf was a student of Chern. Helgason's and Wolf's books ([23, 42], first editions in 1962 and 1967, resp.), containing material on symmetric spaces, would come out of this. In the same year, J.-L. Koszul visited the University of São Paulo and gave a series of lectures on symmetric spaces and homogeneous bounded domains, which notes were published there [31]. Both Borel's and Koszul's notes were added to Bourbaki's archive, as preparatory material towards chapters on semisimple Lie groups and symmetric spaces in the book on Lie groups and Lie algebras, but this has never materialized. In the sixties, symmetric spaces also appeared in book form in [27] and [32]. Borel's notes were finally published in 1998, unedited [4].

The author first learned this material from J. Wolf, and these notes are mostly inspired by the treatment in his book (which is directly based on the Borel notes). Throughout the text, (M, g) or M shall denote a connected Riemannian manifold.

[1] H. Levy had already considered Riemannian manifolds with $\nabla R = 0$, but apparently his work contains errors and does not go very far.

1.1 Definition and basic examples

Any point $x \in M$ is known to admit a *normal neighborhood*, that is, a neighborhood U which is the diffeomorphic image of a neighborhood U_0 of the origin 0_x in the tangent space T_xM under the exponential map $\exp_x : T_xM \to M$. It is clear that U_0 can be taken to be of the form of an open ball $B(0_x, \epsilon)$ for some $\epsilon > 0$. Now it makes sense to define the *geodesic symmetry* at x to be the map $s_x : U \to U$ taking $\exp_x(v)$ to $\exp_x(-v)$ for every $v \in B(0_x, \epsilon)$. Note that s_x reverses geodesics emanating from x.

1.1.1 Proposition *Fix a point x in M. Then the following assertions are equivalent:*

a. The geodesic symmetry s_x at x is a local isometry.

b. There exists a local isometry s of M defined on a neighborhood of x such that $s(x) = x$ and the differential $ds_x = -\mathrm{id}$.

c. There exists an involutive local isometry of M defined on a neighborhood of x which has x as an isolated fixed point.

Proof. (a) is equivalent to (b). Assume the geodesic symmetry at x is a local isometry. We have $s_x(0) = s_x(\exp_x(0_x)) = \exp_x(0_x) = x$. Moreover, let $v \in T_xM$ and consider the geodesic $\gamma(t) = \exp_x(tv)$ for small t. Then $ds_x(v) = \frac{d}{dt}|_{t=0} s_x(\exp_x(tv)) = \frac{d}{dt}|_{t=0} \exp_x(-tv) = -v$. Hence we can take $s = s_x$. Conversely, if s is as in (b), then $s(\exp_x(v)) = \exp_{s(x)}(ds_x(v)) = \exp_x(-v)$ forcing s to be the geodesic symmetry.

(b) implies (c). The fixed point set of a local isometry s at x is a totally geodesic submanifold S of M through x whose tangent space at x is precisely the fixed point set of ds_x in T_xM. Since $ds_x = -\mathrm{id}$, we have $T_xS = \{0\}$ and hence S is discrete at x. Moreover, s^2 is a local isometry with $s^2(x) = x$ and $d(s^2)_x = \mathrm{id}$, hence it must be the identity on a neighborhood of x.

(c) implies (b). Suppose s is an involutive local isometry as in (c). Then $(ds_x)^2 = \mathrm{id}$. Since ds_x is an orthogonal transformation of T_xM, this implies that its eigenvalues are ± 1. However, owing to the fact that x is an isolated fixed point of s, the eigenvalues must be all -1 by the same argument as above. This finishes the proof. $\square$

A Riemannian manifold (M, g) is called a *locally symmetric space* if the assertions of Proposition 1.1.1 hold at every point of M. Furthermore, (M, g) is called a *globally symmetric space* or, simply, a *symmetric space* if the geodesic symmetry s_x is globally defined on M and an isometry for every $x \in M$.

1.1.2 Examples *a.* Euclidean space $\mathbf{R}^n$ is a symmetric space. The geodesic symmetry at the origin is $s_0(y) = -y$ for $y \in \mathbf{R}^n$. More generally, the geodesic symmetry at $x \in \mathbf{R}^n$ is $s_x(y) = 2x - y$. Note that s_x and s_0 are conjugate by the translation $\tau_x : y \mapsto y + x$. This suggests the following example.

b. A Riemannian manifold is called *homogeneous* if it admits a transitive group of isometries. A homogeneous Riemannian manifold is also called a *Riemannian homogeneous space*. A Riemannian manifold is called *locally homogeneous* if given two points x, y in M, there exist neighborhoods U, V of x, y, respectively, and an isometry $f : U \to V$ such that $f(x) = y$. If (M, g) is (resp. locally) homogeneous, the (resp. local) symmetries at different points are all conjugate by (resp. local) isometries among themselves. Therefore it suffices to check that the geodesic symmetry at one single point is a (resp. local) isometry in order to show that M is (resp. locally) symmetric.

c. The canonical metric on the sphere S^n is realized as the induced metric from its embedding as the unit sphere in Euclidean space of one dimension higher. It is then clear that S^n is homogeneous under the group $\mathbf{O}(n+1)$ of orthogonal transformations of $\mathbf{R}^{n+1}$. Anyway, for any $x \in S^n$, the Euclidean reflection on the line $\mathbf{R}x$ is an orthogonal transformation of $\mathbf{R}^{n+1}$ whose restriction to S^n is the geodesic symmetry s_x. Hence S^n is a symmetric space.

d. The real hyperbolic space $\mathbf{R}H^{n+1}$ can be realized as the upper sheet of a two-sheeted hyperboloid in Lorentzian space. Namely, consider the Lorentzian inner product in $\mathbf{R}^{n+1}$ given by

$$\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n,$$

where $x = (x_0, \ldots, x_n)$, $y = (y_0, \ldots, y_n) \in \mathbf{R}^{n+1}$. We will write $\mathbf{R}^{1,n}$ to denote $\mathbf{R}^{n+1}$ with such a Lorentzian inner product. Note that if $x \in \mathbf{R}^{1,n}$ is such that $\langle x, x \rangle < 0$, then the restriction of $\langle , \rangle$ to the orthogonal complement $x^\perp$ is positive-definite. Note also that the equation $\langle x, x \rangle = -1$ defines a two-sheeted hyperboloid in $\mathbf{R}^{1,n}$. Now we can define the *real hyperbolic space* as the following submanifold of $\mathbf{R}^{1,n}$,

$$\mathbf{R}H^n = \{\, x \in \mathbf{R}^{1,n} \mid \langle x, x \rangle = -1 \quad \text{and} \quad x_0 > 0 \,\},$$

equipped with a Riemannian metric g given by the restriction of $\langle , \rangle$ to the tangent spaces at its points. Since the tangent space of the hyperboloid at a point x is given by $x^\perp$, the Riemannian metric g turns out to be well defined. Actually, this submanifold is sometimes called the *hyperboloid model of* $\mathbf{R}P^n$. Of course, as a smooth manifold, $\mathbf{R}H^n$ is diffeomorphic to $\mathbf{R}^n$. It is not difficult to see that $\mathbf{R}H^n$ is homogeneous under the group $\mathbf{O}(1, n)$ of Lorentzian transformations of $\mathbf{R}^{1,n}$. Moreover, $\mathbf{R}H^n$ is a symmetric space, for the geodesic symmetry at a point x is induced by the reflection along the line $\mathbf{R}x$, similar to the case of the sphere.

e. Let G be a compact Lie group. It is known that G admits a bi-invariant metric (cf. subsection 5.6). This means that there exists a Riemannian metric on g such that the left translations $L_g : G \to G$, $L_g(x) = gx$, and right translations $R_g : G \to G$, $R_g(x) = xg$, are isometries for all

$g \in G$. In particular, G is homogeneous. Moreover we claim G is also symmetric. Indeed, let us check that the inversion map $\iota(x) = x^{-1}$ satisfies the conditions in part (b) of Proposition 1.1.1 at the identity 1. Plainly, $\iota(1) = 1$, and $d\iota_1(X) = \frac{d}{dt}|_{t=0}\iota \exp^G(tX) = \frac{d}{dt}|_{t=0} \exp^G(-tX) = -X$ for $X \in T_1G$, where $\exp^G$ denotes the Lie group exponential map. In particular, $d\iota_1$ is a linear isometry. In order to see that $d\iota_g$ is a linear isometry for all $g \in G$, we apply the chain rule to the identity $\iota = R_{g^{-1}} \circ \iota \circ L_{g^{-1}}$ to get $d\iota_g = (dR_{g^{-1}})_1 \circ d\iota_1 \circ (dL_{g^{-1}})_g$ and note that $(dR_{g^{-1}})_1$ and $(dL_{g^{-1}})_g$ are linear isometries. Hence ι is an isometry.

f. A Riemannian manifold locally isometric to a symmetric space is locally symmetric. In particular, if $\tilde{M} \to M$ is a Riemannian covering and $\tilde{M}$ is a symmetric space, then M is locally symmetric. The manifold M does not have to be globally symmetric; examples are given by a surface of genus $g \geq 2$ (covered by $\mathbf{R}H^2$) and most lens spaces (covered by spheres).

We have seen that the class of symmetric spaces is a simultaneous generalization of the classes of spaces of constant curvature and compact Lie groups equipped with bi-invariant metrics. The relation of symmetric spaces to spaces of constant curvature will be made more explicit when we discuss the curvature of symmetric spaces; indeed, there is a local characterization of symmetric spaces in terms of curvature. On the other hand, the relation of symmetric spaces to Lie groups involves the fact that symmetric spaces are a special type of homogeneous spaces, and this is the basis of the structure and classification results that we shall study.

1.2 Transvections

Contemplate a locally or globally symmetric space M. Do the geodesic symmetries generate any kind of group?

Fix a point x in M and a geodesic $\gamma : (-\epsilon, \epsilon) \to M$ through $x = \gamma(0)$. For $t \in (-\epsilon, \epsilon)$, the geodesic symmetry $s_{\gamma(t)}$, which for the moment we denote simply by s_t, is defined and a local isometry. Now the composite map $p_t := s_{\frac{t}{2}} s_0$ is locally defined and a local isometry of M; this is called a *(local) transvection* along γ. Since γ passes through x, we also say that p_t is a transvection at x.

1.2.1 Proposition *Let $\gamma : (-\epsilon, \epsilon) \to M$ be a geodesic through $x = \gamma(0)$.*

a. The transvection p_t induces translation along the curve γ, that is, $p_t(\gamma(t_0)) = \gamma(t+t_0)$. More generally, p_t induces parallel transport on vectors along γ, in the sense that if $v \in T_{\gamma(t_0)}M$ then $X(t) = (dp_{t-t_0})_{\gamma(t_0)}(v)$ is a parallel vector field along γ.

b. The transvections $\{p_t\}$ along γ form a local one-parameter group of local isometries of M, namely, $p_{t+t'} = p_t p_{t'}$ whenever both hand sides are defined.

c. *The transvection p_t depends only on γ but not on the chosen initial point $x = \gamma(0)$. In other words, $s_{\frac{t}{2}} s_0 = s_{t_0+\frac{t}{2}} s_{t_0}$.*

Proof. (a) We have $p_t(\gamma(t_0)) = s_{\frac{t}{2}} s_0(\gamma(t_0)) = s_{\frac{t}{2}}(\gamma(-t_0)) = \gamma(t+t_0)$. In the proof of the second assertion, we are going to use the fact that isometries act on vector fields by push-forward taking parallel vector fields to parallel vector fields. The assertion follows from the fact that if an isometry maps a geodesic to itself, up to a translation in the parameter, then it maps a parallel vector field along that geodesic to itself, up to a translation in the parameter. More formally, assume that $t_0 = 0$ for simplicity of notation. We want to show that $X(t) = (dp_t)_x(v)$ is parallel along γ. Let Y denote the parallel vector field along γ such that $Y(0) = v$. Fix t_1. Then $Z(t) = (dp_{t_1})_{\gamma(t-t_1)}(Y(t-t_1)) \in T_{\gamma(t)}M$ is a parallel vector field along γ. Note that $Z(t_1) = (dp_{t_1})_x(v) = X(t_1)$. Since t_1 is arbitrary, this completes the proof of (a).

(b) An isometry is locally determined by its differential at one point. Moreover, the composition of parallel transports along two adjacent segments of γ equals the parallel transport along the juxtaposed segment, so the result follows from part (a).

(c) Use part (b) to write $p_t = p_{t+2t_0} p_{-2t_0} = s_{\frac{t}{2}+t_0} s_0 s_{-t_0} s_0$. We have already remarked that for a local isometry g, the conjugation $g s_x g^{-1} = s_{gx}$. Applying this to $g = s_0 = g^{-1}$ yields that $s_0 s_{-t_0} s_0 = s_{t_0}$, as desired. □

It follows from Proposition 1.2.1 that each geodesic determines a unique local one-parameter group of transvections along it.

1.2.2 Proposition *A connected locally symmetric space is locally homogeneous. A connected globally symmetric space is homogeneous and complete.*

Proof. Suppose (M, g) is connected and locally symmetric. Declare two points of M to be equivalent if there exists a local isometry of M mapping the first point to the second one. It is enough to prove that the equivalence classes are open. Indeed, the existence of transvections implies that a normal neighborhood of a point is contained in its equivalence class.

If (M, g) is in addition globally symmetric, transvections are global isometries and by this argument M is globally homogeneous. Completeness follows from homogeneity in general. □

It follows from Proposition 1.2.2 and the Hopf-Rinow theorem that a globally symmetric space M is geodesically complete. In this case, the transvections along a geodesic form a full one-parameter group of global isometries of M. Moreover, since any two points of M can be joined by a geodesic arc, the product of any two geodesic symmetries of M is a transvection.

If M is globally symmetric, the group generated by transvections at a fixed point x generate a connected transitive subgroup of the isometry group of M, and we shall see in subsection 2.1 that this group indeed coincides with the identity component of the full isometry group of M in case M has no flat de Rham factor.

1.2.3 Example Consider the unit sphere S^n. We have already remarked that the geodesic symmetry s_x at a point x is the isometry induced by reflection of $\mathbf{R}^{n+1}$ on the line $\mathbf{R}x$. It follows that for $y \neq \pm x$, the transvection $s_y s_x$ is the isometry which rotates the geodesic arc $\overline{xy}$ by an angle twice the angle between x and y, and which is the identity on the orthogonal complement of the plane spanned by x, y (if $y = -x$, then $s_y = s_x$). Every element of $\mathbf{SO}(n+1)$ can be "diagonalized", in the sense that it is a product of 2×2 rotations, and a fixed direction in case n is even, with respect to a suitable basis. It is then clear that the group generated by transvections at x is the full special orthogonal group $\mathbf{SO}(n+1)$.

1.3 Tensorial characterization

We first prove Cartan's criterion allowing to extend a linear isometry, defined on the tangent space to a Riemannian manifold at a point, to an isometry, defined on a normal neighborhood of that point.

We need to introduce some notation. Let M and $\tilde{M}$ be two Riemannian manifolds and let $x \in M$ and $\tilde{x} \in \tilde{M}$. Let $I : T_xM \to T_{\tilde{x}}\tilde{M}$ denote a linear isometry and let $V \subset M$ denote a normal coordinate neighborhood around x such that $\exp_{\tilde{x}}$ is defined in $I(\exp_x^{-1}(V))$. We define a map $\varphi : V \to \tilde{M}$ by setting

$$\varphi(y) = \exp_{\tilde{x}} \circ I \circ \exp_x^{-1}(y). \tag{1.3.1}$$

Note that $d\varphi_x = I$.

For $y \in V$, let $P_\gamma : T_xM \to T_yM$ denote the parallel transport along the unique geodesic $\gamma : [0,1] \to M$ in V from x to y, and let $P_{\tilde{\gamma}} : T_{\tilde{x}}\tilde{M} \to T_{\tilde{y}}\tilde{M}$, where $\tilde{y} = \varphi(y)$, be the parallel transport along $\tilde{\gamma} = \varphi \circ \gamma$. Finally we define

$$I_\gamma : T_yM \to T_{\tilde{y}}\tilde{M} \tag{1.3.2}$$

by setting

$$I_\gamma(u) = P_{\tilde{\gamma}} \circ I \circ P_\gamma^{-1}(u).$$

The purpose of Cartan's Theorem is to give a criterion for φ to be a local isometry. The main ingredient in the proof below is the fact that the Jacobi equation along a geodesic in a locally symmetric space has constant coefficients (with respect to a parallel orthonormal frame).

1.3.3 Theorem (Cartan) *If for every $y \in V$ and every u, v, w in T_qM we have*

$$I_\gamma(R(u,v)w) = \tilde{R}(I_\gamma(u), I_\gamma(v))I_\gamma(w),$$

where R and $\tilde{R}$ are the curvature tensors of M and $\tilde{M}$ respectively, then φ is a local isometry. Moreover $d\varphi_y = I_\gamma$ for every $y \in V$.

Proof. Let y be a point in V and let v be a vector in T_yM. We would like to show that $\|d\varphi_y(v)\| = \|v\|$.

There is a unique Jacobi field J along the geodesic $\gamma : [0,1] \to V \subset M$ joining x and y such that $J(0) = 0$ and $J(1) = v$ since V is a normal neighborhood. It is known that $J(t) = d(\exp_x)_{t\gamma'(0)}(tw)$ where $w = J'(0)$ and hence that $v = d(\exp_x)_{\gamma'(0)}(w)$. By the chain rule we have

$$d\varphi_y(v) = d(\exp_{\tilde{x}})_{I(\gamma'(0))} \circ I \circ d(\exp_x)^{-1}_{\gamma'(0)}(v),$$

and hence

$$d\varphi_y(v) = d(\exp_{\tilde{x}})_{\tilde{\gamma}'(0)}(I(w)),$$

where $\tilde{\gamma} = \varphi \circ \gamma$. Therefore $d\varphi_y(v) = \tilde{J}(1)$ where $\tilde{J}$ is the Jacobi field along $\tilde{\gamma}$ satisfying $\tilde{J}(0) = 0$ and $\tilde{J}'(0) = I(w) = I(J'(0))$.

Let $E_1 = \gamma'$, $E_2, \ldots, E_n$ be an orthonormal frame of parallel vector fields along γ. We write

$$J = \sum_{j=1}^{n} \alpha_j E_j$$

where α_j are real valued functions. The Jacobi equation now implies that the functions α_j are the unique solutions of the system of ordinary differential equations

$$\alpha_j'' + \sum_{k=1}^{n} f_{kj} \alpha_k = 0,$$

where $f_{kj} = \langle R(E_k, \gamma')\,\gamma', E_j \rangle$, satisfying the initial condition $\alpha_j(0) = 0$ and $\alpha_j'(0) = w_j$, where w_j are the coefficients of $w = J'(0)$ with respect to the basis $E_1(0), \ldots, E_n(0)$, i.e., $w = \sum_j w_j E_j(0)$. It is clear that $\|v\|^2 = \sum_j \alpha_j^2(1)$.

Now let $\tilde{E}_1 = \tilde{\gamma}$, $\tilde{E}_2, \ldots, \tilde{E}_n$ denote the parallel vector fields along $\tilde{\gamma}$ such that $\tilde{E}_j(0) = I(E_j(0))$. Notice that $I_{\gamma(t)}(E_j(t)) = \tilde{E}_j(t)$ and $I_{\gamma(t)}(\gamma'(t)) = \tilde{\gamma}'(t)$. Hence

$$\langle \tilde{R}(\tilde{E}_k, \tilde{\gamma}')\,\tilde{\gamma}', \tilde{E}_k \rangle = \langle R(E_k, \gamma')\,\gamma', E_k \rangle$$

by our assumption. Using the Jacobi equation as above we therefore get

$$\tilde{J} = \sum_{j=1}^{n} \alpha_j \tilde{E}_j, \tag{1.3.4}$$

i.e., the coefficients of J and $\tilde{J}$ are the same with respect to the two bases. Here we have used that $\tilde{J}$ satisfies the initial condition $\tilde{J}(0) = 0$ and $\tilde{J}'(0) = \sum_j w_j \tilde{E}_j(0)$. Hence $\|d\varphi_y(v)\|^2 = \sum_j \alpha_j^2(1) = \|v\|^2$. Using (1.3.4), we also have that $d\varphi_y(v) = \tilde{J}(1) = I_\gamma(J(1)) = I_\gamma(v)$. This finishes the proof. $\square$

Next we prove the main result of this section. Consider a Riemannian manifold M with Levi-Cività connection ∇. The characterization of locally symmetric spaces in Theorem 1.3.5 is clearly equivalent to the sectional curvature of M being invariant under parallel transport of tangent 2-planes. The theorem makes clear the degree of generalization that we get by passing from space forms to symmetric spaces.

1.3.5 Theorem *The Riemannian manifold M is locally symmetric if and only if $\nabla R = 0$.*

Proof. Let $x \in M$ and consider the geodesic symmetry s_x. If M is locally symmetric, this is a local isometry at x. Since $(ds_x)_x = -\mathrm{id}$, the equation

$$d(s_x)_x \nabla_u R(v,w)z = \nabla_{d(s_x)_x u} R(d(s_x)_x v, d(s_x)_x w) d(s_x)_x z$$

for u, v, w, $z \in T_xM$ yields that $\nabla R = 0$.

Conversely, assume that $\nabla R = 0$ and let $x \in M$. Take $I = -\mathrm{id}$ and define φ and I_γ as in eqns. (1.3.1) and (1.3.2). Then φ is the geodesic symmetry s_x. Since R is a tensor of degree 4, R_x is invariant under I. Since $\nabla R = 0$, R is preserved by parallel transport, and hence by I_γ. It follows from Theorem 1.3.3 that s_x is a local isometry. Since $x \in M$ is arbitrary, M is locally symmetric. □

1.3.6 Remark The argument in the proof of the first half of Theorem 1.3.5 shows that any "canonical" tensor of even (resp. odd) total degree in a locally symmetric space must be parallel (resp. must vanish).

1.4 Killing fields

Let us recall some facts about Killing fields. A *Killing field* X on a Riemannian manifold M is the infinitesimal generator of a (local) one-parameter group of (local) isometries of M, and it can be characterized by the equation $L_X g = 0$, or, equivalently, that $(\nabla X)_x$ is a skew-symmetric endomorphism of T_xM for every $x \in M$.

1.4.1 Lemma *A Killing vector field X on a Riemannian manifold M is completely determined by the values of X and ∇X at a given point $x \in M$.*

Proof. Fix a point $x \in M$, and denote by $\mathfrak{g}$ the vector space of Killing vector fields on M. The assertion is then equivalent to the linear map $X \in \mathfrak{g} \mapsto ((\nabla X)_x, X_x) \in \mathfrak{so}(T_xM) \oplus T_xM$ being injective. So suppose $X_x = (\nabla X)_x = 0$. For any geodesic γ originating at x, the restriction $J := X \circ \gamma$ is clearly a Jacobi field along γ, and the assumption on X implies that $J(0) = J'(0) = 0$, hence $J \equiv 0$. The manifold M is not necessarily complete, but any point of it can be joined to x by broken geodesic, so that the argument above suffices to conclude that $X \equiv 0$. □

It follows from the identity $L_{[X,Y]} = [L_X, L_Y]$ and Lemma 1.4.1 that the space of Killing fields on a n-dimensional Riemannian manifold forms a finite-dimensional Lie algebra of dimension at most $\frac{1}{2}n(n+1)$. In the case of a complete manifold, it is easily seen that Killing fields are complete (since they have constant length along their integral curves).

Suppose now that M is a locally symmetric space. Fix a base-point x in M. For every one-parameter group of transvections $\{p_t\}$ originating at x, there is a corresponding Killing vector field Y whose value at $y \in M$ is $Y(y) = \frac{d}{dt}|_{t=0} p_t y$; such a Y is called an *infinitesimal transvection* at x.

1.4.2 Proposition *A Killing vector field Y is an infinitesimal transvection at x if and only if $(\nabla Y)_x = 0$. It follows that the bracket of two infinitesimal transvections vanishes at x.*

Proof. Let $\{p_t\}$ be the transvection one-parameter group at x that Y generates and take any curve $\eta(s)$ passing through x at $s = 0$. For the first assertion, it suffices to prove that $\frac{\nabla (Y\circ\eta)}{ds}|_{s=0} = 0$. Since the Levi-Cività connection is torsionless,

$$\frac{\nabla}{ds}\frac{d}{dt}p_t\eta(s) = \frac{\nabla}{dt}\frac{d}{ds}p_t\eta(s) = \frac{\nabla}{dt}(dp_t)_{\eta(s)}\eta'(s). \tag{1.4.3}$$

By Proposition 1.2.1(a), the vector field $(dp_t)_x\eta'(0)$ is parallel along $\gamma(t) = p_t(x)$. Since $Y(\eta(s)) = \frac{d}{dt}|_{t=0}p_t\eta(s)$, evaluating eqn. (1.4.3) at $s = t = 0$ yields one direction of the claim.

Conversely, assume $(\nabla Y)_x = 0$, take γ to be the geodesic with $\gamma(0) = x$, $\gamma'(0) = Y_x$, and consider the infinitesimal transvection Z at x along γ. Then $Y = Z$, due to Lemma 1.4.1.

The last assertion follows from $[Y_1, Y_2] = \nabla_{Y_1}Y_2 - \nabla_{Y_2}Y_1$. □

1.5 Linearization

Recall that the Myers-Steenrod Theorem states that isometry group G of a Riemannian manifold M, equipped with the compact-open topology, has a natural structure of Lie group such that the action of G on M is smooth and represents its Lie algebra $\mathfrak{g}$ as the Lie algebra of Killing vector fields on M. It is also worth recalling that convergence of a sequence of isometries in G in the compact-open topology is equivalent to pointwise convergence in M. Finally, the isotropy group G_x at a point $x \in M$ is compact.

1.5.1 Proposition *Let M be a globally symmetric space with Lie group of isometries G. Fix $x \in M$, write K for the isotropy group at x, and P for the set of transvections at x. Also, denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G, K, and by $\mathfrak{p}$ the set of infinitesimal transvections at x. Then:*

(i) $M = G/K$ *and* $G = KP = PK$.

(ii) $s = \mathrm{Ad}_{s_x}$ *defines an involutive automorphism of* $\mathfrak{g}$, *and* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ *is the* ± 1*-eigenspace decomposition.*

(iii) *The projection* $\pi : G \to M$, *given by* $\pi(g) = gx$, *has differential* $\pi_* : \mathfrak{p} \cong T_xM$.

(iv) *If* $k \in K$ *and* $Y \in \mathfrak{p}$, *then* $\pi_*(\mathrm{Ad}_kY) = dk_x(\pi_*Y)$. *In particular, the inner product on* T_xM *lifts to an* Ad_K*-invariant inner product on* $\mathfrak{p}$.

Proof. (i) G acts transitively on M by Proposition 1.2.2, so M is the homogeneous space G/K. Also, if $g \in G$ and $y = g^{-1}x$, by Hopf-Rinow there is $p \in P$ such that $py = x$. Now $gp^{-1}x = x$, so $gp^{-1} \in K$ and $g = (gp^{-1})p \in KP$, proving $G = KP$. Finally, $PK = KP$ because K is a subgroup of G and $P^{-1} = P$.

(ii) Every Killing field Z on M decomposes as a sum of Killing fields $X+Y$, where $Y \in \mathfrak{p}$ is the infinitesimal transvection such that $Y_x = Z_x$ and $X = Z-Y \in \mathfrak{k}$. Of course $s^2 = 1$ since $s_x^2 = \mathrm{id}_M$. If $k \in K$, then $s_x k s_x^{-1}$ is an isometry of M that fixes x and has the same differential at x as k; hence $s_x k s_x^{-1} = k$. It follows that $s_x = +1$ on $\mathfrak{k}$. Also, s_x maps the geodesic $\gamma_v(t) = \exp_x(tv)$, $v \in T_xM$, to its opposite γ_{-v}, and hence conjugates the transvections along γ_v to the transvections along γ_{-v}. It follows that $s_x = -1$ on $\mathfrak{p}$.

(iii) π is onto and $\ker \pi_* = \mathfrak{k}$.

(iv) We compute

$$\begin{aligned}
\pi_*(\mathrm{Ad}_k Y) &= \pi_*\left(\frac{d}{dt}\Big|_{t=0} k \exp tY k^{-1}\right) \\
&= \frac{d}{dt}\Big|_{t=0} \pi(k \exp tY k^{-1}) \\
&= \frac{d}{dt}\Big|_{t=0} k \exp tY \cdot x \\
&= dk_x(Y_x) \\
&= dk_x(\pi_* Y),
\end{aligned}$$

as desired. □

To every symmetric space M, we have associated a triple $(\mathfrak{g}, s, B)$ where $\mathfrak{g}$ is the Lie algebra of the isometry group of M, $s = \mathrm{Ad}_x$ is an involutive automorphism of $\mathfrak{g}$, where x is a chosen basepoint, the $+1$-eigenspace of s is a Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ that acts faithfully on the the -1-eigenspace $\mathfrak{p}$ of s (since the isotropy representation of K on T_xM is faithful), and B is an Ad_K-invariant, and hence $\mathrm{ad}_{\mathfrak{k}}$-invariant, (positive definite) inner product on $\mathfrak{p}$. The triple $(\mathfrak{g}, s, B)$ is called the *orthogonal involutive Lie algebra* of M at x. Since M is assumed connected, the choice of point x is unimportant.

An abstract *orthogonal involutive Lie algebra* is a triple $(\mathfrak{g}, s, B)$, where $\mathfrak{g}$ is a real finite-dimensional Lie algebra, s is an involutive automorphism of $\mathfrak{g}$, the fixed point set $\mathfrak{k}$ of s does not contain nontrivial ideals of $\mathfrak{g}$ and B is an $\mathrm{ad}_{\mathfrak{k}}$-invariant inner product on the -1-eigenspace $\mathfrak{p}$ of s.

Given an abstract OIL-algebra $(\mathfrak{g}, s, B)$, we construct a simply-connected symmetric space as follows. Let $\tilde{G}$ be the simply-connected Lie group with Lie algebra $\mathfrak{g}$, and let $\tilde{K} \subset \tilde{G}$ be the connected subgroup with Lie algebra $\mathfrak{k}$. There is an involution σ of $\tilde{G}$ such that $d\sigma = s$, and $\tilde{K}$ is the identity component of the fixed point subgroup $\tilde{G}^\sigma$; thus $\tilde{K}$ is closed in $\tilde{G}$. Now $M = \tilde{G}/\tilde{K}$ is a simply-connected (since $\tilde{G}$ is simply-connected and K is connected) homogeneous space, but in general $\tilde{G}$ does not act effectively on M; let $\tilde{Z} = \{g \in \tilde{G} : g : M \to M \text{ is the identity}\} \subset \tilde{K}$ be the kernel of the action. $\tilde{Z}$ is discrete because $\mathfrak{k}$ does not contain nontrivial ideals of $\mathfrak{g}$. Now $M = G'/K'$ where $G' = \tilde{G}/\tilde{Z}$, $K' = \tilde{K}/\tilde{Z}$ have resp. Lie algebras $\mathfrak{g}$, $\mathfrak{k}$. The projection $\pi : G' \to M$ yields $\pi_* : \mathfrak{p} \cong T_xM$, where $x = 1\,K'$, and B defines a K'-invariant inner product on

T_xM that extends to a G'-invariant Riemannian metric on M. Finally, σ induces $\psi : M \to M$ by the rule $\psi(gK') = \sigma(g)K'$, or $\psi(gx) = \sigma(g)x = \pi\sigma(g)$ for $g \in G'$. Note that ψ is well-defined and fixes x. Also, $d\psi_x(Y_x) = \pi_* s(Y) = -Y_x$ for $Y \in \mathfrak{p}$, so $d\psi_x = -\mathrm{id}$. Since $\psi(gg_1K') = \sigma(g)\sigma(g_1)K' = \sigma(g)\psi(g_1K')$, we have $\psi \circ g = \sigma(g) \circ \psi$. By the chain rule $d\psi_{gx} = (d(\sigma(g)))_x \circ (d\psi)_x \circ (dg^{-1})_{gx}$ for every $g \in G'$. It follows that ψ is an isometry of M. Hence M is symmetric.

In particular, suppose that we apply the construction described in the previous paragraph to the orthogonal involutive Lie algebra $(\mathfrak{g}, s, B)$, which is associated to a given symmetric space $M = G/K$ as in Proposition 1.5.1, to obtain $\tilde{M} = \tilde{G}/\tilde{K}$. Since $\tilde{G}$ is simply-connected and has the same Lie algebra as G, there is a covering homomorphism $p : \tilde{G} \to G$. Moreover, $\tilde{K}$ and K also have the same Lie algebra, so $p(\tilde{K}) = K$ and p induces $\bar{p} : \tilde{M} \to M$. Owing to the identity $\pi \circ p = \bar{p} \circ \pi$ and the identifications $T_{\tilde{x}}\tilde{M} \cong \mathfrak{p} \cong T_xM$, $d\bar{p}_{\tilde{x}}$ is the identity, thus an isometry. The homomorphism property of p implies that $\bar{p} \circ g = p(g) \circ \bar{p}$ as maps on $\tilde{M}$, where $g \in \tilde{G}$; since g (resp. $p(g)$) is an isometry of $\tilde{M}$ (resp. M), this shows that $\bar{p}$ is a local isometry. Finally, the completeness of $\tilde{M}$ implies that $\bar{p} : \tilde{M} \to M$ is a Riemannian covering. In particular, if M is taken simply-connected, then it is isometric to $\tilde{M}$.

1.5.2 Example It is known that the group of isometries of the unit sphere is $\mathbf{O}(n+1)$. The isotropy group at, say, $x = (1, 0, \ldots, 0)^t$ is $\mathbf{O}(n)$. Now the associated orthogonal involutive Lie algebra $(\mathfrak{g}, s, B)$ is given by $\mathfrak{g} = \mathfrak{so}(n+1)$, s is conjugation by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix},$$

where I_n is an identity block of order n, its eigenspaces are given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} : A \in \mathfrak{so}(n) \right\}, \qquad \mathfrak{p} = \left\{ Y = \begin{pmatrix} 0 & -v^t \\ v & 0 \end{pmatrix} : v \in \mathbf{R}^n \right\},$$

and $B(Y, Y) = (\mathrm{const})\,\mathrm{tr}(Y^tY) = (\mathrm{const})||v||^2$.

1.5.3 Example Let H be a simply-connected compact connected semisimple Lie group, and denote its Lie algebra by $\mathfrak{h}$. We can define an abstract orthogonal involutive Lie algebra $(\mathfrak{g}, s, B)$ by setting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$, $s(X, Y) = (Y, X)$ for X, $Y \in \mathfrak{h}$; then $\mathfrak{k}$ is the diagonal of $\mathfrak{g}$ and $\mathfrak{p} = \{(X, -X) : X \in \mathfrak{h}\}$; put $B((X, -X), (Y, -Y)) = \lambda\beta(X, Y)$ for X, $Y \in \mathfrak{h}$, where $\lambda < 0$ and β denotes the Killing form of $\mathfrak{h}$.

The simply-connected symmetric space M associated to $(\mathfrak{g}, s, B)$ is $M = H \times H/\Delta_H$, where $\Delta_H = \{(h, h) : h \in H\}$. Note that $H \times H$ acts transitively on H by the rule $(h_1, h_2) \cdot x = h_1 x h_2^{-1}$; the isotropy at $1 \in H$ is Δ_H, hence there is a diffeomorphism $H \times H/\Delta_H \cong H$, $(h_1, h_2)\Delta_H \mapsto h_1h_2^{-1}$. Using this identification $M = H$, the projection $\pi : H \times H \to H$ is $\pi(h_1, h_2) = h_1h_2^{-1}$, the geodesic symmetry $\psi : H \to H$ at 1 is $\psi(x) = \psi((x, 1)\Delta_H) = \sigma(x, 1)\Delta_H = (1, x)\Delta_H = x^{-1}$. Finally $\pi_*(X, Y) = X - Y$ for X, $Y \in \mathfrak{h}$, so $\pi_*(X, -X) = 2X$

and the metric on H is $H \times H$-invariant (i.e. bi-invariant) with value at 1 given by a negative multiple of the Killing form of $\mathfrak{h}$.

1.5.4 Example It is not difficult to show that the full group of isometries of Euclidean space is the semidirect product $G = \mathbf{O}(n) \ltimes \mathbf{R}^n$ (cf. Problem 1.7.11). The isotropy group at the origin is $\mathbf{O}(n)$. Now its associated orthogonal involutive Lie algebra $(\mathfrak{g}, s, B)$ is $\mathfrak{g} = \mathfrak{so}(n) + \mathbf{R}^n$ (semi-direct sum), $s : \mathfrak{g} \to \mathfrak{g}$ is $+1$ on $\mathfrak{so}(n)$ and -1 on $\mathbf{R}^n$, and B is the inner product on $\mathbf{R}^n$. Note that $[\mathfrak{p}, \mathfrak{p}] = 0$ in this example.

More generally, for any subalgebra $\mathfrak{k} \subset \mathfrak{so}(n)$, the semi-direct sum $\mathfrak{g} = \mathfrak{k} + \mathbf{R}^n$ has a similar structure of orthogonal involutive Lie algebra, including the case $\mathfrak{k} = 0$. The associated simply-connected symmetric space is again Euclidean space.

1.6 Complement on simply-connected symmetric spaces

There is a global version of Cartan's Theorem 1.3.3 for complete simply-connected Riemannian manifolds (see section 4), aka the Cartan-Ambrose Theorem, which, in the hypotheses, replaces the geodesics starting from a point by broken geodesics starting from the point, and constructs a global isometry. It follows from the Cartan-Ambrose Theorem that:

- A complete simply-connected locally symmetric space is globally symmetric.
- A complete simply-connected Riemannian manifold is globally symmetric if and only if its curvature tensor is parallel.
- The universal Riemannian covering of a complete locally symmetric space is a globally symmetric space.

Let M be a locally symmetric space. One can also directly associate to M an orthogonal involutive Lie algebra. Namely, fix $x \in M$. The *linear isotropy group* of M at x is defined to be the group K of all linear isometries of the tangent space T_xM that preserve the curvature tensor. Then K has the structure of a Lie group because it is a closed subgroup of the orthogonal group of T_xM. By Theorem 1.3.3, each element of K extends to an isometry of a normal neighborhood U of x. Now K is a group of isometries of U, called the *local isotropy group* of M at x. Let $\mathfrak{k}$ be the Lie algebra of K. Then the action of K on U represents $\mathfrak{k}$ faithfully as a finite-dimensional Lie algebra of vector fields on U. These vector fields vanish at x because $K(x) = \{x\}$. Denote by $\mathfrak{p}$ the set of infinitesimal transvections at x. It follows from Lemma 1.4.1 and Proposition 1.4.2 that $\mathfrak{p}$ is a vector space and the map $Y \in \mathfrak{p} \mapsto Y_x \in T_xM$ is a linear isomorphism.

1.6.1 Proposition *We have*

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

In particular, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ (direct sum of vector spaces) is a Lie algebra of vector fields defined on a normal neighborhood of x. The geodesic symmetry s_x induces an involutive automorphism s of $\mathfrak{g}$ such that $\mathfrak{k}$ and $\mathfrak{p}$ are respectively the ± 1-eigenspaces.

Proof. The first inclusion holds because $\mathfrak{k}$ is a Lie algebra of vector fields. For the second one, let $k \in K$ and $Y \in \mathfrak{p}$. Then Y generates a one-parameter of transvections at x, which we denote by e^{tY}, and $\mathrm{Ad}_k Y$ is the Killing field $\frac{d}{dt}|_{t=0} k e^{tY} k^{-1}$. Using that $d(e^{tY})_x$ is parallel transport along $\gamma(t) = e^{tY} \cdot x$, it is easy to see that $dk_{e^{tY}\cdot x} d(e^{tY})_x dk_x^{-1}$ is parallel transport along $k \cdot \gamma$. Therefore $ke^{tY}k^{-1}$ is the one-parameter group of transvections along $k \cdot \gamma$ and thus $\mathrm{Ad}_k Y \in \mathfrak{p}$, implying the second inclusion. Finally, if Y_1, $Y_2 \in \mathfrak{p}$, then $[Y_1, Y_2]$ is a Killing field generating a one-parameter group $\{g_t\}$ of local isometries fixing x, by Proposition 1.4.2, and obviously preserving R_x. Hence $g_t \in K$ and $[Y_1, Y_2] = \frac{dg_t}{dt} \in \mathfrak{k}$.

Finally, for $Z \in \mathfrak{g}$, set $sZ = \frac{d}{dt}|_{t=0} s_x e^{tZ} s_x^{-1}$. If $X \in \mathfrak{k}$, then $s_x e^{tX} s_x^{-1}$ is a local isometry fixing x with differential at x given by $d(e^{tX})_x$. Hence $s_x e^{tX} s_x^{-1} = e^{tX}$ and $sX = X$. If $Y \in \mathfrak{p}$, then $s_x e^{tY} s_x^{-1}$ is the local transvection at x along $s_x e^{tY} \cdot x = e^{-tY} \cdot x$. Therefore $sY = -Y$. This finishes the proof. □

To a point x in a locally symmetric space M, there is now associated a triple $(\mathfrak{g}, s, B)$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and s are as in Proposition 1.6.1, and B is the inner product of $T_x M$ lifted to $\mathfrak{p}$ under the identification $\mathfrak{p} \cong T_x M$, $Y \mapsto Y_x$. If M is simply-connected, the elements of the local isotropy group K at x and the set P of local transvections at x extend to global isometries of M by Theorem 4.2.1, and hence this construction of orthogonal involutive Lie algebra coincides with that in section 1.5.

To have a more complete picture, for a Riemannian covering $M \to M'$ with M globally symmetric, we would like to know when M' is also globally symmetric.

We first need:

1.6.2 Proposition *The fundamental group of a symmetric space is Abelian.*

Proof. Let M be a symmetric space and fix $x \in M$. By applying a standard curve-shortening process, one shows that any nontrivial element in $\pi_1(M, x)$ can be represented by a closed geodesic through x, say γ. The geodesic symmetry s_x reverses geodesics through x, so $s_x(\gamma(t)) = \gamma(-t)$. Now the homomorphism induced by s_x on the fundamental group level is group inversion. The result follows from noticing that group inversion is a homomorphism only if the underlying group is Abelian. □

The preceding result poses a topological obstruction for a smooth manifold to admit the structure of a symmetric space.

1.6.3 Corollary *A surface of genus $g \geq 2$ does not admit a Riemannian metric with respect to which it is a symmetric space.*

1.6.4 Theorem *Let M be a symmetric space, let G be the transvection group of M, that is, the (connected) subgroup of the isometry group of* $\mathrm{Isom}(M)$ *generated by all transvections, and let Δ be the centralizer of G in* $\mathrm{Isom}(M)$*. If $M \to M'$ is a Riemannian covering with M' symmetric, then $M' = M/\Gamma$ for some discrete subgroup Γ of Δ. Conversely, if Γ is a discrete subgroup of Δ, then $M \to M/\Gamma$ is a Riemannian covering and M/Γ is symmetric.*

Proof. (First half) Assume $p : M \to M'$ is a Riemannian covering with M' symmetric. By Proposition 1.6.2, the covering is Galois, so $M' = M/\Gamma$ for a discrete subgroup Γ of $\mathrm{Isom}(M)$. Let $s_{x'}$ be the symmetry at $x' \in M'$ and take $x \in M$ projecting to x'. Again in view of Proposition 1.6.2, $(s_{x'})_\# p_\# (\pi_1(M)) = p_\# \pi_1(M)$, so there is a unique lift of $s_{x'}p : M \to M'$ to a smooth map $f : M \to M$ taking x to itself. Since $df_x = -\mathrm{id}$, f must be the geodesic symmetry at x, that is, $s_{x'}p = ps_x$. It follows that s_x maps fibers of p to fibers of p. Now for $\gamma \in \Gamma$, $s_x \gamma s_x^{-1}$ preserves each fiber and hence is an element of Γ. We have shown that every geodesic symmetry of M normalizes Γ. In particular, G normalizes Γ. But Γ is discrete and G is connected, so G centralizes Γ.

For the second half, see [42, Theorem 8.3.11]. □

1.7 Problems

1.7.1 Problem Let M be a complete connected Riemannian manifold with vanishing sectional curvature. Deduce from Cartan-Ambrose theorem that, for every $x \in M$, $\exp_x : T_xM \to M$ is a smooth covering.

1.7.2 Problem Let M be a symmetric space and $x \in M$. Prove that the geodesic symmetry s_x normalizes the group generated by transvections at x.

1.7.3 Problem Prove that every geodesic loop in a symmetric space is a closed geodesic. (Hint: Consider the one-parameter group of transvections along the geodesic.)

1.7.4 Problem Let G be a compact connected Lie group equipped with a bi-invariant metric. Show that the left translations in G are transvections if and only if G is Abelian.

1.7.5 Problem Let σ be an involution of a Lie group G, and let K be a subgroup of G which is open in the fixed point set G^σ, $(G^\sigma)^0 \subset K \subset G^\sigma$, and such that $\mathrm{Ad}_G(K)$ is a compact subgroup of $\mathrm{Aut}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G. Check that $M = G/K$ carries a structure of symmetric space.

1.7.6 Problem Let M be a symmetric space as in Problem 1.7.5, and denote by x_0 the basepoint of M. Show that gx_0 lies in cut-locus of x_0 for every $g \in G^\sigma \setminus K$.

1.7.7 Problem Let M be a compact symmetric space. For each $x \in M$, denote the geodesic symmetry at x by s_x. Fix two points x, $y \in M$ and prove that the following assertions are equivalent:

a. $s_x(y) = y$.

b. There is a closed geodesic γ in M, and x, y are antipodal points along γ.

c. There exists a transvection p of M such that $p(x) = y$ and $p^2(x) = x$.

Further, show that these conditions imply that y belongs to the cut-locus of x.

1.7.8 Problem With the notation of Problem 1.7.7, prove that the following assertions are equivalent:

a. y is an isolated fixed point of s_x.

b. $s_y = s_x$.

c. There exists a transvection p of M such that $p(x) = y$ and $p^2 = \mathrm{id}$.

If these conditions are satisfied, one says that y is a *pole* of x [14].

1.7.9 Problem View a compact connected Lie group equipped with a bi-invariant Riemannian metric as a symmetric space and show that the poles of the identity element 1 (cf. Problem 1.7.8) are the central elements that are square roots of 1.

1.7.10 Problem Let (M, g) be a connected Riemannian manifold and consider the underlying metric space structure (M, d). Prove that any isometry f of (M, g) is *distance-preserving*, that is, it satisfies the condition that $d(f(x), f(y)) = d(x, y)$ for every x, $y \in M$.

1.7.11 Problem Describe the isometry group G of $\mathbf{R}^n$:

a. Show that G is generated by orthogonal transformations and translations.

b. Show that G is isomorphic to the semidirect product $\mathbf{O}(n) \ltimes \mathbf{R}^n$, where

$$(B, w) \cdot (A, v) = (BA, Bv + w)$$

for A, $B \in \mathbf{O}(n)$ and v, $w \in \mathbf{R}^n$.

(Hint: Use the result of the previous exercise.)

1.7.12 Problem Prove that every isometry of the unit sphere S^n of Euclidean space $\mathbf{R}^{n+1}$ is the restriction of a linear orthogonal transformation of $\mathbf{R}^{n+1}$. Deduce that the isometry group of S^n is isomorphic to $\mathbf{O}(n+1)$. What is the isometry group of real projective space $\mathbf{R}P^n$?

1.7.13 Problem Prove that every isometry of the hyperboloid model of $\mathbf{R}H^n$ is the restriction of a linear Lorentzian orthochronous (time-preserving) transformation of $\mathbf{R}^{1,n}$. Deduce that the isometry group of $\mathbf{R}H^n$ is isomorphic to $\mathbf{O}_0(1,n)$.

1.7.14 Problem Let $(\mathfrak{g}, s, B)$ and $(\mathfrak{g}', s', B')$ be two orthogonal involutive Lie algebras such that $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}'$, $s = s'|_{\mathfrak{g}}$, $\mathfrak{p} = \mathfrak{p}'$ and $B = B'$. Prove that the corresponding simply-connected symmetric spaces M and M' are isometric. (There is essentially only one nontrivial concrete example; can you guess it?)

2 Lecture 2: Classification

The enumeration of all the symmetric Riemann spaces is not a simple problem. Cartan first observed that if a locally symmetric space is decomposed into a product, then each factor is locally symmetric. The problem is thus reduced to the irreducible case. Cartan then proposed two different lines of attack. The first method is the determination of the subgroups of the orthogonal group that can be holonomy groups of an irreducible symmetric space. For a point p in a Riemannian manifold M, the *holonomy group of M at p* is the group of all linear isometries of T_pM generated by parallel translation along loops at p. In case of $\nabla R \equiv 0$, such a subgroup must leave the curvature tensor invariant, and this imposes strong restrictions. Cartan proved a number of important results, but did not fully carry out this method since a simpler way became available.[2] The second method brings the classification of locally symmetric spaces into the realm of group theory. In particular, he notes that the search for irreducible locally symmetric spaces amounts to that of real forms of complex simple Lie algebras, a problem he himself had already solved in 1914. In the sequel we expose the basic ideas behind this method.

2.1 Decomposition theorem

The first result shows that if a symmetric space is decomposed into a product of Riemannian manifolds, then each factor is also a symmetric space. We first prove a number of lemmata.

Let M be a symmetric space with associated orthogonal involutive algebra $(\mathfrak{g}, s, B)$. Denote by G the identity component of the isometry group of M and write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ under s.

2.1.1 Lemma *If G is semisimple, then it is generated by the transvections of M at the basepoint.*

[2]Much later holonomy groups would be used to study Riemannian geometry in a more general setting. In 1952 Georges de Rham proved the de Rham decomposition theorem, a principle for splitting a Riemannian manifold into a Cartesian product of Riemannian manifolds by splitting the tangent bundle into irreducible spaces under the action of the local holonomy groups. Later, in 1953, Marcel Berger classified the possible irreducible holonomies.

Proof. It suffices to prove that $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$. Indeed, set $\mathfrak{h} = [\mathfrak{p}, \mathfrak{p}] + \mathfrak{p}$; we prove that this is an ideal of $\mathfrak{g}$. In fact, $[\mathfrak{k}, \mathfrak{h}] \subset \mathfrak{h}$ because $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{k}, [\mathfrak{p}, \mathfrak{p}]] \subset [[\mathfrak{k}, \mathfrak{p}], \mathfrak{p}] \subset [\mathfrak{p}, \mathfrak{p}]$ by the Jacobi identity. Also, $[\mathfrak{p}, \mathfrak{h}] \subset \mathfrak{h}$ because $[\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]] \subset [\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$.

Since $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is an ideal, there exists an ideal $\mathfrak{u}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{u}$ (direct sum of ideals). Note that s is automorphism of $\mathfrak{g}$ and preserves $\mathfrak{p}$, so it also preserves $[\mathfrak{p}, \mathfrak{p}]$. Now $s(\mathfrak{h}) = \mathfrak{h}$ and hence $s(\mathfrak{u}) = \mathfrak{u}$. It follows that $\mathfrak{u} = \mathfrak{u} \cap \mathfrak{k} + \mathfrak{u} \cap \mathfrak{p}$. Since $\mathfrak{u} \cap \mathfrak{p} \subset \mathfrak{u} \cap \mathfrak{h} = 0$, we get $\mathfrak{u} \subset \mathfrak{k}$. However, $\mathfrak{k}$ does not contains nontrivial ideals of $\mathfrak{g}$, thus $\mathfrak{u} = 0$ and $\mathfrak{h} = \mathfrak{g}$. □

It follows from Lemma 2.1.1 that a symmetric space M with semisimple isometry group (equivalently, without a flat factor, according to Corollary 2.1.5 below) has a canonical presentation as a homogeneous space, namely, $M = G/K$ where G is the transvection group (a connected Lie group) and K is the isotropy group at a point.

2.1.2 Lemma *Denote by β the Killing form of $\mathfrak{g}$. Then:*

a. $\beta(\mathfrak{k}, \mathfrak{p}) = 0$

b. $\beta|_{\mathfrak{k}}$ *is negative-definite.*

c. if $\mathfrak{a}$ and $\mathfrak{b}$ are β-orthogonal subspaces of $\mathfrak{p}$ and $\mathfrak{b}$ is $\mathrm{ad}_{\mathfrak{k}}$-invariant then $[\mathfrak{a}, \mathfrak{b}] = 0$.

Proof. (a) Owing to the s-invariance of β, $\beta(X, Y) = \beta(sX, sY) = \beta(X, -Y) = -\beta(X, Y)$ for $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$.

(b) Let $X \in \mathfrak{k}$. Since ad_X leaves B invariant and B is positive-definite, $\mathrm{ad}_X|\mathfrak{p} \to \mathfrak{p}$ is semisimple with purely imaginary eigenvalues, therefore $\mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}_X^2) \leq 0$. Using Theorem 5.5.1, the compactness of $\mathfrak{k}$ gives that $\mathrm{tr}_{\mathfrak{k}}(\mathrm{ad}_X^2) \leq 0$. Now $\beta(X, X) = \mathrm{tr}_{\mathfrak{k}}(\mathrm{ad}_X^2) + \mathrm{tr}_{\mathfrak{p}}(\mathrm{ad}_X^2) \leq 0$, and equality holds if and only if $\mathrm{ad}_X = 0$, namely, X is central in $\mathfrak{g}$. In this case, the multiples of X form an ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$, hence $X = 0$.

(c) Let $X \in \mathfrak{a}$, $Y \in \mathfrak{b}$ and $Z = [X, Y]$. Then $Z \in \mathfrak{k}$ and $\beta(Z, Z) = \beta([X, Y], Z) = \beta(X, [Y, Z]) = 0$, where the last equality follows from $[Y, Z] \in \mathfrak{b}$ and the assumptions. Hence $Z = 0$ by (b). □

An OIL-algebra $(\mathfrak{g}, s, B)$ is called *Euclidean* if $[\mathfrak{p}, \mathfrak{p}] = 0$, and it is called *irreducible* if it is not Euclidean and the adjoint representation of $\mathfrak{k}$ on $\mathfrak{p}$ is irreducible.

2.1.3 Proposition (Decomposition) *Let $(\mathfrak{g}, s, B)$ be an OIL-algebra. Then there is a decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$$

into a direct sum of ideals such that:

a. each $\mathfrak{g}_i$ is s-invariant;

b. $(\mathfrak{g}_0, s|_{\mathfrak{g}_0}, B|_{\mathfrak{p}_0})$ *is Euclidean;*

c. $(\mathfrak{g}_i, s|_{\mathfrak{g}_i}, B|_{\mathfrak{p}_i})$ *is irreducible for* $i = 1, \ldots, r$.

Proof. Since B is nondegenerate, we can define a linear map $A : \mathfrak{p} \to \mathfrak{p}$ by putting $B(AX, Y) = \beta(X, Y)$ for X, $Y \in \mathfrak{p}$. Due to the symmetry of B and β, we have $B(AX, Y) = B(X, AY)$, thus A has real eigenvalues $\lambda_0 = 0$, $\lambda_1, \ldots, \lambda_t$ with corresponding B-orthogonal eigenspace decomposition $\mathfrak{p} = \mathfrak{q}_0 + \mathfrak{q}_1 + \cdots \mathfrak{q}_t$. It immediately follows that this decomposition is β-orthogonal and

$$\beta|_{\mathfrak{q}_i} = \lambda_i B|_{\mathfrak{q}_i} \quad \text{for } i = 1, \ldots, t, \qquad \text{and} \qquad \beta(\mathfrak{q}_0, \mathfrak{p}) = 0.$$

Since B and β are $\mathrm{ad}_{\mathfrak{k}}$-invariant, A commutes elementwise with $\mathrm{ad}_{\mathfrak{k}}$, and thus $\mathrm{ad}_{\mathfrak{k}}$ preserves each $\mathfrak{q}_i$. Now each $\mathfrak{q}_i$ with $i > 0$ can be decomposed into $\mathrm{ad}_{\mathfrak{k}}$-irreducible subspaces yielding a B-orthogonal decomposition $\sum_{i=1}^t \mathfrak{q}_i = \sum_{i=1}^r \mathfrak{p}_i$, where $[\mathfrak{k}, \mathfrak{p}_i] \subset \mathfrak{p}_i$. Put $\mathfrak{p}_0 = \mathfrak{q}_0$. Clearly $\beta(\mathfrak{p}_i, \mathfrak{p}_j) = 0$ for i, $j > 0$, $i \neq j$, so Lemma 2.1.2(c) yields that

$$[\mathfrak{p}_i, \mathfrak{p}_j] = 0 \quad \text{for } i,\ j > 0,\ i \neq j \qquad \text{and} \qquad [\mathfrak{p}_0, \mathfrak{p}] = 0. \tag{2.1.4}$$

Set now $\mathfrak{g}_i = [\mathfrak{p}_i, \mathfrak{p}_i] + \mathfrak{p}_i$ for $i > 0$. It follows from the $\mathrm{ad}_{\mathfrak{k}}$-invariance of $\mathfrak{p}_i$, (2.1.4) and the Jacobi identity that the $\mathfrak{g}_i$ are ideals of $\mathfrak{g}$. Suppose $i > 0$. Since β is a nonzero multiple of B on $\mathfrak{p}_i$, β is nondegenerate of $\mathfrak{p}_i$ and hence on $\mathfrak{g}_i$ by Lemma 2.1.2(a) and (b). Since $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$, $\beta|_{\mathfrak{g}_i}$ is its Killing form. Hence $\mathfrak{g}_i$ is semisimple.

Let now $\tilde{\mathfrak{g}} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$, s-invariant semisimple ideal of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \tilde{\mathfrak{g}}$ where $\mathfrak{g}_0$ is the centralizer of $\tilde{\mathfrak{g}}$. Clearly $s(\mathfrak{g}_0) = \mathfrak{g}_0$, $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{p}_0$ and $[\mathfrak{p}_0, \mathfrak{p}_0] = 0$. □

2.1.5 Corollary *If* $\mathfrak{g}$ *is simple then* $(\mathfrak{g}, s, B)$ *is irreducible. If* $(\mathfrak{g}, s, B)$ *is irreducible then* $\mathfrak{k}$ *is a maximal subalgebra of* $\mathfrak{g}$ *and its own normalizer in* $\mathfrak{g}$*.* $\mathfrak{g}$ *is semisimple if and only if* $(\mathfrak{g}, s, B)$ *has no Euclidean factor.*

2.1.6 Remark With just a little bit more work one can show that the decomposition in Proposition 2.1.3 is unique, up to permutation of the simple factors.

2.1.7 Proposition *The Euclidean OIL-algebras are precisely the OIL-algebras* $(\mathfrak{g}, s, B)$ *where* $\mathfrak{g} = \mathfrak{k} + \mathbf{R}^n$ *(semi-direct sum),* $\mathfrak{k} \subset \mathfrak{so}(n)$*,* s *is* $+1$ *on* $\mathfrak{k}$ *and* -1 *on* $\mathbf{R}^n$*, and* B *is the standard inner product on* $\mathbf{R}^n$*.*

Proof. Suppose $(\mathfrak{g}, s, B)$ is Euclidean. Since $[\mathfrak{p}, \mathfrak{p}] = 0$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a semi-direct sum. Let $n = \dim \mathfrak{p}$ and isometrically identify $\mathfrak{p} = \mathbf{R}^n$ so that B is the standard inner product. Now $\mathfrak{k} \subset \mathfrak{so}(n)$, since it is effective on $\mathfrak{p}$. □

The associated simply-connected symmetric space in Proposition 2.1.7 is (flat) Euclidean space, and its quotients are flat tori.

2.1.8 Remark It follows from Theorem 1.3.3 that the holonomy group of a symmetric space M at a point x is contained in the local isotropy group at x defined in subsection 1.6. If M has no flat factor, then $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]$ (Lemma 2.1.1) and thus the identity component of holonomy group (i.e. the *restricted holonomy group*) coincides with the identity component of the isotropy group at x.

2.2 Cartan duality

Cartan duality associates to an OIL-algebra $(\mathfrak{g}, s, B)$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, another OIL-algebra $(\mathfrak{g}, s, B)^* = (\mathfrak{g}^*, s^*, B^*)$ which we define now. Consider the complexification $\mathfrak{g}^c = \mathfrak{k}^c + \mathfrak{p}^c$ and extend s $\mathbf{C}$-linearly to an automorphism s^c of $\mathfrak{g}^c$. We put $\mathfrak{g}^* = \mathfrak{k} + \mathfrak{p}^*$, where $\mathfrak{p}^* = \sqrt{-1}\mathfrak{p}$, and note that it is real subalgebra of $\mathfrak{g}^c$, invariant under s^c. Let s^* be the restriction of s^c to $\mathfrak{g}^*$ and set $B^*(\sqrt{-1}X, \sqrt{-1}Y) = B(X, Y)$ for X, $Y \in \mathfrak{p}$. Note that $X \mapsto \sqrt{-1}X$ for $X \in \mathfrak{p}$ defines an equivalence between the $\mathrm{ad}_{\mathfrak{k}}$-representations $\mathfrak{p}$ and $\mathfrak{p}^*$ which maps B to B^*.

2.2.1 Proposition *Let $(\mathfrak{g}, s, B)$ be an OIL-algebra. Then:*

a. If $(\mathfrak{g}, s, B) = \sum_{i=0}^{r}(\mathfrak{g}_i, s_i, B_i)$ is the decomposition of Proposition 2.1.3, then

$$(\mathfrak{g}, s, B)^* = \sum_{i=0}^{r}(\mathfrak{g}_i, s_i, B_i)^*$$

is the corresponding decomposition.

b. If $(\mathfrak{g}, s, B)$ is irreducible, then precisely only of $\mathfrak{g}$, $\mathfrak{g}^*$ is compact.

Proof. (a) is clear. We prove (b). We have $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where β is negative definite on $\mathfrak{k}$, $\beta(\mathfrak{k}, \mathfrak{p}) = 0$ and $\beta = \lambda B$ on $\mathfrak{p}$ where $\lambda \neq 0$. Since β, β^* are restrictions of the Killing form of $\mathfrak{g}^c$,

$$\begin{aligned}
\beta^*(\sqrt{-1}X, \sqrt{-1}Y) &= \beta^c(\sqrt{-1}X, \sqrt{-1}Y) \\
&= -\beta^c(X, Y) \\
&= -\beta(X, Y) \\
&= -\lambda B(X, Y) \\
&= -\lambda B^*(\sqrt{-1}X, \sqrt{-1}Y)
\end{aligned}$$

for X, $Y \in \mathfrak{p}$. Just one of $\pm\lambda$ is negative, so just one of $\beta|_{\mathfrak{p}}$, $\beta^*|_{\mathfrak{p}^*}$ is negative definite, so just one of $\mathfrak{g}$, $\mathfrak{g}^*$ is compact. □

2.3 The irreducible case

2.3.1 Lemma *Let $\mathfrak{g}$ be a real simple Lie algebra. Then the complexified Lie algebra $\mathfrak{g}^c$ is not simple if and only if $\mathfrak{g}$ is the realification of a complex simple Lie algebra.*

Proof. Assume that $\mathfrak{g}$ is the realification of a complex Lie algebra $\mathfrak{h}$. Then there is complex structure J on the real Lie algebra[3] $\mathfrak{g}$ such that $J[X,Y] = [JX,Y] = [X,JY]$ for X, $Y \in \mathfrak{g}$. The $\mathbf{C}$-linear extension of J to $\mathfrak{g}^c$ admits eigenvalues $\pm\sqrt{-1}$ and corresponding eigenspace decomposition $\mathfrak{g}^c = \mathfrak{a}_{\sqrt{-1}} + \mathfrak{a}_{-\sqrt{-1}}$, where $\mathfrak{a}_{\pm\sqrt{-1}} = \{\frac{1}{2}(Z \mp \sqrt{-1}JZ) : Z \in \mathfrak{g}^c\}$. It is easy to see that $\mathfrak{g}^c = \mathfrak{a}_{\sqrt{-1}} + \mathfrak{a}_{-\sqrt{-1}}$ is a direct sum of ideals. In particular we see that $\mathfrak{g}^c$ is not simple, which proves half the lemma. Note that $\mathfrak{a}_{\sqrt{-1}}$ is isomorphic as a complex Lie algebra via $\frac{1}{2}(Z - \sqrt{-1}JZ) \mapsto Z$ to $\mathfrak{h}$, and $\mathfrak{a}_{-\sqrt{-1}}$ is isomorphic as a complex Lie algebra via $\frac{1}{2}(Z + \sqrt{-1}JZ) \mapsto Z$ to $\mathfrak{h}$ endowed with the conjugate complex structure.

Conversely assume that the complexification $\mathfrak{g}^c$ can be written as a direct sum of simple ideals $\mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r$ for some $r > 1$, where the $\mathfrak{h}_i$ are complex simple Lie algebras. Let $\pi_i : \mathfrak{g} \to \mathfrak{h}_i$ be the composition of the inclusion map $\mathfrak{g} \to \mathfrak{g}^c$ followed by the projection $\mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r \to \mathfrak{h}_i$. We claim that $\pi_i(\mathfrak{g}) \neq 0$ for all i. In fact, if Z is a nonzero element of $\mathfrak{h}_i$, we write $Z = X + \sqrt{-1}Y$ for some X, $Y \in \mathfrak{g}$, and then $Z = \pi_i(X) + \sqrt{-1}\pi_i(Y)$, which implies that either $\pi_i(X) \neq 0$ or $\pi_i(Y) \neq 0$, and this proves the claim. Since $\mathfrak{g}$ is simple, we have that π_i is injective and then the real dimension of $\mathfrak{h}_i$ cannot be less than the real dimension of $\mathfrak{g}$, namely

$$\dim_{\mathbf{R}} \mathfrak{h}_i \geq \dim_{\mathbf{R}} \mathfrak{g} = \frac{1}{2} \dim_{\mathbf{R}} \mathfrak{g}^c.$$

This implies that $r = 2$ and that $\dim_{\mathbf{R}} \mathfrak{h}_i = \dim_{\mathbf{R}} \mathfrak{g}$. Now $\pi : \mathfrak{g} \to \mathfrak{h}_i$ is an isomorphism and we can transfer the complex structure from $\mathfrak{h}_i$ to $\mathfrak{g}$, which completes the proof. $\square$

2.3.2 Proposition *The irreducible OIL-algebras $(\mathfrak{g}, s, B)$ fall into four pairwise disjoint classes, as follows:*

> Class 1: $\mathfrak{g}$ *is a compact simple Lie algebra, s is an involutive automorphism of $\mathfrak{g}$, and $B = \lambda\beta$ with $\lambda < 0$.*
>
> Class 2: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ *where $\mathfrak{h}$ is a compact simple Lie algebra, $s(X,Y) = (Y,X)$ for $(X,Y) \in \mathfrak{g}$, and $B = \lambda\beta$ with $\lambda < 0$.*
>
> Class 3: $\mathfrak{g}$ *is a noncompact simple Lie algebra with $\mathfrak{g}^c$ simple, $\mathfrak{k}$ is a maximal compact subalgebra, and $B = \lambda\beta$ with $\lambda > 0$.*
>
> Class 4: $\mathfrak{g}$ *is the realification of a complex simple Lie algebra, $\mathfrak{k}$ is a compact real form and a maximal compact subalgebra, s is complex conjugation over $\mathfrak{k}$, and $B = \lambda\beta$ with $\lambda > 0$.*

Duality exchanges classes 1 and 3, 2 and 4.

Proof. We first remark that the algebras $(\mathfrak{g}, s, B)$ of the four classes are irreducible, otherwise by Proposition 2.1.3 there would be a s-invariant direct

[3] That is, an endomorphism $J : \mathfrak{g} \to \mathfrak{g}$ such that $J^2 = -\mathrm{id}$.

sum of ideals $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, but this is clearly impossible in classes 1, 3 and 4 because $\mathfrak{g}$ a simple, and also impossible in class 2 as it is not difficult to see. By diagonalizing B with respect to β, in the irreducible case we always have $B = \lambda\beta$ on $\mathfrak{p}$ with $\lambda < 0$ for compact $\mathfrak{g}$ and $\lambda > 0$ for noncompact $\mathfrak{g}$. If $(\mathfrak{g}, s, B)$ is irreducible then $\mathfrak{k}$ is a maximal subalgebra by Corollary 2.1.5; in classes 3 and 4, $\mathfrak{k}$ is compact by Lemma 2.1.2(b). and Theorem 5.5.1(c).

Next suppose that $(\mathfrak{g}, s, B)$ is irreducible. Assume first that $\mathfrak{g}$ is compact. Then $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ where the $\mathfrak{g}_i$ are compact simple. Since s is an automorphism, it must permute the simple ideals $\mathfrak{g}_i$. Since s is involutive, the orbit of each $\mathfrak{g}_i$ under the cyclic group generated by s contains at most two elements. By irreducibility, $\mathfrak{g}$ does not admit an s-invariant ideal, so there is only one such orbit. It follows that either $\mathfrak{g}$ is simple or $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where s exchanges the summands. In the first case $(\mathfrak{g}, s, B)$ is in class 1. In the second case, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic under s and $(\mathfrak{g}, s, B)$ is in class 2.

Now assume that $\mathfrak{g}$ is noncompact. The dual $(\mathfrak{g}^*, s^*, B^*)$ is such that $\mathfrak{g}^*$ is compact and so it falls into classes 1 or 2. If $\mathfrak{g}^*$ is simple then $\mathfrak{g}^c = (\mathfrak{g}^*)^c$ is also simple, as $\mathfrak{g}^*$ is compact. Since $\mathfrak{g}$ is a real form of $\mathfrak{g}^c$, it must be simple. Therefore $(\mathfrak{g}, s, B)$ is in class 3. On the other hand, if $\mathfrak{g}^* = \mathfrak{h} \oplus \mathfrak{h}$ where $\mathfrak{h}$ is a compact simple Lie algebra and $s^*(X, Y) = (Y, X)$, then $\mathfrak{g}^c = (\mathfrak{g}^*)^c = \mathfrak{h}^c \oplus \mathfrak{h}^c$ and therefore $\mathfrak{g}^c$ is not simple. The same argument as in the previous paragraph shows that if $\mathfrak{g}$ is not simple, then $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_1$ is a noncompact simple Lie algebra and s exchanges the summands; but then the $(+1)$-eigenspace $\mathfrak{k}$ of s is the diagonal in $\mathfrak{g}_1 \oplus \mathfrak{g}_1$, hence isomorphic to $\mathfrak{g}_1$ which is a contradiction since $\mathfrak{g}_1$ is noncompact. Therefore $\mathfrak{g}$ has to be simple. By Lemma 2.3.1, $\mathfrak{g}$ is the realification of a complex Lie algebra. Let J be the underlying complex structure of $\mathfrak{g}$. Since $\mathfrak{k}$ is compact, $J\mathfrak{k} \not\subset \mathfrak{k}$ (see Problem 2.6.6), and then $\mathfrak{g} = \mathfrak{k} + J\mathfrak{k}$ by maximality of $\mathfrak{k}$. Recall that $\mathfrak{g}^* = \mathfrak{h} \oplus \mathfrak{h}$ and $\mathfrak{k}$ is the diagonal; since

$$\dim_{\mathbf{R}} \mathfrak{k} = \dim_{\mathbf{R}} \mathfrak{h} = \frac{1}{2} \dim_{\mathbf{R}} \mathfrak{g}^* = \frac{1}{2} \dim_{\mathbf{R}} \mathfrak{g} = \dim_{\mathbf{C}} \mathfrak{g},$$

we get that $\mathfrak{k}$ is a real form of $\mathfrak{g}$ and $\mathfrak{k} \cap J\mathfrak{k} = 0$. Due to Problem 2.6.1, $\beta(\mathfrak{k}, J\mathfrak{k}) = 0$, so $J\mathfrak{k} = \mathfrak{p}$ and s is complex conjugation of $\mathfrak{g}$ over $\mathfrak{k}$, thus $(\mathfrak{g}, s, B)$ is in class 4. □

2.3.3 Remark In particular, in classes 3 and 4, the subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is *compactly embedded*, in the sense that the subgroup generated by e^{ad_X} for $X \in \mathfrak{k}$ has compact closure in $GL(\mathfrak{g})$.

An OIL-algebra $(\mathfrak{g}, s, B)$ is called of *compact type* (resp. *noncompact type*) if the Killing form of $\mathfrak{g}$ is negative (resp. positive) definite on the -1-eigenspace of s. A symmetric space is called of *compact type* (resp. *noncompact type*) if its OIL-algebra is of compact (resp. noncompact type). Of course, the irreducible OIL-algebras in class 1 or 2 (resp. 3 or 4) are of compact (resp. noncompact type).

We shall proceed with the classification of irreducible OIL-algebras by discussing classes 1 and 2: this amounts to the classification of compact simple Lie algebras and their involutive automorphisms.

The compact simple Lie algebras are listed in a table in subsection 5.7. From that table, one obtains the irreducible symmetric spaces of compact type in class 2. In order to list the irreducible symmetric spaces of compact type in class 1, one needs to classify involutive automorphisms s of each Lie algebra $\mathfrak{g}$ in the table.

Suppose first s in an inner automorphism of $\mathfrak{g}$, $s = \mathrm{Ad}_g$ for some $g \in G$, where G is any Lie group with Lie algebra $\mathfrak{g}$. We can always take g in a given maximal torus of G.[4] In particular, the fixed point space $\mathfrak{k}$ will always be a subalgebra of maximal rank of $\mathfrak{g}$ (that is, $\mathfrak{k}$ contains a Cartan subalgebra of $\mathfrak{g}$). In case of classical Lie algebras, we can find canonical forms for s just using Linear Algebra, as follows.

$\mathbf{A}_n$: $g \in \mathbf{SU}(n+1)$ is an element of order 2 in the adjoint group which belongs to the maximal torus of diagonal matrices. The eigenvalues of g are ± 1, so it is conjugate to $\begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$, where $p + q = n + 1$. The associated symmetric space is the complex Grassmannian $\mathbf{SU}(p+q)/\mathbf{S}(\mathbf{U}(p) \times \mathbf{U}(q)) = Gr_q(\mathbf{C}^{p+q})$.

$\mathbf{B}_n$: This case is similar to $\mathbf{A}_n$ and we get the real Grassmannian

$$\mathbf{SO}(2p + 2q + 1)/\mathbf{S}(\mathbf{O}(2p + 1) \times \mathbf{O}(2q)) = Gr_{2q}(\mathbf{R}^{2p+2q+1}),$$

where $n = p + q$.

$\mathbf{C}_n$: The center of $\mathbf{Sp}(n)$ is $\pm I$. If $g^2 = I$, then we get the quaternionic Grassmannian $\mathbf{Sp}(p+q)/(\mathbf{Sp}(p)\times\mathbf{Sp}(q)) = Gr_q(\mathbf{H}^{p+q})$, as above, where $p+q = n$, In case $g^2 = -I$, we realize $\mathbf{Sp}(n)$ as the subgroup of $\mathbf{SU}(2n)$ consisting of matrices of the form $\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$. Then g is conjugate to $J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, so its centralizer consists of matrices with $B = 0$, that is, it is isomorphic to $\mathbf{U}(n)$; hence we get the symmetric space $\mathbf{Sp}(n)/\mathbf{U}(n)$.

$\mathbf{D}_n$: The center of $\mathbf{SO}(2n)$ is $\pm I$. If $g^2 = I$, then we get the real Grassmannian $\mathbf{SO}(2p+2q)/\mathbf{S}(\mathbf{O}(2p)\times\mathbf{O}(2q)) = Gr_{2q}(\mathbf{R}^{2p+2q})$, as above, where $p+q = n$. If $g^2 = -I$, g is conjugate to J_n and we get the symmetric space $\mathbf{SO}(2n)/\mathbf{U}(n)$.

In the case of exceptional Lie algebras, such simple matrix representations are not available and one instead resorts to root systems.[5] We quote the global classification (the rank will be introduced in subsection 2.5):

2.3.4 Theorem *The irreducible symmetric spaces of class 1 and inner type are listed as follows:*

[4] Any two maximal tori of G are conjugate, a consequence of Proposition 2.5.5.

[5] We have $s = \mathrm{Ad}_{\exp X}$ where X belongs to a given CSA $\mathfrak{t}$ of $\mathfrak{g}$. A conjugation of X, using the Weyl group, can bring $\frac{1}{2\pi\sqrt{-1}}X$ to the closed positive Weyl chamber $\bar{\mathcal{C}} \subset \sqrt{-1}\mathfrak{t}$. Finally, a translation by an element in the central lattice (namely, $\exp^{-1}(Z(G)) = 2\pi\sqrt{-1}L_{rt}^*$, where L_{rt} is the root lattice) does not alter s and can make $0 \leq \frac{1}{2\pi\sqrt{-1}}\alpha(X) \leq 1$ for all roots α, so it brings $\frac{1}{2\pi\sqrt{-1}}X$ to the simplex with vertices $\{v_0, v_1, \ldots, v_n\}$, where

$$v_0 = 0, \ \alpha_i(v_j) = 0 \text{ for } i \neq j, \ \alpha_i(v_i) = 1/m_i,$$

where $\alpha_1, \ldots, \alpha_n$ are simple roots and $\sum_{i=1}^n m_i\alpha_i$ is the highest root. The fact that s is an involution implies that we can take $X = v_i$ with $m_i = 2$ or $X = \frac{1}{2}v_i$ with $m_i = 1$ for some i. See [32, Theorem 3.1, p.121, v. 2] or [42, Theorem 8.10.8].

G/K	*Dimension*	*Rank*
$\mathbf{SU}(p+q)/\mathbf{S}(\mathbf{U}(p)\times\mathbf{U}(q))$	$2pq$	$\min\{p,q\}$
$\mathbf{SO}(2p+q)/\mathbf{SO}(2p)\times\mathbf{SO}(q)$	$2pq$	$\min\{2p,q\}$
$\mathbf{Sp}(p+q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q)$	$4pq$	$\min\{p,q\}$
$\mathbf{Sp}(n)/\mathbf{U}(n)$	$n(n+1)$	n
$\mathbf{SO}(2n)/\mathbf{U}(n)$	$n(n-1)$	$[\frac{1}{2}n]$
$\mathbf{G}_2/\mathbf{SO}(4)$	8	2
$\mathbf{F}_4/(\mathbf{Sp}(3)\times\mathbf{Sp}(1)/\mathbf{Z}_2)$	28	4
$\mathbf{F}_4/\mathbf{Spin}(9)$	16	1
$\mathbf{E}_6/(\mathbf{Spin}(10)\times\mathbf{SO}(2)/\mathbf{Z}_2)$	32	2
$\mathbf{E}_6/(\mathbf{SU}(6)\times\mathbf{SU}(2)/\mathbf{Z}_2)$	40	4
$\mathbf{E}_7/(\mathbf{E}_6\times\mathbf{SO}(2)/\mathbf{Z}_3)$	54	3
$\mathbf{E}_7/(\mathbf{SU}(8)/\mathbf{Z}_4)$	70	7
$\mathbf{E}_7/(\mathbf{Spin}(12)\times\mathbf{SU}(2)/\mathbf{Z}_2)$	64	4
$\mathbf{E}_8/(\mathbf{Spin}(16)/\mathbf{Z}_2)$	128	8
$\mathbf{E}_8/(\mathbf{E}_7\times\mathbf{SU}(2)/\mathbf{Z}_2)$	112	4

In order to deal with outer automorhisms, one first shows that the group $\mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ of outer automorphisms modulo inner automorphisms of a compact simple Lie algebra $\mathfrak{g}$ is canonically isomorphic to the group of symmetries of the Dynkin diagram of $\mathfrak{g}$. An inspection of the diagrams (cf. subsection 5.7) shows that the only types that admit outer automorphisms are types $\mathbf{A}$, $\mathbf{D}$, $\mathbf{E}$, where the symmetry group is given by $\mathbf{Z}_2$, unless we are in case $\mathbf{D}_4$, in which case it is the dihedral group of order 6; in particular, in all cases the conjugation class of automorphisms of order 2 is unique. A canonical representative s_0 is given as follows:

$\mathbf{A}_n$: Here $\mathfrak{g}=\mathfrak{su}(n+1)$ consists of skew-Hermitian matrices of trace zero and $s_0X=\bar{X}=-X^t$. On the group level, it is given by $g\mapsto\bar{g}=(g^t)^{-1}$ and we get the symmetric space $\mathbf{SU}(n)/\mathbf{SO}(n)$.

$\mathbf{D}_n$: Here $\mathfrak{g}=\mathfrak{so}(2n)$ consists of real skew-symmetric matrices and s_0 is given by conjugation with the matrix $I_{1,2n-1}=\begin{pmatrix}-1 & \\ & I_{2n-1}\end{pmatrix}$. We have the sphere $\mathbf{SO}(2n)/\mathbf{SO}(2n-1)$ as corresponding symmetric space.

$\mathbf{E}_6$: Here the description is more involved and we just quote the result. There is a symmetric space $\mathbf{E}_6/\mathbf{F}_4$ of outer type which is the Cartan dual of $\mathbf{E}_{6(-26)}/\mathbf{F}_4$, where $\mathbf{E}_{6(-26)}$ (a certain noncompact real form of ${\mathbf{E}_6}^c$) is the group of collineations of the Cayley projective plane, and $\mathbf{F}_4$ is its maximal compact subgroup and the isometry group of the Cayley projective plane.[6]

[6] Alternatively, there is a description in terms of Jordan algebras [7]. The *Albert algebra* $\mathcal{J}$ is the 27-dimensional real Jordan algebra consisting of 3×3 Hermitian matrices

$$\begin{pmatrix}\xi_1 & x_3 & \bar{x}_2\\ \bar{x}_3 & \xi_2 & x_1\\ x_2 & \bar{x}_1 & \xi_3\end{pmatrix}$$

with octonionic entries, and the (commutative) multiplication being defined by

$$X\circ Y=\frac{1}{2}(XY+YX)$$

Conversely, let s be an arbitrary outer automorphism of $\mathfrak{g}$. We carry out the classification in each case.

$\mathbf{A}_n$: Then $s = s_0 \circ \mathrm{Ad}_g$ for some $g \in \mathbf{SU}(n+1)$. Since $s_0 \circ \mathrm{Ad}_g = \mathrm{Ad}_{\bar{g}} \circ s_0$, we have

$$\mathrm{id} = s^2 = (s_0 \circ \mathrm{Ad}_g)^2 = \mathrm{Ad}_{\bar{g}} \circ \sigma_0^2 \circ \mathrm{Ad}_g = \mathrm{Ad}_{\bar{g}g},$$

which implies that $\bar{g}g$ is in the center of $\mathbf{SU}(n+1)$, so $\bar{g}g = cI_{n+1}$ for some $c \in \mathbf{C}$, where I_{n+1} denotes the identity matrix of order $n+1$. Using that $\bar{g} = (g^t)^{-1}$, we get $g = cg^t$ and then $g = c(cg^t)^t = c^2 g$, therefore $c = \pm 1$.

If $c = 1$ then $g = g^t$, so there is $h \in \mathbf{SO}(n+1)$ such that hgh^{-1} is a diagonal matrix $d \in \mathbf{SU}(n+1)$, and we can replace s by its conjugate

$$\mathrm{Ad}_h \circ s \circ \mathrm{Ad}_h^{-1} = s_0 \circ \mathrm{Ad}_d,$$

and assume g diagonal. Clearly we can choose a diagonal matrix $b \in \mathbf{SU}(n+1)$ with $b^2 = g$. Replacing s by its conjugate

$$\mathrm{Ad}_b \circ s_0 \circ \mathrm{Ad}_g \circ \mathrm{Ad}_b^{-1} = s_0 \circ \mathrm{Ad}_b^{-1} \circ \mathrm{Ad}_g \circ \mathrm{Ad}_b^{-1} = s_0,$$

we may assume $s = s_0$. This gives the symmetric space $\mathbf{SU}(n+1)/\mathbf{SO}(n+1)$.

If $c = -1$ then $g = -g^t$, so there is $h \in \mathbf{SU}(n+1)$ such that hgh^t equals

$$J_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

where I_m denotes an $m \times m$ identity block and $2m = n+1$. Replacing s by its conjugate

$$\mathrm{Ad}_{h^t}^{-1} \circ s \circ \mathrm{Ad}_{h^t} = s_0 \circ \mathrm{Ad}_h \circ \mathrm{Ad}_g \circ \mathrm{Ad}_{h^t} = s_0 \circ \mathrm{Ad}_{J_m},$$

we may assume $g = J_m$. The fixed point group of $s_0 \circ \mathrm{Ad}_{J_m}$ in $\mathbf{SU}(2m)$ consists of the matrices $g \in \mathbf{SU}(2m)$ satisfying $J_m \bar{g} J_m^{-1} = g$. Since this relation is equivalent to $g^t J_m g = J_m$, that fixed point group is isomorphic to $\mathbf{Sp}(m)$. Hence in this case s defines the symmetric space $\mathbf{SU}(2m)/\mathbf{Sp}(m)$.

$\mathbf{D}_n$: By passing to a conjugate if $n = 4$, we can suppose that $s = s_0 \circ \mathrm{Ad}_g$ for some $g \in \mathbf{SO}(2n)$. Now $s = \mathrm{Ad}_h$ where $h = I_{1,2n-1}\, g \in \mathbf{O}(2n)$ and $\det h = -1$. Also, $s^2 = \mathrm{id}$ implies that h^2 centralizes $\mathbf{SO}(2n)$, so $h^2 = \pm I_{2n}$.

If $h^2 = -I_{2n}$ then $h^t = -h$ and the eigenvalues of h are pure imaginary, implying $\det h = 1$, so we must have $h^2 = I_{2n}$. Now $h^t = h$ and the eigenvalues of h are all ± 1. We thus have that h is conjugated in $\mathbf{SO}(2n)$ to

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix},$$

---

where XY is the ordinary matrix product. The derivation algebra of $\mathcal{J}$ is a Lie algebra isomorphic to $\mathfrak{f}_4$. Let $\mathfrak{p}$ denote the 26-dimensional space of left multiplications L_X by elements $X \in \mathcal{J}$ of trace zero. From general theory, $[D, L_X] = L_{DX}$ for $D \in \mathfrak{f}_4$ and $X \in \mathcal{J}$; moreover, $[L_X, L_Y]$ is a derivation of $\mathcal{J}$ for all X, $Y \in \mathfrak{p}$. It can be shown that $\mathfrak{f}_4 + \mathfrak{p}$ is a Lie algebra isomorphic to $\mathfrak{e}_6$. Now we have an involutive decomposition $\mathfrak{e}_6 = \mathfrak{f}_4 + \mathfrak{p}$ where $\mathfrak{f}_4$ is compact, and hence we have an OIL-algebra. Since the rank of $\mathfrak{f}_4$ is less than the rank of $\mathfrak{e}_6$, the corresponding involution of $\mathfrak{e}_6$ is of outer type.

where $p+q=2n$ and p, q are odd. This gives the real Grassmann manifold $\mathbf{SO}(p+q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$.

Before dealing with $\mathbf{E}_6$, we prove a few lemmata.

2.3.5 Lemma *Let $\mathfrak{g}$ be compact semisimple Lie algebra and let σ be an involutive automorphism. Given a Cartan subalgebra $\mathfrak{s}$ of the fixed point set $\mathfrak{g}^\sigma$, there is a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing $\mathfrak{s}$ which is invariant under σ; further, σ preserves some Weyl chamber in $\mathfrak{t}$.*

Proof. By Problem 2.6.3, $\mathfrak{g}^\sigma\neq 0$. Consider the ±1-eigenspace decomposition $\mathfrak{g}=\mathfrak{g}^\sigma+\mathfrak{g}^{-\sigma}$. The centralizer $\mathfrak{t}=Z_{\mathfrak{g}}(\mathfrak{s})$ is σ-invariant and compact, $\mathfrak{t}=\mathfrak{s}+\mathfrak{z}$, where $\mathfrak{z}=Z_{\mathfrak{g}^{-\sigma}}(\mathfrak{s})$. Therefore it decomposes as a direct sum of ideals $\mathfrak{t}=Z(\mathfrak{t})+[\mathfrak{t},\mathfrak{t}]$, where $Z(\mathfrak{t})$ is its center, and the derived algebra $[\mathfrak{t},\mathfrak{t}]$ is compact and semisimple. But $[\mathfrak{t},\mathfrak{t}]=[\mathfrak{z},\mathfrak{z}]\subset\mathfrak{t}\cap\mathfrak{g}^\sigma=\mathfrak{s}$ is Abelian. Therefore $[\mathfrak{t},\mathfrak{t}]=0$ and $\mathfrak{t}$ is thus a Cartan subalgebra of $\mathfrak{g}$. Since $Z_{\mathfrak{g}}(\mathfrak{s})$ is the union of all CSA containing $\mathfrak{s}$, this shows that $\mathfrak{s}$ is contained in exactly one CSA of $\mathfrak{g}$. In particular, a generic element[7] of $\mathfrak{s}$ will be a regular element of $\mathfrak{t}$. Since σ fixes a regular element of $\mathfrak{t}$, it must fix the Weyl chamber containing that element. □

2.3.6 Lemma *Let $\mathfrak{g}$ be a compact semisimple Lie algebra, denote the adjoint group by G, let σ be an involutive automorphism of $\mathfrak{g}$, let $\mathfrak{u}$ be a CSA of $\mathfrak{g}^\sigma$, and denote the associated maximal torus in G^σ by U. Then every involutive automorphism $\varphi=\sigma\circ\mathrm{Ad}_g$ with $g\in G$ is conjugated under an inner automorphism to an automorphism of the form $\sigma\circ\mathrm{Ad}_u$ for some $u\in U$. In particular, G^σ and G^φ have the same rank.*

Proof. Let $\mathfrak{v}$ be a CSA of $\mathfrak{g}^\varphi$. Due to Lemma 2.3.5, $\mathfrak{u}$ and $\mathfrak{v}$ are respectively contained in CSA's $\mathfrak{t}$ and $\mathfrak{s}$ of $\mathfrak{g}$, respectively σ- and φ-invariant. By conjugacy of CSA, there is $h\in G$ such that $\mathfrak{t}=\mathrm{Ad}_h\mathfrak{s}$. Now $\mathfrak{t}$ is preserved by $\varphi':=\mathrm{Ad}_h\varphi\mathrm{Ad}_h^{-1}$. Again by Lemma 2.3.5, σ and φ' preserve Weyl chambers $\mathcal{C}$ and $\mathcal{C}'$ in $\mathfrak{t}$. Since the Weyl group acts transitively on the Weyl chambers, there is $n\in N_G(T)$ such that $\mathrm{Ad}_n\mathcal{C}'=\mathcal{C}$, where T is the maximal torus associated to $\mathfrak{t}$. Now σ and $\varphi'':=\mathrm{Ad}_n\varphi'\mathrm{Ad}_n^{-1}$ both preserve $\mathcal{C}$, and we have:

$$\begin{aligned}\sigma^{-1}\varphi''&=\sigma^{-1}\mathrm{Ad}_{nh}\varphi\mathrm{Ad}_{(nh)^{-1}}\\&=\mathrm{Ad}_{\sigma^{-1}(nh)}\sigma^{-1}\varphi\mathrm{Ad}_{(nh)^{-1}}\\&=\mathrm{Ad}_x,\end{aligned}$$

where $x=\sigma^{-1}(nh)g\varphi(nh)^{-1}$. It follows that x centralizes T, so $x\in T$. We have shown that φ is conjugated to $\varphi''=\sigma\mathrm{Ad}_x$ with $x\in T$. To finish, we need only solve

$$\sigma\mathrm{Ad}_x=\mathrm{Ad}_t\sigma\mathrm{Ad}_u\mathrm{Ad}_t^{-1}=\sigma\mathrm{Ad}_{\sigma(t)ut^{-1}}$$

for $t\in T$ and $u\in U$. But this follows from Problem 2.6.4. □

[7] Say, the infinitesimal generator of a one-parameter group dense in the torus generated by $\mathfrak{s}$.

Now we can carry out the classification of symmetric spaces of outer type of $\mathbf{E}_6$. Recall that the group of outer automorphisms of $\mathfrak{e}_6$ is $\mathbf{Z}_2$, with a generator represented by s_0, which has fixed point set $\mathfrak{f}_4$. Any other given outer automorphism can be assumed of the form $s = s_0 \circ \mathrm{Ad}_g$ for some $g \in \mathbf{F}_4$, owing to Lemma 2.3.6. In particular, s and s_0 are commuting involutions of $\mathfrak{e}_6$, defining involutions σ and σ_0 of $\mathbf{E}_6$ and corresponding symmetric spaces $\mathbf{E}_6/K$ and $\mathbf{E}_6/\mathbf{F}_4$. Now σ defines an involution of $\mathbf{F}_4$ and σ_0 defines an involution of K; further, the fixed point sets of $\sigma\sigma_0$ on $\mathbf{F}_4$ and on K coincide. The only symmetric spaces of $\mathbf{F}_4$ are $\mathbf{F}_4/\mathbf{Spin}(9)$ and $\mathbf{F}_4/\mathbf{Sp}(3)\mathbf{Sp}(1)$, by the classification of irreducible symmetric spaces of compact type and inner type. Suppose we are in the first case; then K contains $\mathbf{Spin}(9)$ and has rank 4. Since K is a maximal subgroup of $\mathbf{E}_6$, we get $\dim K > \dim \mathbf{Spin}(9) = 36$, but an enumeration of compact Lie group satisfying these conditions shows $K = \mathbf{F}_4$. Therefore we may assume we are in the second case, that is σ_0 defines $\mathbf{F}_4/\mathbf{Sp}(3)\mathbf{Sp}(1)$. Now K has rank 4, contains $\mathbf{Sp}(3)\mathbf{Sp}(1)$ and $K/\mathbf{Sp}(3)\mathbf{Sp}(1)$ is symmetric. It follows that $K = \mathbf{Sp}(4)/\mathbf{Z}_2$. Finally, we obtain the symmetric space $\mathbf{E}_6/(\mathbf{Sp}(4)/\mathbf{Z}_2)$.

We collect the results in the case of outer automorphism:

2.3.7 Theorem *The irreducible symmetric spaces of class 1 and outer type are listed as follows:*

G/K	*Dimension*	*Rank*
$\mathbf{SU}(n)$	n^2-1	$n-1$
$\mathbf{Spin}(n)$	$\frac{1}{2}n(n-1)$	$[\frac{n}{2}]$
$\mathbf{Sp}(n)$	$2n^2+n$	n
$\mathbf{G}_2$	14	2
$\mathbf{F}_4$	52	4
$\mathbf{E}_6$	78	6
$\mathbf{E}_7$	133	7
$\mathbf{E}_8$	248	8
$\mathbf{SU}(n)/\mathbf{SO}(n)$	$\frac{1}{2}(n-1)(n+2)$	$n-1$
$\mathbf{SU}(2n)/\mathbf{Sp}(n)$	$(n-1)(2n+1)$	$n-1$
$\dfrac{\mathbf{SO}(2p+2q+2)}{\mathbf{SO}(2p+1)\times\mathbf{SO}(2q+1)}$	$(2p+1)(2q+1)$	$\min\{p,q\}$
$\mathbf{E}_6/\mathbf{F}_4$	26	2
$\mathbf{E}_6/(\mathbf{Sp}(4)/\mathbf{Z}_2)$	42	6

2.3.8 Theorem *Every simply-connected symmetric space is isometric to the Riemannian product of an Euclidean factor with irreducible symmetric spaces of compact type given by the tables in Theorems 2.3.4, 2.3.7, 5.7.1, and their Cartan duals of noncompact type.*

2.4 Noncompact real forms of complex simple Lie algebras

It all starts with the existence of compact real forms of complex semisimple Lie algebras. Cartan first checked it case by case, without realizing its importance, and Hermann Weyl [41, Satz 6, p. 375] gave a general proof, relying of the full strength of the Killing-Cartan structure theory of complex semisimple Lie algebras via a subtle analysis of structure constants and the construction of the so called Cartan-Weyl basis.[8] In [9] Cartan refers to Weyl's proof and asks whether a simpler argument is possible, for "*une telle démonstration permettrait de simplifier notablement l'exposition de la théorie des groupes simples.*" He then describes an unsuccessful attempt of his to prove it. His guess is that a basis that diagonalizes the Killing form and minimizes the sum of the squares of the absolute values of the structure constants would span a compact real form. This idea was taken up much later, and pushed through to a proof by Richardson [39].

In order to obtain uniqueness of compact real forms, up to inner automorphism, Cartan uses the following tool. This is a striking example of what he means by Differential Geometry aiding Lie group theory.

2.4.1 Theorem (Cartan (1929)) *Let M be a Hadamard manifold (i.e. a complete simply-connected Riemannian manifold of nonpositive sectional curvature), and let H be a compact group of isometries of M. Then H has a fixed point.*

Proof. Recall that the distance function to a point on a nonpositively curved manifold is convex (this follows e.g. from the formula for the second variation of length). Consider the orbit Hp for any fixed $p \in M$. Fix a Haar measure μ on H. Now the function

$$F(x) = \int_H d^2(hp, x)\, d\mu(h)$$

is strictly convex (due to the exponent 2 in the distance function), so it has a unique point of minimum $\bar{p} \in M$, called the *center of mass* of Hp. For any $h' \in H$,

$$\begin{aligned} F(h'x) &= \int_H d^2(hp, h'x)\, d\mu(h) \\ &= \int_H d^2((h')^{-1}hp, x)\, d\mu(h) \\ &= \int_H d^2((h''p, x)\, d\mu(h'') \\ &= F(x), \end{aligned}$$

where we have used left-invariance of the Haar measure in the next to last equality. By the uniqueness of the center of mass, $\bar{p}$ is a fixed point of H. $\square$

2.4.2 Proposition *Let $\mathfrak{g}$ be a noncompact real simple Lie algebra. Then there exist s and B such that $(\mathfrak{g}, s, B)$ is an orthogonal involutive Lie algebra.*

[8] In modern terms, this involves, among other things, Lie's theorem, Engel's theorem, the existence of Cartan subalgebras and the root space decomposition of a semisimple Lie algebra.

Proof. Suppose first the complexification $\mathfrak{g}^c$ of $\mathfrak{g}$ is not simple. Then $\mathfrak{g}$ is a complex Lie algebra viewed as real, owing to Lemma 2.3.1. We take s to be complex conjugation of $\mathfrak{g}$ over a compact real form $\mathfrak{k}$. The Killing form β of $\mathfrak{g}$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p} = \sqrt{-1}\mathfrak{k}$, so we can take $B = \beta|_{\mathfrak{p}}$. We get an OIL-algebra in class 4.

Consider now the case $\mathfrak{g}^c$ is simple. It admits a compact real form $\mathfrak{g}_u$, which is also simple. Let G^c be the adjoint group of $\mathfrak{g}^c$, and denote by G_u the connected subgroup of G^c with Lie algebra $\mathfrak{g}_u$. Note that G_u is a compact Lie group and $M = G^c/G_u$ is a symmetric space of nonpositive curvature by Corollary 3.1.4 below, where the metric is induced from the Killing form of $\mathfrak{g}^c$.

We claim that M is simply-connected. Recall from Proposition 1.5.1(i) that $G^c = G_u \exp[\sqrt{-1}\mathfrak{g}_u]$. Consider the inner product on $\mathfrak{g}^c$ given by the Killing form on $\sqrt{-1}\mathfrak{g}_u$ and its negative on $\mathfrak{g}_u$. Then $\mathfrak{g}_u$ consists of skew-symmetric endomorphisms, and $\sqrt{-1}\mathfrak{g}_u$ consists of symmetric endomorphisms of $\mathfrak{g}^c$. Now G_u consists of orthogonal matrices. It is known that the exponential is a diffeomorphism from $\sqrt{-1}\mathfrak{g}_u$ onto its image, which consists of positive definite symmetric matrices. It follows that $G_u \cap \exp[\sqrt{-1}\mathfrak{g}_u] = \{1\}$ and $G_u \times \sqrt{-1}\mathfrak{g}_u \to G^c$, $(g, X) \mapsto g \exp X$ is a diffeomorphism. In particular, $M \approx \sqrt{-1}\mathfrak{g}_u$.

Denote the complex conjugation of $\mathfrak{g}^c$ over $\mathfrak{g}$ by σ. Then σ is an automorphism of $\mathfrak{g}^c$, viewed as a real Lie algebra, and induces an involutive automorphism of G^c, still denoted σ, which fixes G_u pointwise. Any automorphism of G^c defines an isometry of M. Let x, $y \in M$ be two points interchanged by σ. Then σ fixes the midpoint z of the unique geodesic joining x and y. It follows that the isotropy group of G^c at z is a compact subgroup of G^c which is invariant under σ. In particular, σ induces an involutive automorhism on its Lie algebra, denoted by $\mathfrak{g}^*$. Therefore $\mathfrak{g}^* = \mathfrak{g}^* \cap \mathfrak{g} + \mathfrak{g}^* \cap \sqrt{-1}\mathfrak{g}$ is an involutive decomposition.

As a conjugate of $\mathfrak{g}_u$, also $\mathfrak{g}^*$ is a real form of $\mathfrak{g}^c$, $\mathfrak{g}^c = \mathfrak{g}^* + \sqrt{-1}\mathfrak{g}^*$. It follows that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{g}^*$ and $\mathfrak{p} = \mathfrak{g} \cap \sqrt{-1}\mathfrak{g}^*$. Since $\mathfrak{g}^* = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ is an involutive decomposition (under σ), also $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ is an involutive decomposition, that is, the ± 1-eigenspaces of an involutive automorphism s, and B can be taken to be the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{p}$. In this case we get an OIL-algebra in class 3. $\square$

2.4.3 Proposition *Let $\mathfrak{g}$ be a noncompact real simple Lie algebra. Then any two maximal compactly embedded subalgebras of $\mathfrak{g}$ are conjugated under an inner automorphism.*

Proof. Let G be the adjoint group of $\mathfrak{g}$ and use Proposition 2.4.2 to find a compact connected subgroup K of G and a symmetric space $M := G/K$ in class 3 or 4, hence of nonpositive curvature, due Corollary 3.1.4 below. Let $\mathfrak{h}$ be any compactly embedded subalgebra of $\mathfrak{g}$ and denote by H the associated connected subgroup of G. Note that H is compact, and it obviously acts by isometries on M. By Theorem 2.4.1, H has a fixed point $x \in M$. Write $x = gx_0$ for some $g \in G$, where x_0 is the basepoint. Then $g^{-1}Hgx_0 = g^{-1}Hx = g^{-1}x = x_0$, that is, $g^{-1}Hg \subset K$. It follows that $\mathrm{Ad}_{g^{-1}}\mathfrak{h} \subset \mathfrak{k}$. $\square$

2.4.4 Corollary *A compact real form of a complex semisimple Lie algebra is unique, up to inner automorphism.*

Proof. One needs only note that the compact real form is compactly embedded and a maximal subalgebra. $\square$

Incidentally, the s in Proposition 2.4.2 is called a *Cartan involution* of $\mathfrak{g}$, and the associated ± 1-decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is called a *Cartan decomposition* of $\mathfrak{g}$ (for the existence and uniqueness of Cartan decompositions, see also [26, 6.18, 6.19]). Recall this means that the Killing form β of $\mathfrak{g}$ satisfies $\beta|_{\mathfrak{k}}$ is negative definite and $\beta|_{\mathfrak{p}}$ is positive definite.

2.4.5 Remark As a consequence of the Proposition 2.4.2, the determination of OIL-algebras in class 3 is tantamount to the classification of noncompact real forms of complex simple Lie algebras. Since the determination of OIL-algebras in class 2 is tantamount to the classification of compact real forms of complex simple Lie algebras, we note that the classification of irreducible symmetric spaces is equivalent to the classification of real forms of complex simple Lie algebras.

2.4.6 Theorem *Every connected semisimple Lie group G with finite center admits a maximal compact subgroup K, which is unique, up to conjugation. Further, K is connected and contains the center of G. Finally, G is diffeomorphic to $K \times \mathbf{R}^n$ for some n.*

Proof. Let $\mathfrak{g}$ denote the Lie algebra of G. Let $\mathfrak{k}$ be a maximal compactly embedded subalgebra of $\mathfrak{g}$ as in Proposition 2.4.2, and write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Cartan decomposition.

For the adjoint group $\bar{G} := \mathrm{Ad}(G)$, as in the proof of Proposition 2.4.2, one shows that $\bar{K} \times \mathfrak{p} \to \bar{G}$, $(k, X) \mapsto k \exp^{\bar{G}} X$ defines a diffeomorphism, where $\bar{K} = \exp^{\bar{G}}[\mathfrak{k}]$ is compact. In case of a covering group $\pi : G \to \bar{G}$, let $K = \pi^{-1}(\bar{K})$. The covering is finite, as the center $Z(G)$ of G is finite, so K is compact and $Z(G) \subset K$. Now $\bar{K} = K/Z(G)$ and $K \times \mathfrak{p} \to G$, $(k, X) \mapsto k \exp^G X$ defines a diffeomorphism; in particular K is connected. K is a maximal subgroup of G, since K normalizes $P = \exp[\mathfrak{p}]$, and it is unique up to inner automorphism, by Proposition 2.4.3. $\square$

2.4.7 Corollary *A symmetric space of noncompact type is simply-connected, and the isotropy subgroup of its transvection group at a point is connected and a maximal compact subgroup.*

2.4.8 Corollary *A connected semisimple Lie group with finite center has the homotopy type of a compact Lie group.*

2.5 Restricted roots

Recall from Propositions 2.4.2 and 2.4.3 that for a given real noncompact simple Lie algebra $\mathfrak{g}$, there exists a structure (s, B) of OIL-algebra in $\mathfrak{g}$, unique up to inner automorhism, such that $(\mathfrak{g}, s, B)$ is irreducible in class 3 or 4.

Throughout this section, we let $(\mathfrak{g}, s, B)$ be an irreducible OIL-algebra in class 3 or 4, i.e. $\mathfrak{g}$ is a real noncompact simple Lie algebra, $\mathfrak{k} \subset \mathfrak{g}$ is a maximal compact subalgebra and $B = \lambda\beta$ with $\lambda > 0$. We construct a (non-reduced) root system associated to $(\mathfrak{g}, s, B)$; this root system can be used to characterize the symmetric space, and in Lecture 3 it will be shown to reflect some geometric properties of the associated symmetric space.

Note that $B_s(X, Y) = -\beta(X, sY)$ for X, $Y \in \mathfrak{g}$ defines a positive definite inner product on $\mathfrak{g}$.

2.5.1 Lemma *We have* $(\mathrm{ad}_X)^* = -\mathrm{ad}_{sX}$ *for* $X \in \mathfrak{g}$*, where the adjoint homomorphism is with respect to* B_s*.*

Proof. For X, Y, $Z \in \mathfrak{g}$,

$$\begin{aligned} B_s(\mathrm{ad}_X Y, Z) &= -\beta([X, Y], sZ) \\ &= \beta(Y, [X, sZ]) \\ &= \beta(sY, [sX, Z]) \\ &= -B_s(Y, \mathrm{ad}_{sX} Z), \end{aligned}$$

as wished. □

Of course any subalgebra of $\mathfrak{p}$ must be Abelian since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. A maximal Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ is called a *Cartan subspace* of $\mathfrak{p}$.

Fix a maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. Now $\{\mathrm{ad}_H : H \in \mathfrak{a}\}$ is a family of self-adjoint endomorphisms of $\mathfrak{g}$ whose members pairwise commute. It follows that $\mathfrak{g}$ is the vector space orthogonal direct sum of simultaneous eigenspaces, with real eigenvalues. Let

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} : \mathrm{ad}_H X = \lambda(H)X, \text{ for } H \in \mathfrak{a}\}.$$

Then $\lambda \in \mathfrak{a}^*$ and we denote

$$\Delta(\mathfrak{g}, \mathfrak{a}) = \{\lambda \in \mathfrak{a}^* : \mathfrak{g}_\lambda \neq 0 \text{ and } \lambda \neq 0\} \quad (\textit{system of restricted roots}).$$

If $\lambda \in \Delta(\mathfrak{g}, \mathfrak{a})$, the number $\dim \mathfrak{g}_\lambda$ is called the *multiplicity* of λ.

2.5.2 Proposition *a. There is a* $\mathrm{ad}_{\mathfrak{a}}$*-invariant* B*-orthogonal vector space direct sum decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\lambda \in \Delta(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\lambda.$$

b. $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.

c. $s[\mathfrak{g}_\lambda] = \mathfrak{g}_{-\lambda}$*; in particular,* $\lambda \in \Delta(\mathfrak{g}, \mathfrak{a})$ *if and only if* $-\lambda \in \Delta(\mathfrak{g}, \mathfrak{a})$.

d. $\mathfrak{g}_0 = \mathfrak{m} + \mathfrak{a}$*, where* $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$.

Proof. (a) This is the common eigenspace decomposition of ad_H for $H \in \mathfrak{a}$.
(b) Let $H \in \mathfrak{a}$, $X \in \mathfrak{g}_\lambda$, $Y \in \mathfrak{g}_\mu$. Then, using the Jacobi identity,

$$\mathrm{ad}_H[X,Y] = [\mathrm{ad}_H X, Y] + [X, \mathrm{ad}_H Y] = (\lambda(H) + \mu(H))[X,Y],$$

so $[X,Y] \in \mathfrak{g}_{\lambda+\mu}$ (possibly zero).
(c) Let $H \in \mathfrak{a}$, $X \in \mathfrak{g}_\lambda$. Then, using that s is an automorphism and that $sH = -H$,

$$[\mathrm{ad}_H, sX] = s[\mathrm{ad}_{-H}, X] = s\lambda(-H)X = -\lambda(H)sX,$$

so $sX \in \mathfrak{g}_{-\lambda}$.
(d)

$$\begin{aligned}
\mathfrak{g}_0 &= Z_{\mathfrak{g}}(\mathfrak{a}) \\
&= Z_{\mathfrak{k}}(\mathfrak{a}) + Z_{\mathfrak{p}}(\mathfrak{a}) \qquad \text{(since } \mathfrak{a} \subset \mathfrak{p}\text{)} \\
&= \mathfrak{m} + \mathfrak{a} \qquad \text{(since } \mathfrak{a} \text{ is maximal Abelian in } \mathfrak{p}\text{)},
\end{aligned}$$

as wished. □

One can show the set $\Delta(\mathfrak{g}, \mathfrak{a}) \subset \mathfrak{a}^*$ introduced above is a (non-reduced) root system (cf. subsection 5.7). Fix a basis of $\mathfrak{a}^*$. Then we have an associated lexicographic order. This defines a notion of positivity, and we have a system of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{a})$. Let $\mathfrak{n} = \sum_{\lambda \in \Delta^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\lambda$. Thanks to Proposition 2.5.2(b), $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$ and $\mathfrak{a} + \mathfrak{n}$ is a solvable subalgebra of $\mathfrak{g}$.

2.5.3 Theorem (Iwasawa decomposition) *There is a direct sum of vector spaces* $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$.

Proof. Given $X \in \mathfrak{g}$, use Proposition 2.5.2(a) and (d) to find $X_0 \in \mathfrak{m}$, $H \in \mathfrak{a}$ and $X_\lambda \in \mathfrak{g}_\lambda$ such that

$$\begin{aligned}
X &= X_0 + H + \sum_{\lambda \in \Delta(\mathfrak{g},\mathfrak{a})} X_\lambda \\
&= \left(X_0 + \sum_{\lambda \in \Delta^+(\mathfrak{g},\mathfrak{a})} X_\lambda + sX_\lambda \right) + H + \left(\sum_{\lambda \in \Delta^+(\mathfrak{g},\mathfrak{a})} X_\lambda - sX_\lambda \right);
\end{aligned}$$

note that the last line is in $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, which shows that the vector space sum is all of $\mathfrak{g}$. To see that the sum is direct, it suffices to consider $X \in \mathfrak{k} \cap (\mathfrak{a} + \mathfrak{n})$. Note that $X = sX \in \mathfrak{a} + s\mathfrak{n}$. Since $\mathfrak{a} + \mathfrak{n} + s\mathfrak{n}$ is a direct sum, we obtain $X \in \mathfrak{a} \subset \mathfrak{p}$. Now $X \in \mathfrak{k} \cap \mathfrak{p} = 0$ implies $X = 0$. □

2.5.4 Theorem (Global Iwasawa decomposition) *Let G be a noncompact real semisimple connected Lie group with finite center, and fix an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ of the Lie algebra of G. Let K, A, N be the connected subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, respectively. Then the multiplication map $K \times A \times N \to G$ is a global diffeomorphism.*

Proof. Let $\bar{G} = \mathrm{Ad}(G)$ be the adjoint group, viewed as a closed subgroup of $\mathbf{GL}(\mathfrak{g})$, and let $\bar{K}$, $\bar{A}$ and $\bar{N}$ be the subgroups of $\bar{G}$ corresponding to K, A and N.

By semisimplicity, indeed $\bar{G} \subset \mathbf{SL}(n, \mathbf{R})$. The elements of $\bar{K}$ are special orthogonal matrices, those of A are diagonal matrices with positive entries, and those of N are upper triangular matrices with 1's along the diagonal. Hence the result for $\bar{G}$ follows from the result for $SL(n, \mathbf{R})$ (cf. Problem 2.6.9).

Now we have a commutative diagram

$$\begin{array}{ccc} K \times A \times N & \longrightarrow & G \\ \downarrow & & \downarrow \\ \bar{K} \times \bar{A} \times \bar{N} & \longrightarrow & \bar{G} \end{array}$$

where the horizontal arrows denote the multiplication maps, and the vertical arrows denote covering maps. Note that $\bar{A}$ and $\bar{N}$ are simply-connected, and $\bar{G} = G/Z(G)$, where $Z(G) \subset \bar{K}$ (cf. proof of Theorem 2.4.6). Therefore the result for $\bar{G}$ can be lifted to G. $\square$

The construction of the restricted root system of $\mathfrak{g}$ is independent of choice of Cartan subspace of $\mathfrak{p}$, as we show now. Let K be the connected subgroup of $\mathrm{GL}(\mathfrak{g})$ generated by $\exp \mathrm{ad}_X$ for $X \in \mathfrak{k}$.

2.5.5 Proposition *Let $\mathfrak{a}$ and $\mathfrak{a}'$ be two Cartan subspaces of $\mathfrak{p}$. Then there exists $k \in K$ such that $\mathrm{Ad}_k \mathfrak{a} = \mathfrak{a}'$.*

Proof. It follows from Proposition 2.5.2 that there exists $X \in \mathfrak{a}$ such that the centralizer of X in $\mathfrak{p}$ is exactly $\mathfrak{a}$ (namely, any element in the complement of the union of the kernels of the restricted roots). Given $Y \in \mathfrak{p}$, consider the continuous real-valued function $f(k) = \beta(\mathrm{Ad}_k Y, X)$ for $k \in K$, where β is the Killing form of $\mathfrak{g}$. Recall K is compact, and choose a point of minimum $k_0 \in K$. Then, for $Z \in \mathfrak{k}$,

$$0 = \frac{d}{dt}\Big|_{t=0} \beta(\mathrm{Ad}_{\exp(tZ)k_0} Y, X) = \beta([Z, \mathrm{Ad}_{k_0} Y], X) = \beta(Z, [\mathrm{Ad}_{k_0} Y, X]).$$

Since β is negative definite on $\mathfrak{k}$ and $Z \in \mathfrak{k}$ is arbitrary, $[\mathrm{Ad}_{k_0} Y, X] = 0$. By the choice of X, $\mathrm{Ad}_{k_0} Y \in \mathfrak{a}$. In particular, if $Y \in \mathfrak{a}'$ is taken so that its centralizer of in $\mathfrak{p}$ is exactly $\mathfrak{a}'$, then $\mathfrak{a} \subset \mathrm{Ad}_{k_0} \mathfrak{a}'$. In particular, $\dim \mathfrak{a} \leq \dim \mathfrak{a}'$. By symmetry, the dimensions must be equal and hence $\mathfrak{a} = \mathrm{Ad}_{k_0} \mathfrak{a}'$. $\square$

Next, we want to explain that the restricted roots of $(\mathfrak{g}, \mathfrak{a})$ are literally restrictions of roots of $\mathfrak{g}$ with respect to a Cartan subalgebra containing $\mathfrak{a}$.

2.5.6 Proposition *If $\mathfrak{t}$ is a maximal Abelian subalgebra of $\mathfrak{m}$, then $\mathfrak{h}^c$ is a Cartan subalgebra of $\mathfrak{g}^c$, where $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$.*

Proof. It is clear that $\mathfrak{h}^c$ is Abelian. Suppose now $Z = X + \sqrt{-1}Y$ centralizes $\mathfrak{h}^c$, where $X, Y \in \mathfrak{g}$. Then X and Y centralize $\mathfrak{h}$. In particular, $[X, \mathfrak{a}] = 0$ and $[sX, \mathfrak{a}] = s[X, \mathfrak{a}] = 0$, implying X, $sX \in \mathfrak{m} + \mathfrak{a}$. Now $X + sX \in \mathfrak{k}$ implies $X + sX \in \mathfrak{m}$, so $[X + sX, \mathfrak{t}] = [X, \mathfrak{t}] + s[X, \mathfrak{t}] = 0$ yields $X + sX \in \mathfrak{t}$. Similarly, $X - sX \in \mathfrak{p}$ gives $X - sX \in \mathfrak{a}$. Therefore $2X = (X + sX) + (X - sX) \in \mathfrak{t} + \mathfrak{a} = \mathfrak{h}$. In a similar way, $Y \in \mathfrak{h}$ and thus $Z \in \mathfrak{h}^c$. This proves that $\mathfrak{h}^c$ is maximal Abelian.

Finally, if $H \in \mathfrak{t} \subset \mathfrak{k}$ then $\mathrm{ad}_H^* = -\mathrm{ad}_{sH} = -\mathrm{ad}_H$, so ad_H is diagonalizable over $\mathbf{C}$. On the other hand, if $H \in \mathfrak{a}$ then ad_H is already diagonal. We have seen that the real and imaginary parts of any $H \in \mathfrak{h}^c$, under the adjoint representation, are semisimple endomorphisms. Since they commute, this proves that ad_H is semisimple for all $H \in \mathfrak{h}^c$. □

2.5.7 Corollary *The roots $\alpha \in \Delta(\mathfrak{g}^c, \mathfrak{h}^c)$ take real values on $\sqrt{-1}\mathfrak{t} + \mathfrak{a}$. If $\mathfrak{m} = 0$, then $\mathfrak{g}$ is a normal real form of $\mathfrak{g}^c$ and all the multiplicities of the restricted roots are 1.*

Proof. If $H \in \mathfrak{a}$, then $\mathrm{ad}_H : \mathfrak{g} \to \mathfrak{g}$ is self-adjoint and has thus real eigenvalues; the same is of course true for $\mathrm{ad}_H : \mathfrak{g}^c \to \mathfrak{g}^c$. On the other hand, if $H \in \mathfrak{t}$, then $\mathrm{ad}_H : \mathfrak{g} \to \mathfrak{g}$ is skew-adjoint and so has purely-imaginary eigenvalues.

If $\mathfrak{m} = 0$, then $\mathfrak{t} = 0$ so the roots are real-valued on $\mathfrak{h}$. Hence $\mathfrak{g}$ contains the real form $\mathfrak{h}$ of a Cartan subalgebra $\mathfrak{h}^c \subset \mathfrak{g}^c$ where the roots are real. Also, $\mathfrak{h} = \mathfrak{a}$ and $\mathfrak{g}_\lambda$ is a real form of $(\mathfrak{g}^c)_\alpha$ for $\alpha \in \Delta(\mathfrak{g}^c, \mathfrak{h}^c)$ and $\lambda = \alpha|\mathfrak{a}$. □

In the special case in which $\mathfrak{m} = 0$, we have $\mathfrak{h} = \mathfrak{a}$ and the rank of M equals the rank of G; the associated symmetric space G/K (and its dual) are called of *maximal rank*.

In general, we compare the decomposition of $\mathfrak{g}$ into restricted root spaces, with respect to $\mathfrak{a}$, with the decomposition of $\mathfrak{g}^c$ into complex root spaces, with respect to $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ (cf. Proposition 2.5.6). We have

$$\begin{aligned} \mathfrak{g}^c &= \mathfrak{h}^c + \sum_{\alpha \in \Delta(\mathfrak{g}^c, \mathfrak{h}^c)} (\mathfrak{g}^c)_\alpha \\ &= \mathfrak{t}^c + \mathfrak{a}^c + \sum_{\alpha|\mathfrak{a} \neq 0} (\mathfrak{g}^c)_\alpha + \sum_{\alpha|\mathfrak{a} = 0} (\mathfrak{g}^c)_\alpha . \end{aligned}$$

If $\alpha|\mathfrak{a} \neq 0$, then $((\mathfrak{g}^c)_\alpha + \sigma(\mathfrak{g}^c)_\alpha)) \cap \mathfrak{g} \subset \mathfrak{g}_\lambda$ for $\lambda = \alpha|\mathfrak{a} \in \Delta(\mathfrak{g}, \mathfrak{a})$, where σ denotes the conjugate-linear conjugation of $\mathfrak{g}^c$ over $\mathfrak{g}$. Hence

$$(\mathfrak{g}^c)_\alpha \subset (\mathfrak{g}_\lambda)^c.$$

On the other hand, if $\alpha|\mathfrak{a} = 0$, then $(\mathfrak{g}^c)_\alpha \subset Z_{\mathfrak{a}}(\mathfrak{g}^c) = \mathfrak{m}^c + \mathfrak{a}^c$. Now unidimensionality and s-invariance of $(\mathfrak{g}^c)_\alpha$ imply

$$(\mathfrak{g}^c)_\alpha \subset \mathfrak{m}^c.$$

We finally get

$$\mathfrak{m}^c = \mathfrak{t}^c + \sum_{\alpha|\mathfrak{a}=0} (\mathfrak{g}^c)_\alpha \quad \text{and} \quad (\mathfrak{g}_\lambda)^c = \sum_{\alpha|\mathfrak{a}=\lambda} (\mathfrak{g}^c)_\alpha.$$

Finally, we discuss some peculiarities in the case in which $(\mathfrak{g}, s, B)$ is in class 4. Then $\mathfrak{g}$ admits an ad-invariant complex structure J. We have $\mathfrak{g} = \mathfrak{k} + J\mathfrak{k}$ relative to s, and $\mathfrak{k}$ is a maximal compact subalgebra and a real form of $\mathfrak{g}$. Let $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ be as above. Since $\mathfrak{a}$ centralizes $\mathfrak{h}$, so does $J\mathfrak{a}$. We have $J\mathfrak{a} \subset \mathfrak{k}$, and this implies $J\mathfrak{a} \subset \mathfrak{t}$. Similarly, one checks that $J\mathfrak{t} \subset \mathfrak{a}$ and then $J\mathfrak{a} = \mathfrak{t}$. Now $\mathfrak{h} = J\mathfrak{a} + \mathfrak{a}$ is a (complex) Cartan subalgebra of $(\mathfrak{g}, J)$. Consider

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h} + \sum_{\alpha\in\Delta(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha \qquad \text{(complex decomposition)} \\ &= \mathfrak{t} + \mathfrak{a} + \sum_{\alpha\in\Delta(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha \qquad \text{(real decomposition)} \\ &= \mathfrak{t} + \mathfrak{a} + \sum_{\alpha|\mathfrak{a}\in\Delta(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha. \end{aligned}$$

It follows that $\mathfrak{m} = \mathfrak{t}$ and all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are nonzero on $\mathfrak{a}$. Since the real dimension of $\mathfrak{g}_\alpha$ is 2, all the multiplicities are 2. The restriction map $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mapsto \alpha|\mathfrak{a} \in \Delta(\mathfrak{g}, \mathfrak{a})$ is a bijection and an isometry, and $\Delta(\mathfrak{g}, \mathfrak{a})$ is a reduced root system. Also, the rank of $\mathfrak{g}$ as a complex Lie algebra equals the rank of the symmetric space G/K.

2.5.8 Example For the symmetric space $\mathbf{SL}(n, \mathbf{R})/\mathbf{SO}(n)$ of class 3x, we have $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{R})$, $sX = -X^t$ and $\mathfrak{k} = \mathfrak{so}(n)$. The vector space $\mathfrak{p}$ consists of the n by n traceless real symmetric matrices. A maximal Abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ is given by the diagonal matrices, namely,

$$\mathfrak{a} = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} ; \ a_1 + \cdots + a_n = 0, \quad a_i \in \mathbf{R} \right\}.$$

Hence the rank is $n - 1$. The centralizer of a diagonal matrix with pairwise different entries is diagonal, so $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a}) = 0$. Therefore $\mathfrak{t} = 0$ and $\mathfrak{h} = \mathfrak{a}$. It follows that $\mathfrak{sl}(n, \mathbf{R})$ is a normal real form of $\mathfrak{sl}(n, \mathbf{C})$ and $\mathbf{SL}(n, \mathbf{R})/\mathbf{SO}(n)$ is a symmetric space of maximal rank. Let $\theta_i(H)$ denote the ith diagonal element of $H \in \mathfrak{a}$. Then

$$\Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm(\theta_i - \theta_j) : 1 \le i < j \le n\}$$

and the multiplicities are 1.

We close this section combining the restricted root decomposition with the decomposition into eigenspaces of the involution. By Proposition 2.5.2(c), $\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}$ is s-invariant, thus it decomposes into the sum of its intersections with $\mathfrak{k}$ and $\mathfrak{p}$. For $\lambda \in \Delta^+(\mathfrak{g}, \mathfrak{a})$, set

$$\mathfrak{k}_\lambda = (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}) \cap \mathfrak{k} \quad \text{and} \quad \mathfrak{p}_\lambda = (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}) \cap \mathfrak{p}.$$

2.5.9 Proposition *a. There are vector space direct sum decompositions*

$$\mathfrak{k} = \mathfrak{m} + \sum_{\lambda \in \Delta^+(\mathfrak{g},\mathfrak{a})} \mathfrak{k}_\lambda \quad \textit{and} \quad \mathfrak{p} = \mathfrak{a} + \sum_{\lambda \in \Delta^+(\mathfrak{g},\mathfrak{a})} \mathfrak{p}_\lambda.$$

b. We have $\dim \mathfrak{k}_\lambda = \dim \mathfrak{p}_\lambda$*, and it equals the multiplicity* m_λ *of* λ*.*

c. We have

$$\mathfrak{k}_\lambda = \{X \in \mathfrak{k} \mid \operatorname{ad}_H^2 X = \lambda(H)^2 X \textit{ for all } H \in \mathfrak{a}\}$$

and

$$\mathfrak{p}_\lambda = \{Y \in \mathfrak{p} \mid \operatorname{ad}_H^2 Y = \lambda(H)^2 Y \textit{ for all } H \in \mathfrak{a}\}.$$

d. We have

$$[\mathfrak{k}_\lambda, \mathfrak{k}_\mu] \subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu}, \quad [\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu}, \quad [\mathfrak{p}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu}.$$

e. Let $X \in \mathfrak{k}_\lambda$*,* $Y \in \mathfrak{p}_\lambda$*,* $H \in \mathfrak{a}$ *be nonzero vectors. Then* $\lambda(H) = 0$ *if and only if* $[H, X] = 0$ *if and only if* $[H, Y] = 0$*.*

Proof. (a) This follows from Proposition 2.5.2(a), since $\mathfrak{k}_\lambda + \mathfrak{p}_\lambda = \mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}$.

(b) Given λ, take $H \in \mathfrak{a} \setminus \ker \lambda$. Then ad_H sends $\mathfrak{k}_\lambda$ into $\mathfrak{p}_\lambda$ and $\mathfrak{p}_\lambda$ into $\mathfrak{k}_\lambda$, injectively. The assertion about the dimensions now follows from $\mathfrak{k}_\lambda + \mathfrak{p}_\lambda = \mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}$.

(c) ad_H^2 preserves $\mathfrak{k}$ and $\mathfrak{p}$ for $H \in \mathfrak{a}$, so the desired result again follows from $\mathfrak{k}_\lambda + \mathfrak{p}_\lambda = \mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda}$.

(d) This is a calculation analogous to Proposition 2.5.2(b).

(e) If $\lambda(H) = 0$ then $\operatorname{ad}_H^2 X = 0$ by part (c). Using Lemma 2.5.1:

$$0 = B_s(\operatorname{ad}_H^2 X, X) = B_s(\operatorname{ad}_H X, \operatorname{ad}_H X).$$

Hence $\operatorname{ad}_H X = 0$ by definiteness of B_s. The converse is clear, and the result for Y is analogous. □

2.5.10 Remark The results in Proposition 2.5.9 can also be phrased in the case of a symmetric space of compact type by using Cartan duality. On the maximal Abelian subalgebra $\mathfrak{a}_u := \sqrt{-1}\mathfrak{a} \subset \sqrt{-1}\mathfrak{p}$, the restricted roots take purely imaginary values.

2.5.11 Example Here is an example of using the restricted root system to do calculations in a symmetric space. We shall use the resulting formula in section 3.

The projections

$$\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda} \to \mathfrak{k}_\lambda, \qquad \mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda} \to \mathfrak{p}_\lambda$$

are respectively given by

$$Z \mapsto \frac{1}{2}(Z + sZ), \quad Z \mapsto \frac{1}{2}(Z - sZ).$$

Taking a basis $\{Z^i_\lambda\}_{i=1}^{m_\lambda} \subset \mathfrak{g}_\lambda$ for $\lambda \in \Delta^+(\mathfrak{g},\mathfrak{a})$ and setting $Z^i_{-\lambda} = sZ^i_\lambda$, we have, dropping the "i" from the notation,

$$[H, Z_\lambda + Z_{-\lambda}] = \lambda(H)(Z_\lambda - Z_{-\lambda}), \qquad [H, Z_\lambda - Z_{-\lambda}] = \lambda(H)(Z_\lambda + Z_{-\lambda}).$$

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}_u$ be the Cartan dual of $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and take $\mathfrak{a}_u = \sqrt{-1}\mathfrak{a}$. Then

$$[\sqrt{-1}H, Z_\lambda + Z_{-\lambda}] = \lambda(H)\sqrt{-1}(Z_\lambda - Z_{-\lambda})$$

and

$$[\sqrt{-1}H, \sqrt{-1}(Z_\lambda - Z_{-\lambda})] = -\lambda(H)(Z_\lambda + Z_{-\lambda}).$$

Now

$$X_\lambda = Z_\lambda + Z_{-\lambda} \in \mathfrak{k}_\lambda, \qquad Y_\lambda = \sqrt{-1}(Z_\lambda - Z_{-\lambda}) \in (\mathfrak{p}_u)_\lambda$$

and $H' = \sqrt{-1}H \in \mathfrak{a}_u$ satisfy

$$[H', X_\lambda] = -\sqrt{-1}\lambda(H')Y_\lambda, \qquad [H', Y_\lambda] = \sqrt{-1}\lambda(H')X_\lambda.$$

Put $\lambda(H') = \sqrt{-1}t$ for $t \in \mathbf{R}$. Then

$$\begin{aligned}
\mathrm{Ad}_{\exp H'} Y_\lambda &= e^{\mathrm{ad}_{H'}} Y_\lambda \\
&= \sum_{k=0}^\infty \frac{1}{k!}\mathrm{ad}^k_{H'} Y_k \\
&= \sum_{k=0}^\infty \frac{1}{(2k)!}\lambda(H')^{2k} Y_\lambda + \sqrt{-1}\sum_{k=0}^\infty \frac{1}{(2k+1)!}\lambda(H')^{2k+1} X_\lambda \\
&= \sum_{k=0}^\infty \frac{1}{(2k)!}(-1)^k t^{2k} Y_\lambda - \sum_{k=0}^\infty \frac{1}{(2k+1)!}(-1)^k t^{2k+1} X_\lambda \\
&= \cos t\, Y_\lambda - \sin t\, X_\lambda \\
&= \cos t\, Y_\lambda + \frac{\sin t}{t}\mathrm{ad}_{H'} Y_\lambda \\
&= \cos\sqrt{-\lambda(H')^2}\, Y_\lambda + \frac{\sin\sqrt{-\lambda(H')^2}}{\sqrt{-\lambda(H')^2}}\mathrm{ad}_{H'} Y_\lambda. \qquad (2.5.12)
\end{aligned}$$

Note that, in formula (2.5.12), Y_λ can be considered as an arbitrary element of $(\mathfrak{p}_u)_\lambda$.

2.5.13 Remark A real simple Lie algebra is completely determined by the abstract root system together with the multiplicities (see [23, Ex. 9, Ch. X, p. 535]). Equivalently, the classification of irreducible symmetric spaces is equivalent to the determination of the possible abstract root systems together with their multiplicities, and the construction of a symmetric space (and its dual) for each possibility.

2.6 Problems

2.6.1 Problem Let $\mathfrak{g}$ be a complex Lie algebra and denote its Killing form by β. Prove that the Killing form of the realification of $\mathfrak{g}$ is twice the real part of β.

2.6.2 Problem Let $\mathfrak{g}$ be a complex simple Lie algebra. Prove that the realification $\mathfrak{g}^r$ is a real simple Lie algebra. (Hint: Show that any ideal of $\mathfrak{g}^r$ is invariant under the complex structure.)

2.6.3 Problem Let $\mathfrak{g}$ be a non-Abelian Lie algebra and let σ be an involutive automorphism of $\mathfrak{g}$. Show that the fixed point set $\mathfrak{g}^\sigma$ is not zero.

2.6.4 Problem Let T be a torus, and let σ be an involutive automorphism of T with fixed point set U. Prove that the map $\Phi : T \times U \to T$ given by $\Phi(t, u) = \sigma(t)ut^{-1}$ is surjective. (Hint: Show it is open and closed by computing its differential.)

2.6.5 Problem Explain with $\mathfrak{sl}(2, \mathbf{R})$ is not a compact Lie algebra.

2.6.6 Problem Prove that a complex compact Lie algebra is trivial.

2.6.7 Problem Let G be a connected semisimple Lie group whose Lie algebra has a complex structure. Check that G has finite center.

2.6.8 Problem Let $\mathfrak{g}$ be a noncompact real semisimple Lie algebra, let s be a Cartan involution, write $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ for the Cartan decomposition. An s-invariant CSA $\mathfrak{h}$ of $\mathfrak{g}$ is called *maximally compact* (resp. *maximally noncompact*) if $\mathfrak{h} \cap \mathfrak{k}$ (resp. $\mathfrak{h} \cap \mathfrak{p}$) is a maximal Abelian subalgebra of $\mathfrak{k}$ (resp. $\mathfrak{p}$).

a. Prove that two maximally compact s-invariant CSA of $\mathfrak{g}$ are conjugate under an inner automorphism of $\mathfrak{g}$ induced by $\mathfrak{k}$. (Hint: Use Lemma 2.3.5.)

b. Prove that two maximally noncompact s-invariant CSA are conjugate under an inner automorphism of $\mathfrak{g}$ induced by $\mathfrak{k}$. (Hint: Use Proposition 2.5.5 and the conjugacy of maximal tori in a compact connected Lie group.)

2.6.9 Problem Derive the global Iwasawa decomposition for $G = \mathbf{SL}(n, \mathbf{R})$ from the Gram-Schmidt orthonormalization process.

2.6.10 Problem Let G/K be a symmetric space of maximal rank, where G is semisimple. Show that $\dim M = \frac{1}{2}(\dim G + \operatorname{rank} G) = \operatorname{rank} G + \dim K$.

2.6.11 Problem Let $M = G/K$ be a symmetric space of noncompact type, where G is the transvection group. Show that K is a maximal compact subgroup of G.

2.6.12 Problem Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition of a real semisimple Lie algebra. Let K be the connected subgroup of $\mathrm{GL}(\mathfrak{g})$ generated by $\exp \mathrm{ad}_X$ for $X \in \mathfrak{k}$. Assume $\mathfrak{a}_1$, $\mathfrak{a}_2$ are two Cartan subspaces of $\mathfrak{p}$ such that $H \in \mathfrak{a}_1 \cap \mathfrak{a}_2$. Show that there exists $k_0 \in K$ such that $\mathrm{Ad}_{k_0}\mathfrak{a}_1 = \mathfrak{a}_2$ and $\mathrm{Ad}_{k_0}H = H$. (Hint: Adapt the idea of the proof of Proposition 2.5.5.)

2.6.13 Problem (i) Generalize Example 1.5.2 to construct the orthogonal involutive Lie algebra associated to the Grassmann manifold $Gr(k, \mathbf{R}^n)$ of unoriented k-planes in $\mathbf{R}^n$.

(ii) Explain the Riemannian covering $Gr^+(k, \mathbf{R}^n) \to Gr(k, \mathbf{R}^n)$, in terms of the canonical presentation of the symmetric spaces as homogeneous spaces, where $Gr^+(k, \mathbf{R}^n)$ denotes Grassmann manifold $Gr(k, \mathbf{R}^n)$ of oriented k-planes in $\mathbf{R}^n$.

2.6.14 Problem (i) Identify $\mathbf{SO}(n)/\mathbf{U}(n)$ with the space of orthogonal complex structures on $\mathbf{R}^{2n}$, where $\mathbf{U}(n)$ is embedded into $\mathbf{SO}(2n)$ as the subgroup of matrices of the form $\left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right)$.

(ii) What is the involution s of $\mathfrak{so}(2n)$ whose fixed point set is $\mathfrak{u}(n)$? What is the (-1)-eigenspace of s?

2.6.15 Problem (i) Identify $\mathbf{SL}(n, \mathbf{R})/\mathbf{SO}(n)$ with the convex cone of $n \times n$ positive definite symmetric matrices (cf. Example 2.5.8). .

(ii) Show that $\mathbf{SL}(2, \mathbf{R})/\mathbf{SO}(2)$ is isometric to the real hyperbolic plane $\mathbf{R}H^2$.

2.6.16 Problem Show that a symmetric space of dimension at most 3 must have constant curvature.

2.6.17 Problem Prove that the only 4-dimensional compact symmetric spaces are S^4 and $\mathbf{C}P^2$.

3 Lecture 3: Geometry

The Borel lectures seem to be the starting point for the wide dissemination of Élie's Cartan's theory of symmetric spaces and, in the late 1950's and through the 1960´s, they started to received more attention. It was realized that symmetric spaces help unify and explain in a general way various phenomena in classical geometries, in addition to its applications to functions of several complex variables, number theory and topology.

For one thing, an interesting connection between symmetric spaces and holonomy was noticed: the de Rham decomposition theorem (1952) and Berger's classification of holonomy groups (1953). It then became clear that almost all holonomy groups occurred for symmetric spaces and therefore gave good approximating geometries to most holonomy groups. An even more interesting question also came out of this, namely, what about those few holonomy groups that do not occur for symmetric spaces?

In the early 1950's Bott devised the concept of variational completeness for isometric group actions, which roughly translates to the absence of conjugate points in the quotient, and developed powerful Morse-theoretic methods to compute the homology and cohomology of their orbits. These methods were put to use in the study of the topology of symmetric spaces (together with Samelson), as their isotropy representations are variationally complete, and in the proof (in 1957) of the celebrated Bott periodicity theorem for the stable homotopy groups of the classical groups, where loop spaces are interpreted as symmetric spaces.

As later developments in the geometric theory, we would also like to quote various rigidity theorems in the realm of symmetric spaces, like e.g. Mostow's rigidity and higher rank rigidity of Ballmann and Burns-Spatzier.

Symmetric spaces play a central role in modern differential geometry, with a list of connections and ramifications in different areas of Mathematics and Mathematical Physics that is so long to compilate, that it is better left to the interested reader to search and investigate, according to his/her tastes.

3.1 Curvature

The calculation of the curvature of symmetric spaces was already known to Cartan. Let M be a locally symmetric space with Levi-Cività connection ∇ and curvature tensor R. We use the sign convention that

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for vector fields X, Y, Z on M.

3.1.1 Lemma *Let M be a locally symmetric space and fix $x \in M$. Let X be an infinitesimal transvection at x and let Y be any vector field defined on a neighborhood of x. Then $(\nabla_X Y)_x = (L_X Y)_x$.*

Proof. Let $\{p_t\}$ denote the local one-parameter group of local transvection generated by X. Since p_t induces parallel transport of vectors along the geodesic $\gamma(t) = p_t(x)$, we have

$$\begin{aligned}
(L_X Y)_x &= \frac{d}{dt}\Big|_{t=0} dp_{-t}(X_{\gamma(t)}) \\
&= \frac{d}{dt}\Big|_{t=0} P^{\gamma}_{0,t}(X_{\gamma(t)}) \\
&= (\nabla_X Y)_x,
\end{aligned}$$

where $P^{\gamma}_{t_1,t_0}$ denotes parallel transport along γ from t_0 to t_1. $\square$

3.1.2 Proposition *Let M be a locally symmetric space, $x \in M$ and X, Y, $Z \in T_xM$. Then*

$$R_x(X,Y)Z = -[[X,Y],Z]_x, \tag{3.1.3}$$

where on the right-hand side we use the infinitesimal transvection induced by the corresponding tangent vector.

Proof. Let γ be the geodesic with initial speed $X_x \in T_xM$. Since Y is a Killing field, its restriction along γ is a Jacobi field. Therefore, using $\frac{\nabla}{dt}$ to denote the covariant derivative along γ,

$$\begin{aligned} R(X,Y)X &= \left(\frac{\nabla}{dt}\frac{\nabla}{dt}Y\right)\Big|_{t=0} \\ &= \left(\frac{\nabla}{dt}[X,Y]_{\gamma(t)}\right)_{t=0} \qquad \text{(by Lemma 3.1.1)} \\ &= [X,[X,Y]]_{\gamma(0)} \qquad \text{(idem)} \\ &= -[[X,Y],X]_x. \end{aligned}$$

The proof is finished by recalling that the sectional curvature determines the curvature tensor and that the right-hand side of (3.1.3) has the symmetries of the curvature tensor. □

3.1.4 Corollary *Let M be an irreducible locally symmetric space, $x \in M$, and consider the OIL-algebra $(\mathfrak{g}, s, B)$ at x. Recall that $B = \lambda\beta|_{\mathfrak{p}\times\mathfrak{p}}$ for some $\lambda \neq 0$, where β is the Killing form of $\mathfrak{g}$. Then the sectional curvature of the 2-plane spanned by an orthonormal pair X, $Y \in T_xM \cong \mathfrak{p}$ is given by*

$$K(X,Y) = \lambda\beta([X,Y],[X,Y]).$$

Proof. Using formula (3.1.3) and the ad-invariance of the Killing form, we have

$$K(X,Y) = -B(R(X,Y)X,Y) = \lambda\beta([[X,Y],X],Y) = \lambda\beta([[X,Y],[X,Y]]),$$

as desired. □

It follows from this corollary that Cartan duality changes the signs of the sectional curvatures of the tangent planes. Indeed symmetric spaces of compact (resp. noncompact) type has nonnegative (resp. nonpositive) curvature.

3.1.5 Corollary *Let M be a locally symmetric space and construct the simply-connected symmetric space $\tilde{M}$ associated to the OIL-algebra of M. Given $x \in M$, $\tilde{x} \in \tilde{M}$, there exist neighborhoods U, $\tilde{U}$ of x, $\tilde{x}$ and an isometry $\varphi : \tilde{U} \to U$ such that $\varphi(\tilde{x}) = x$. If M is complete, then φ extends to the universal Riemannian covering.*

Proof. Let $\tilde{R}$ and R be the curvature tensors of $\tilde{M}$ and M. By formula (3.1.3), there exists a linear isometry $I : T_{\tilde{x}}\tilde{M} \to T_xM$ which sends $\tilde{R}_{\tilde{x}}$ to R_x. Since $\tilde{R}$ and R are parallel, I extends to an isometry $\varphi : \tilde{U} \to U$ of normal neighborhoods by Theorem 1.3.3. If M is complete, φ further extends to a universal Riemannian covering by Theorem 4.2.1. □

3.2 Totally geodesic submanifolds

Let M be a Riemannian manifold. Recall that an isometric immersion $f : N \to M$ is called *totally geodesic* if the geodesics of N are geodesics of M. A necessary and sufficient condition is that the second fundamental form vanishes identically. In stark contrast to general Riemannian manifolds, we shall see that symmetric spaces posses an abundance of totally geodesic submanifolds.

Let M be a locally symmetric space. Fix a base-point $x \in M$ and denote by $(\mathfrak{g}, s, B)$ the associated OIL-algebra. Recall that the geodesics of M through x are of the form $t \mapsto \exp(tX) \cdot x$ for $X \in T_x M$.

3.2.1 Proposition *Let M be a locally symmetric space. Then every totally geodesic submanifold N has an induced structure of locally symmetric space.*

Proof. Let $x \in N$. It is obvious that the geodesic symmetry s_x of M locally leaves N invariant. □

A subspace $\mathfrak{s}$ of a Lie algebra is called a *Lie triple system* if $[[X, Y], Z] \in \mathfrak{s}$ for every X, Y, $Z \in \mathfrak{s}$.

3.2.2 Theorem *Let M be a symmetric space. The connected complete totally geodesic submanifolds of M passing through x are precisely of the form* $\exp[\mathfrak{s}] \cdot x$, *where $\mathfrak{s} \subset \mathfrak{p}$ is a Lie triple system.*

Proof. Suppose N is a totally geodesic submanifold of M passing through x. Let $\mathfrak{s} \subset \mathfrak{p}$ be the subspace corresponding to $T_x N \subset T_x M$. Due to total-geodesicness, the curvature tensor of M restricts to that of N. By Proposition 3.1.2, we get $[[\mathfrak{s}, \mathfrak{s}], \mathfrak{s}] \subset \mathfrak{s}$, hence $\mathfrak{s}$ is a Lie triple system. If N is complete and connected, every one of its points can be joined to x by a (minimizing) geodesic. It follows that $N = \exp[\mathfrak{s}] \cdot x$.

Conversely, suppose $\mathfrak{s} \subset \mathfrak{p}$ is a Lie triple system. Then $\mathfrak{b} = [\mathfrak{s}, \mathfrak{s}] + \mathfrak{s}$ is a subalgebra of $\mathfrak{g}$. Denote by B the associated connected subgroup of G. Then the orbit $N = B(x)$ is a connected homogeneous submanifold of M such that the induced Riemannian metric is B-invariant. If $X \in \mathfrak{s}$, then the geodesic $t \mapsto \exp(tX) \cdot x$ of M is contained in N. It follows that N is totally geodesic at x; hence, owing to homogeneity, it is totally geodesic everywhere. It is now obvious that N is complete and $N = \exp[\mathfrak{s}] \cdot x$. □

Examples of complete totally geodesic submanifolds are more interesting if they are closed.

3.2.3 Proposition *A maximal connected complete totally geodesic submanifold of a symmetric space is properly embedded.*

Proof. Let $M = G/K$ be a symmetric space where $K = G_x$, and let $N = \exp[\mathfrak{s}] \cdot x$ be a totally geodesic submanifold as in Theorem 3.2.2. Put $\tilde{\mathfrak{b}} = N_{\mathfrak{k}}(\mathfrak{s}) + \mathfrak{s}$, where $N_{\mathfrak{k}}(\mathfrak{s})$ is the normalizer of $\mathfrak{s}$ in $\mathfrak{k}$. Maximality of the LTS $\mathfrak{s}$ implies that $\tilde{\mathfrak{b}}$ is a self-normalizing Lie subalgebra of $\mathfrak{g}$. It follows that the corresponding connected subgroup $\tilde{B}$ is the connected normalizer of $\tilde{\mathfrak{b}}$ in G. In particular, $\tilde{B}$ is a closed subgroup of G. Since $N = \tilde{B}(x)$, the desired result follows. $\square$

Although the determination of totally geodesic submanifolds is reduced to an algebraic problem, it has only been accomplished in low rank or under additional hypothesis. In particular, the work of Chen and Nagano [13] is noteworthy for the geometric ideas introduced (see also [25], for an approach based on restricted root systems).

A symmetric space admits a totally geodesic submanifold of codimension one only in case it has constant curvature. The minimal codimension of a totally geodesic submanifold of a symmetric space was investigated by Onishchik and, recently, it has been computed for almost all irreducible symmetric spaces by Berndt and Olmos [3].

3.3 Maximal flats and rank

Let $M = G/K$ be a symmetric space where G is the connected isometry group of M and $(\mathfrak{g}, s, B)$ is the OIL-algebra at $x \in M$.

A complete connected totally geodesic flat submanifold of M will be simply called a *flat*. A flat is said to be *maximal* if it is not properly contained in another flat. It follows from Theorem 3.2.2 and Proposition 3.1.2 that a maximal flat through x has the form $F = \exp[\mathfrak{a}]x$ where $\mathfrak{a}$ is Cartan subalgebra of $\mathfrak{p}$. It follows from Proposition 2.5.5 that all the maximal flats of M are conjugate and hence have the same dimension. This number is called the *rank* of the symmetric space.

3.3.1 Proposition *A maximal flat F of M is a properly embedded submanifold. If M is of compact (resp. noncompact) type, then F is isometric to a flat torus (resp. flat Euclidean space).*

Proof. We have $F = Ax$ and $A = \exp[\mathfrak{a}]$ is the connected Abelian subgroup of G with Lie algebra some Cartan subspace $\mathfrak{a}$. F is properly embedded if A is closed in G. In fact, the closure $\bar{A}$ is also a connected Abelian group with Lie algebra contained in $\mathfrak{p}$ (because the involution σ of G satisfies $\sigma(g) = g^{-1}$ for $g \in A$ and thus for $g \in \bar{A}$). Hence $\bar{A} = A$. If M is of compact type, F is a compact homogeneous flat manifold, thus isometric to a torus. If M is of noncompact type, the exponential map $\exp_x : T_x\mathbf{R}^n \to M$ is a diffeomorphism and the result follows (cf. Corollary 2.4.7). $\square$

3.4 Conjugate locus

In this section, we describe the conjugate locus of a point in a symmetric space of compact type $M = G/K$. By homogeneity, it suffices to study the conjugate locus of the basepoint x_0. Furthermore, it is clear that the conjugate locus of x_0 is K-invariant, so it is completely determined by the conjugate locus along a maximal flat passing through x_0.

First we describe Jacobi fields on a locally symmetric space. It is convenient to introduce the functions

$$c_\alpha(t) = \begin{cases} \cos(\sqrt{\alpha}t) & \text{if } \alpha > 0, \\ 1 & \text{if } \alpha = 0, \\ \cosh(\sqrt{-\alpha}t) & \text{if } \alpha < 0, \end{cases} \quad \text{and} \quad s_\alpha(t) = \begin{cases} \frac{\sin(\sqrt{\alpha}t)}{\sqrt{\alpha}} & \text{if } \alpha > 0, \\ t & \text{if } \alpha = 0, \\ \frac{\sinh(\sqrt{-\alpha}t)}{\sqrt{-\alpha}} & \text{if } \alpha < 0. \end{cases}$$

Let γ be a geodesic with initial speed $v \in T_{x_0}M$. Since the curvature tensor is parallel along γ, the Jacobi equation

$$-Y'' + R(\gamma', Y)\gamma' = 0$$

has constant coefficients when expressed in a parallel frame along γ. The self-adjoint endomorphism $u \mapsto -R(v,u)v$ of $T_{x_0}M$ has real eigenvalues $\alpha_0 = 0$, $\alpha_1, \ldots, \alpha_{n-1}$ with corresponding orthonormal eigenbasis $v_0 = v$, $v_1, \ldots, v_{n-1}$, so the Jacobi equation splits into n independent constant coefficient second-order linear ordinary differential equations

$$y_i'' + \alpha_i y_i = 0,$$

where $\alpha_i = -\langle R(v,v_i)v, v_i\rangle$. Hence a basis of solutions of the Jacobi equation can be given as

$$Y_i(t) = c_{\alpha_i}(t)E_i(t), \qquad Z_i(t) = s_{\alpha_i}(t)E_i(t)$$

where $\{E_0 = \gamma', E_1, \ldots, E_{n-1}\}$ is a parallel orthonormal frame along γ.

3.4.1 Proposition (Variational completeness) *Let M be a symmetric space and $\gamma : [0, +\infty) \to M$ be a geodesic ray. If $\gamma(t_0)$, $t_0 > 0$, is a conjugate point to $\gamma(0)$ along γ, then there exists a nontrivial one-parameter group of transvections $\{p_t\}$ fixing both $\gamma(0)$ and $\gamma(t_0)$.*

Proof. By the discussion above, there exists a Jacobi field along γ of the form $J(t) = s_\alpha(t)E(t)$ and $J(t_0) = 0$. Since $s_\alpha(t_0) = 0$, we have $\alpha = (m\pi/t_0)^2$ for some integer $m \neq 0$ (note that we must have $\alpha > 0$). There exists $t_1 \in (0, t_0)$ such that $c_\alpha(t_1) = 0$. This means $J'(t_1) = 0$. Let $\{p_t\}$ be the one-parameter group of transvections induced by $J(t_1)$, and set $Z = \frac{d}{dt}|_{t=0} p_t$. Since the restriction of Z along γ is a Jacobi field and $(\nabla Z)_{\gamma(t_1)} = 0 = J'(t_1)$, $Z_{\gamma(t_1)} = J(t_1)$, we get $Z \circ \gamma = J$. In particular, $Z(\gamma(0)) = Z(\gamma(t_0)) = 0$. Using that $\{p_t\}$ is a one-parameter group of isometries, we finally see that $p_t(\gamma(0)) = \gamma(0)$ and $p_t(\gamma(t_0)) = \gamma(t_0)$ for all t. □

3.4.2 Remark The ideas in Proposition 3.4.1 provide another proof that a symmetric space M of noncompact type is simply-connected. In fact, suppose M is not simply-connected. Then in a nontrivial free homotopy class of loops we can find a closed geodesic γ (of minimal length in that class). Now there is a parallel vector field $E(t)$ along γ such that $J(t) = \cosh(\sqrt{-\alpha}t)E(t)$ is a

Jacobi field along γ; in particular, $||J||$ is unbounded. On the other hand, since $J'(0) = 0$, J must be the restriction along γ of the infinitesimal transvection $Z \in \mathfrak{p}$ with $Z_{\gamma(0)} = J(0)$; therefore $||J||$ is bounded, as γ is a closed curve, a contradiction.

3.4.3 Remark Proposition 3.4.1 already implies that the conjugate locus of a point x is a symmetric space of compact type $M = G/K$, where $K = G_x$, consists of singular points of the K-action. In fact, since $x = \gamma(0)$ is a fixed point of K, the geodesic γ is orthogonal to every K-orbit it meets, in particular to Ky, where $y = \gamma(t_0)$. The existence of the nontrivial one-parameter of transvections $\{p_t\}$ shows that the isotropy subgroup K_y does not fix the normal vector $-\gamma'(t_0)$. Hence y is a singular point.

We now give a more precise, algebraic description of the conjugate locus.

3.4.4 Theorem *Let $M = G/K$ be a symmetric space of compact type where G is the connected group of isometries and K is the isotropy subgroup at a basepoint x. Fix a maximal flat F_u passing through x and let $T_xF_u = \mathfrak{a}_u$. Then the conjugate locus of x consists precisely of the points of the form $(k \exp H)x$, where $k \in K$ and $H \in \mathfrak{a}_u$ satisfies $\lambda(H) = m\pi\sqrt{-1}$ for some $\lambda \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ and some integer $m \neq 0$.*

Proof. Let y be a point conjugate to x along some geodesic. Without loss of generality, we may assume that $y = (\exp H)x$ for some nonzero $H \in \mathfrak{a}_u$. Let $t_0 = ||H|| > 0$. The Jacobi operator $-R(\frac{H}{t_0}, X)\frac{H}{t_0} = -\frac{1}{t_0^2}\mathrm{ad}_H^2 X$ for $X \in T_xM \cong \mathfrak{p} = \mathfrak{a}_u \oplus \sum_\lambda \mathfrak{p}_\lambda$, and its nonzero eigenvalues are $-\lambda(H)^2/t_0^2$ for $\lambda \in \Delta^+(\mathfrak{g}, \mathfrak{a})$. By the argument in the proof of Proposition 3.4.1, our conjugate point must satisfy $-\lambda(H)^2/t_0^2 = (m\pi/t_0)^2$ for some $\lambda \in \Delta^+(\mathfrak{g}, \mathfrak{a})$ and some integer $m \neq 0$. The desired result follows. $\square$

3.4.5 Remark In Theorem 3.4.4, the Jacobi fields associated to a conjugate point $y = (\exp H)x$ of x along $t \mapsto \exp_x(t\frac{H}{||H||})x$, for $H \in \mathfrak{a}_u$, have the form $J(t) = s_\alpha(t)E(t)$, where $\alpha = (m\pi/||H||)^2 = -\lambda(H/||H||)^2$, and $E(0) \in \mathfrak{p}_\lambda$ is an eigenvector of the Jacobi operator $-\mathrm{ad}^2_{\frac{H}{||H||}}$ with eigenvalue α. It follows that the multiplicity of y as a conjugate point is $\sum m_\lambda$, where λ runs through all positive restricted roots such that $\frac{1}{\pi\sqrt{-1}}\lambda(H)$ is a non-zero integer.

3.5 Cut locus

We next explain the result that, in a simply-connected symmetric space of compact type, the cut locus coincides with the first conjugate locus, a result probably first published by Crittenden [17].

Let $M = G/K$ be a symmetric space of compact type where G is the connected group of isometries and K is the isotropy subgroup at a basepoint x. Fix a maximal flat F_u passing through x and let $T_xF_u = \mathfrak{a}_u$. It is clear that the cut locus of x is K-invariant, so it suffices to describe it along F_u.

3.5.1 Proposition *For a unit vector $H \in \mathfrak{a}_u$, suppose $y = \exp(t_0 H)x$ is a cut point of x along $t \mapsto \exp(tH)x$. Then either y is the first conjugate point of x along γ or there exists a unit vector $H' \in \mathfrak{a}_u$, $H' \neq H$, such that $y = \exp(t_0 H')x$.*

Proof. It is known that if y is not the first conjugate point of x along γ, then y is not conjugate to x and there exists a unit vector $Y \in \mathfrak{p}_u$, $Y \neq H$, such that $y = \exp(t_0 Y)x$. We will prove that $[H, Y] = 0$. In fact, suppose $[H, Y] \neq 0$. Then $Y \notin \mathfrak{a}_u$. Since $\exp(t_0 H)x = \exp(t_0 Y)x$, we have $\exp(-t_0 H)\exp(t_0 Y) = k$ for some $k \in K$. Now

$$\exp(-t_0 H)\exp(sY)\exp(t_0 H) = k\exp(sY)k^{-1}$$

for all $s \in \mathbf{R}$, so

$$\mathrm{Ad}_{\exp(-t_0 H)} Y = \mathrm{Ad}_k Y.$$

Write $Y = Y_0 + \sum_\lambda Y_\lambda \in \mathfrak{a} + \sum_\lambda (\mathfrak{p}_u)_\lambda$. Then $Y_\lambda \neq 0$ for some λ. By equation (2.5.12)

$$\mathrm{Ad}_{\exp(-t_0 H)} Y_\lambda = \cos\sqrt{-\lambda(t_0 H)^2}\, Y_\lambda + \frac{\sin\sqrt{-\lambda(t_0 H)^2}}{\sqrt{-\lambda(t_0 H)^2}}\, \mathrm{ad}_{-t_0 H} Y_\lambda.$$

Since $\mathrm{Ad}_k Y \in \mathfrak{p}$, we must have $\lambda(t_0 H) = m\pi\sqrt{-1}$ for some integer m. Also, $[H, Y] \neq 0$ implies that $m \neq 0$. However, owing to Theorem 3.4.4, this contradicts the fact that y is not a conjugate point of x along γ. Thus we get $[H, Y] = 0$. Now H and Y lie in a Cartan subspace $\mathfrak{a}'_u$. There exists $k' \in K$ such that $\mathrm{Ad}_{k'} H = H$ (in particular $k' y = y$) and $\mathrm{Ad}_{k'} \mathfrak{a}'_u = \mathfrak{a}_u$ (cf. Problem 2.6.12). We are done by taking $H' = \mathrm{Ad}_k Y$. $\square$

Put

$$(\mathfrak{a}_u)_K = \{H \in \mathfrak{a}_u \mid \exp H \in K\}.$$

Then $(\mathfrak{a}_u)_K$ is a lattice in $\mathfrak{a}_u$ and F_u is isometric to the quotient $\mathfrak{a}_u/(\mathfrak{a}_u)_K$. It follows from Proposition 3.5.1 that a cut point of x is either a first conjugate point or a cut point along a maximal flat. The tangential cut locus of x in $T_x F_u = \mathfrak{a}_u$ is easy to compute; it is determined by the hyperplanes in $(\mathfrak{a}_u)_K$ which are equidistant from 0 and a point in $(\mathfrak{a}_u)_K$, that is it is the boundary of the convex set D_x given by the intersection of the half-spaces

$$\{H' \in \mathfrak{a}_u \ :\ |\langle H', H\rangle| \leq ||H||^2/2\} \tag{3.5.2}$$

for $H \in (\mathfrak{a}_u)_K$.

3.5.3 Lemma *Let $M = G/K$ be a symmetric space of compact type, where G is simply-connected and K is a connected. Then $(\mathfrak{a}_u)_K$ is the lattice generated by*

$$A(\lambda) = 2\pi\sqrt{-1}\frac{A_\lambda}{||\lambda||^2},$$

where $\lambda \in \Delta(\mathfrak{g}, \mathfrak{a})$, and $A_\lambda \in \mathfrak{a} = \sqrt{-1}\mathfrak{a}_u$ is defined by $B(A_\lambda, H) = \lambda(H)$ for all $H \in \mathfrak{a}$.

Proof. (Sketch) Since G is simply-connected and K is connected, one shows that $K = G^\sigma$, where $d\sigma = s$ [23, Theorem 8.2]. It follows that $\exp H \in K$ if and only if $\exp(-H) = \sigma(\exp H) = \exp H$ if and only if $\exp(2H) = 1$, showing that $(\mathfrak{a}_u)_K = \frac{1}{2}(\mathfrak{a}_u)_1$, where

$$(\mathfrak{a}_u)_1 = \{H \in \mathfrak{a}_u \ : \ \exp H = 1\}.$$

One needs to analyze the relation between the maximal flat in G/K and the maximal torus in G, and recall the description of the unit lattice in a simply-connected Lie group. We skip the technical details and refer the reader to [23, Corollary 7.8 and Theorem 8.5] or [4]. □

3.5.4 Theorem *Let $M = G/K$ be a simply-connected symmetric space of compact type. Then the cut locus of a point coincides with its first conjugate locus.*

Proof. In view of Proposition 3.5.1, it suffices to show that the cut locus of the basepoint x along the maximal flat coincides with the first conjugate locus.

In Lemma 3.5.3, note that $\lambda(A_\lambda) = 2\pi\sqrt{-1}$. It follows that the tangential cut locus along $\mathfrak{a}_u$, as given by (3.5.2), is the boundary of

$$\bigcap_{\lambda\in\Delta(\mathfrak{g},\mathfrak{a})} \{H' \in \mathfrak{a}_u \ : \ |\lambda(H')| \leq \pi\}.$$

Due to Theorem 3.4.4, this is also the first tangential conjugate locus of x, as wished. □

3.5.5 Remark Let $y = (\exp H)x$, for $H \in \mathfrak{a}_u$ be a conjugate point to x along $\gamma(t) = \exp(t\frac{H}{||H||})x$, where $\frac{1}{\pi}\sqrt{-1}\lambda(H)$ is a non-zero integer for exactly one $\lambda \in \Delta^+(\mathfrak{g},\mathfrak{a})$. Then the multiplicity of y is exactly m_λ, the associated Jacobi fields have the form $J(t) = s_\alpha(t)E(t)$, where $\alpha = -\lambda(H/||H||)^2 > 0$, and E is the parallel extension along γ of $E(0) = Y_\lambda \in \mathfrak{p}_\lambda$. We can write $Y_\lambda = \frac{1}{m\pi}\mathrm{ad}_H X_\lambda$ for some $X_\lambda \in \mathfrak{k}_\lambda$. This is another way to see the variational completeness of Proposition 3.4.1, namely, J is induced by the Killing field $X_\lambda \in \mathfrak{k}_\lambda$ along γ. In particular $X_\lambda y = 0$. Indeed the isotropy algebra of y under the isotropy action of K is $\mathfrak{m} + \mathfrak{k}_\lambda$ (the isotropy algebra of a generic point in F_u is $\mathfrak{m}$).

Next, view the first conjugate locus of x in $\mathfrak{a}_u \subset T_xM$ as the closure of the union of the sets C_λ consisting of points the form $(\exp H)x$, where $\frac{1}{\pi}\sqrt{-1}\lambda(H) = \pm 1$ and $\frac{1}{\pi}\sqrt{-1}\mu(H)$ is not an integer for $\mu \neq \lambda$. The isotropy group of any point in $\exp C_\lambda$, denoted $K_{[\lambda]}$, has Lie algebra $\mathfrak{m}+\mathfrak{k}_\lambda$. The first conjugate locus $Conj^1(x)$ is contained in the closure of $\cup_\lambda K(\exp C_\lambda)$.

We can estimate the codimension of $Conj^1(x)$, as a metric space, as follows. For each λ, there is a product decomposition $K(\exp C_\lambda]) \approx K/N_K(K_{[\lambda]}) \times (\exp C_\lambda)$, where $N_K(K_{[\lambda]})$ denotes the normalizer of $K_{[\lambda]}$ in K. Therefore $\dim K(\exp C_\lambda) = \dim K - \dim N_K(K_{[\lambda]}) + \dim C_\lambda$. It is easy to see that $N_K(K_{[\lambda]})$ has as Lie algebra the normalizer in $\mathfrak{g}$ of $\mathfrak{m} + \mathfrak{k}_\lambda$, but the later is self-normalizing. Therefore $\dim K(\exp C_\lambda) = \dim \mathfrak{k} - (\dim \mathfrak{m} + \dim \mathfrak{k}_\lambda) + \dim \mathfrak{a} - 1 = (\dim \mathfrak{k} - \dim \mathfrak{m}) - m_\lambda + \dim \mathfrak{a} - 1 = (\dim \mathfrak{p} - \dim \mathfrak{a}) - m_\lambda + \dim \mathfrak{a} - 1 = \dim \mathfrak{p} - m_\lambda - 1$. It follows that $\cup_\lambda K(\exp C_\lambda)$ has codimension $1 + \min_\lambda m_\lambda$ in $\mathfrak{p}$.

3.5.6 Remark Theorem 3.5.4 can be used to prove that $\pi_2(M)$ is trivial for a symmetric space of compact type all of whose multiplicities are bigger than one. In fact, we can pass to the universal covering and assume M simply-connected. Fix a basepoint $x \in M$ and suppose $\dim M = n$. It is known that M is obtained from $\mathrm{Cut}(x)$ by attaching an n-dimensional cell in T_xM via the exponential map. Since $\mathrm{Cut}(x)$ coincides with the first conjugate locus of x, by Remark 3.5.5 the codimension of $\mathrm{Cut}(x)$ in M is at least $1 + \min_\lambda m_\lambda \geq 3$. Hence the image of a given continuous map $S^2 \to M$ can be first homotopically deformed so as to avoid $\mathrm{Cut}(x)$, and then contracted to a point, proving that $\pi_2(M)$ is trivial. In particular, $H_2(M, \mathbf{Z}) = 0$ if M is simply-connected, by the Hurewicz theorem.

3.6 Isometric actions on symmetric spaces

Bott introduced the concept of variational completeness in [5], and in [6] together with Samelson. Roughly speaking, an isometric action of a compact[9] Lie group G on a complete Riemannian manifold M is variationally complete if it produces enough Jacobi fields along geodesics to determine the multiplicities of the focal points to the orbits. More precisely, for any geodesic γ orthogonal to the G-orbits, and for every Jacobi field J along γ that generates a variation of γ through geodesics orthogonal to the G-orbits, and such that J vanishes at one point, there exists a Killing field induced by G whose restriction along γ coincides with J (Proposition 3.4.1 is a special case, as it applies to the isotropy action of K on M, where $\{x\}$ is an orbit, and its focal points are the conjugate points to x). The motivation of Bott and Samelson to consider variationally complete actions of G on M was to construct an explicit basis of cycles in the $\mathbf{Z}_2$-homology of the path space $\Omega(M; x, N)$, where N is an arbitrary G-orbit, $x \in M$, and the paths start at x and end at a point in N. In modern terminology, we can state their result as follows:

3.6.1 Theorem (Bott-Samelson) *The orbits of a variationally complete action are taut submanifolds (with respect to $\mathbf{Z}_2$-coefficients).*

Here a submanifold N of M is called *taut* if, for every nonfocal point x, the energy functional $E : \Omega(M; x, N) \to \mathbf{R}$, $E(\gamma) = \frac{1}{2}\int ||\gamma'||^2 ds$, is a perfect Morse function, that is, every critical point (geodesic) of E corresponds to a basis element of $H_*(\Omega(M; x, N))$. Indeed, Bott and Samelson provide an algorithm to construct an explicit cycle for each critical point. In the same paper, for a symmetric space G/K, they prove that the isotropy action of K on G/K, the $K \times K$-action on G by left and right-multiplication, and the linear isotropy action of K on $T_{x_0}(G/K) \cong \mathfrak{p}$ are variationally complete. Soon thereafter, Hermann [24] found a more general family of variationally complete actions on symmetric spaces. Namely, if K and H are both symmetric subgroups of the compact Lie group G, then the action of H on G/K is variationally complete.

[9] We can replace the assumption of compactness of the group by the properness of the action.

L. Conlon was a student of Bott. In [16], he notes that a sufficient condition for variational completeness is (in modern jargon) hyperpolarity. An isometric action of G on M is called *hyperpolar* if there exists an isometrically immersed submanifold Σ, flat with respect to the induced metric, that meets all G-orbits, and meets them always orthogonally; such a Σ is called a *section* of the action. If we do not require flatness of the section in this definition, the action is called simply *polar*.

3.6.2 Theorem (Conlon) *A hyperpolar action of a compact Lie group G on a complete Riemannian manifold M is variationally complete.*

Proof. Let $N = Gx$ be a fixed orbit and let y be a focal point of N (that is, a critical value of the normal exponential map) along a geodesic $\gamma : [0, \ell] \to M$ with $\gamma(0) = x$ and $\gamma(\ell) = y$. Then there exists a Jacobi field J along γ satisfying $J(0) \in T_xN$, $J'(0) + A_{\gamma'(0)}J(0) \in \nu_xN$ and $J(\ell) = 0$; denote by V the space of Jacobi fields satisfying the first two of these conditions, and note that $\dim V = \dim M$.

Fix $s_0 \in (0, \ell)$ such that $z = \gamma(s_0)$ is a regular point for the action of G and z is not a focal point of N. There exists a unique section Σ passing through z. Of course, Σ is flat and contains the image of γ. Since z is not a focal point of N, the map $J \in V \mapsto J(s_0) \in T_zM$ is a linear isomorphism.

Decompose $J = J^V + J^H$ where J^H is the orthogonal projection of J on Σ. Due to the total-geodesicness of Σ, both J^V and J^H are Jacobi fields along γ. Since J^H vanishes at $s = 0$ and $s = \ell$ and Σ is flat, we have $J^H \equiv 0$. Since z is a regular point, $J^V(s_0) \in T_z(Gz)$. Let $X \in \mathfrak{g}$ be such that $X \cdot z = J^V(s_0)$. Owing to $X \circ \gamma \in V$, we have $X \circ \gamma = J^V = J$, finishing the proof. □

The Bott-Samelson and Hermann examples are in fact hyperpolar, as noted by Conlon. Here we show:

3.6.3 Theorem *The isotropy action of a symmetric space is hyperpolar, with maximal flats as embedded sections.*

Proof. Let $M = G/K$ be a symmetric space, write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ under s as usual and let $\mathfrak{a} \subset \mathfrak{p}$ be a Cartan subspace. Then $F = \exp[\mathfrak{a}]$ is a maximal flat. The isotropy action of K on M is given by $k \cdot (gK) = (kg)K$. We shall prove that F is a section for this action. Of course F is flat.

We first note that any point x of M lies in a maximal flat. Since all maximal flats are conjugate via the isotropy action, this shows that the orbit Kx meets F. It remains to prove that whenever a K-orbit meets F, it meets perpendicularly.

Suppose $x \in F$. We need to check that T_xF and $T(Kx)$ are orthogonal. We have $x = ax_0$ for some $a \in A = \exp[\mathfrak{a}]$. We may assume G/K is irreducible and the Riemannian metric is induced by an Ad-invariant inner product on $\mathfrak{g}$. Now $K \times K$ acts on G by left and right translations, K acts on by left translations and the projection $\pi : G \to G/K$ is an equivariant Riemannian submersion, so

$$T_x(Kx) = d\pi_a(\mathfrak{k} \cdot a + a \cdot \mathfrak{k}),$$

which implies

$$a^{-1} \cdot T_x(Kx) = \pi_*(\mathrm{Ad}_{a^{-1}}\mathfrak{k}),$$

where $\pi_* : \mathfrak{g} \to \mathfrak{p}$ is the projection. Taking orthogonal complements, we obtain

$$a^{-1} \cdot \nu_x(Kx) = \mathfrak{p} \cap \mathrm{Ad}_{a^{-1}}\mathfrak{p}.$$

But the right hand-side contains

$$\mathfrak{p} \cap \mathrm{Ad}_{a^{-1}}\mathfrak{a} = \mathfrak{p} \cap \mathfrak{a} = \mathfrak{a}.$$

Hence

$$T_xF = a \cdot \mathfrak{a} \subset \nu_x(Kx),$$

as desired. □

The most basic example of a (linear) polar action occurs with the diagonalization of real symmetric matrices. Here the orthogonal group $\mathbf{O}(n)$ acts on the space of $n \times n$ real symmetric spaces and a section is given by the subspace of diagonal matrices. Indeed the linear case — polar representations — has been considered in a geometric vein by Szenthe [40]. In [34] the submanifold geometry of orbits of polar representation is discussed and they are shown to be exactly the homogeneous isoparametric submanifolds of Euclidean space; see also the book [35] and, especially, [2] for a more modern treatment.

In [18] we find the following classification result:

3.6.4 Theorem (Dadok) *A polar representation of a compact connected Lie group is orbit-equivalent (i.e. has the same orbits, under a suitable isometric identification of the target spaces) to the isotropy representation of a symmetric space of compact type.*

Note that a symmetric space and its Cartan dual have equivalent isotropy representations.

It was proved in [21], by means of classification, that a variationally complete representation is orbit-equivalent to the isotropy representation of a symmetric space, and hence is polar. In [19], a geometric proof of this result was provided.

3.6.5 Theorem (Di Scala-Olmos) *A variationally complete representation of a compact Lie group G on an Euclidean space V is polar.*

Proof. Let $p \in V$ be a regular point so that $N = Gp$ is a principal orbit. A standard argument shows that $\Sigma := \nu_pN$ meets all orbits (a minimizing geodesic from any given orbit to N must meet N orthogonally, and hence has a G-translate entirely contained in Σ, which will also meet the given orbit).

Choose $v \in \nu_pN$ such that the Weingarten operator A_v has all eigenvalues nonzero. This is possible, since $A_p = -\mathrm{id}$, and indeed the set of such vectors is open and dense in ν_pN. Consider the geodesic $\gamma(s) = p + sv$, normal to N, and fix $s_1 > 0$ such that $N_1 = Gq$, $q = \gamma(s_1)$, is also a principal orbit. Due to the assumption of variational completeness, q is not a focal point of N along γ. We will show that $T_pN = T_qN_1$ as subspaces of V.

Each eigenvector $u \in T_pN$ of A_v, with corresponding eigenvalue $\lambda \neq 0$, gives rise to a Jacobi field $J(s) = (1 - \lambda s)u$ along the geodesic $\gamma(s) = p + sv$, associated to the variation $\gamma_t(s) = c(t) + s\hat{v}(t)$, where c is a smooth curve in N with $c(0) = p$ and $c'(0) = u$, and $\hat{v}$ is the parallel extension of v to a normal vector field along c. Since $J(0) = u \in T_pN$ and $J(\frac{1}{\lambda}) = 0$, the assumption of variational completeness yields a Killing vector field X induced by G such that $X \circ \gamma = J$. In particular, $J(s) \in T_{\gamma(s)}(G\gamma(s))$ for all s. Since q is not a focal point of N along γ, $s_1 \neq \frac{1}{\lambda}$ and therefore $u \in T_qN_1$. As the eigenvectors of A_v span T_pN, this shows $T_qN_1 = T_pN$.

We have seen that Σ is orthogonal to all principal orbits passing through an open and dense subset of itself. By a continuity argument, Σ is orthogonal to every orbit it meets. This finishes the proof. $\square$

Now the classes of polar and variationally complete representations coincide, and they also coincide, up to orbit-equivalence, with the class of isotropy representations of symmetric spaces. In [22], Theorem 3.6.5 was extended to compact symmetric spaces. So the classes of hyperpolar and variationally complete actions coincide also for compact symmetric spaces.

3.6.6 Theorem (Gorodski-Thorbergsson) *A variationally complete action of a compact Lie group on a compact symmetric space is hyperpolar.*

Later, Theorem 3.6.6 was generalized to variationally complete actions on nonnegatively curved complete Riemannian manifolds [33].

We close our discussion with a brief account on the classification of polar and hyperpolar actions on symmetric spaces. Podestà and Thorbergsson [36] classified polar actions on compact symmetric spaces of rank one (although they missed one action on the Cayley projective plane, which was found in [20]). In his PhD thesis [28], A. Kollross classified hyperpolar actions on irreducible symmetric spaces of compact type, up to orbit-equivalence; they turn out to be just the Hermann examples and the actions with cohomogeneity (i.e. codimension of principal orbits) one. No example of a polar, non-hyperpolar action on an irreducible symmetric space of compact type and rank greater than one was known. Several papers by Podestà, Thorbergsson, L. Biliotti, A. Gori and Kollross culminated with the work of Lytchak and Kollross [30], in which they confirmed that no such example exists. Recently Kollross [29] has investigated the case of reducible symmetric spaces of compact type, and since long J. Berndt and his collaborators have studied the case of symmetric spaces of noncompact type [1, 38], where there is a greater richness of examples and the investigation still has a lot to go.

3.7 Problems

3.7.1 Problem Recall that a Riemannian manifold is called an *Einstein manifold* if the Ricci tensor is proportional to the metric tensor.

Show that a symmetric space is an Einstein manifold if and only if it is Euclidean or irreducible.

3.7.2 Problem Let σ be an involutive automorphism of a compact connected Lie group G. Denote by K the closed subgroup of fixed points of σ. Equip the Lie algebra $\mathfrak{g}$ of G with an ad-invariant inner product and consider the associated bi-invariant Riemannian metric on G and G-invariant Riemannian metric on the symmetric space $M = G/K$. Prove that the map

$$f : M \to G, \qquad f(gK) = g\sigma(g^{-1})$$

defines, up to a multiplicative constant, an isometric embedding of M as a totally geodesic closed submanifold of G (the *Cartan embedding*).

3.7.3 Problem Let $M = G/K$ be a symmetric space of non-compact type, where G is connected and $K = G_x$ for a point $x \in M$. Consider a connected subgroup H of K with Lie algebra $\mathfrak{h}$. Prove that the fixed point set of H in M is $(\exp \mathfrak{s})x$, where $\mathfrak{s}$ is the centralizer of $\mathfrak{h}$ in $\mathfrak{p}$. Further, check that $\mathfrak{s}$ is a Lie triple system.

3.7.4 Problem Let G be a compact connected Lie group endowed with a bi-invariant metric. Check that the notions of rank *qua* compact Lie group *qua* symmetric space coincide for G.

3.7.5 Problem Let M be a symmetric space of rank one. Prove that M has positive (resp. negative) sectional curvature if it is compact (resp. noncompact) and

$$\frac{\kappa_{\mathrm{MAX}}}{\kappa_{\mathrm{MIN}}} = 1 \text{ or } 4,$$

where κ_{MAX} and κ_{MIN} denote the maximum and the minimum of the sectional curvatures of 2-planes tangent to M. (Hint: There are at most two positive restricted roots.)

3.7.6 Problem Let $M = G/K$ be a symmetric space and write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ under the involution as usual. Prove that the isotropy representation of K on $\mathfrak{p}$ is polar, with the Cartan subspaces as sections.

4 Appendix: The Cartan-Ambrose theorem

Possibly the main application of the Cartan-Ambrose theorem is in the basic theory of symmetric spaces. The proof herein is adapted from [12]. We retain the notation of section 1.3.

4.1 Preliminaries

A *broken geodesic* in a Riemannian manifold M is a continuous curve $\gamma : [0,1] \to M$ such that there is a partition $t_0 = 0 < t_1 < \cdots < t_k < t_{k+1} = 1$ with the property that $\gamma|[t_i, t_{i+1}]$ is a geodesic for $i = 0, \ldots, k$. Let complete Riemannian manifolds M and $\tilde{M}$ be given and let $x \in M$ and $\tilde{x} \in \tilde{M}$. Let $I : T_x M \to T_{\tilde{x}} \tilde{M}$ be a linear isometry. Then we associate to a broken geodesic γ in M starting at x a broken geodesic $\tilde{\gamma}$ in $\tilde{M}$ starting at $\tilde{x}$ as follows: We first define $\tilde{\gamma}$ on $[0, t_1]$ by setting $\tilde{\gamma}(t) = \exp_{\tilde{x}} tI(\gamma'(0))$. Assume now that we have defined $\tilde{\gamma}$ on $[0, t_i]$ and that $t_i < 1$. Then we define $\tilde{\gamma}$ on $[t_i, t_{i+1}]$ by setting $\tilde{\gamma}(t) = \exp_{\tilde{\gamma}(t_i)}(t - t_i) I_{{}_i\gamma}(\gamma'(t_i+))$, where $I_{{}_i\gamma} = P_{{}_i\tilde{\gamma}} \circ I \circ P_{{}_i\gamma}^{-1}$, and $P_{{}_i\gamma}$ and $P_{{}_i\tilde{\gamma}}$ are the parallel translations along ${}_i\gamma = \gamma|[0, t_i]$ and ${}_i\tilde{\gamma} = \tilde{\gamma}|[0, t_i]$ respectively. Finally we can more generally define $I_\eta : T_{\eta(1)} M \to T_{\tilde{\eta}(1)} M$ for any broken geodesic $\eta : [0,1] \to M$ starting at x by setting $I_\eta = P_{\tilde{\eta}} \circ I \circ P_\eta^{-1}$.

We will need the following lemma in the proof of Theorem 4.2.1.

4.1.1 Lemma *Let M be a complete Riemannian manifold and let γ_0 and γ_1 be broken geodesics in M which are defined on $[0,1]$, join x and y, and are homotopic to each other. Then there is a homotopy $\Gamma : [0,1] \times [0,1] \to M$ between γ_0 and γ_1 such that $\gamma_s|[t_i, t_{i+1}]$ is a geodesic for $i = 0, \ldots, k$ and all s where $\gamma_s(t) = \Gamma(s,t)$ and $t_0 = 0 < t_1 < \cdots < t_{k+1} = 1$ is a partition of $[0,1]$.*

Proof. Let $\kappa : [0,1] \times [0,1] \to M$ be a continuous homotopy between γ_0 and γ_1, i.e., $\kappa_0 = \gamma_0$, $\kappa_1 = \gamma_1$, $\kappa(s,0) = x$, and $\kappa(s,1) = y$. Let ℓ denote the radius of a closed ball around x containing the image of the homotopy κ. Let $r > 0$ be the minimum of the injectivity radius on the closed ball $\overline{B_\ell(x)}$. Let $t_0 = 0 < t_1 < \cdots < t_{k+1} = 1$ be a partition with the property that $\gamma_0|[t_i, t_{i+1}]$ and $\gamma_1|[t_i, t_{i+1}]$ are geodesics for all $i = 0, \ldots, k$, and such that $\kappa_s|[t_i, t_{i+1}]$ is contained in $B_r(\kappa_s(t_i))$ for all $i = 0, \ldots, k$ and all s in $[0,1]$. Now we define the homotopy $\Gamma : [0,1] \times [0,1] \to M$ by setting $\gamma_s|[t_i, t_{i+1}]$ equal to the unique shortest geodesic between $\kappa_s(t_i)$ and $\kappa_s(t_{i+1})$ for all $i = 0, \ldots, k$ and all s in $[0,1]$. The homotopy Γ has by construction the properties asked for in the claim of the lemma. $\square$

4.2 Statement and proof of the theorem

We retain the notation of subsection 4.1.

4.2.1 Theorem (Cartan-Ambrose) *Assume that M and $\tilde{M}$ are complete Riemannian manifolds, M is simply connected, and that*

$$I_\gamma(R(u,v)w) = \tilde{R}(I_\gamma u, I_\gamma v) I_\gamma w$$

for all $u, v, w \in T_{\gamma(t)} M$ and all broken geodesics γ starting in x. We define a map $\Phi : M \to \tilde{M}$ by setting $\Phi(y) = \exp_{\tilde{x}}(I\gamma'(0))$ where $\gamma : [0,1] \to M$ is a geodesic joining x and y. Then Φ is well-defined, a local isometry, and a covering map. In particular, Φ is an isometry if $\tilde{M}$ is also simply connected.

Proof. Since M and $\tilde{M}$ are complete, the map Φ is clearly onto if it is well-defined. We prove the following claim which implies that Φ is well-defined: For all broken geodesics γ_0, $\gamma_1 : [0,1] \to M$ starting at x with $\gamma_0(1) = \gamma_1(1)$, we have $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$ and $I_{\gamma_0} = I_{\gamma_1}$. We may assume that γ_0 and γ_1 have common break points at $t_1, \ldots, t_k$, simply by adding some of them if necessary.

The first case is when γ_0 and γ_1 are both contained in a normal coordinate neighborhood V around x. Then Theorem 1.3.3 implies that $\varphi = \exp_{\tilde{x}} \circ I \circ \exp_x^{-1}|_V$ is an isometry and, for $j = 0$, 1, that $d\varphi_{\gamma_j(t_1)} = I_{{}_1\gamma_j}$. In this proof it will be convenient to consider ${}_i\theta_j = \gamma_j|[t_i, t_{i+1}]$ and ${}_i\tilde{\theta}_j = \tilde{\gamma}_j|[t_i, t_{i+1}]$, and to notice that $I_{{}_{i+1}\gamma_j} = P_{{}_i\tilde{\theta}_j} \circ I_{{}_i\gamma_j} \circ P^{-1}_{{}_i\theta_j}$. Since φ is an isometry,

$$\varphi(\gamma_j(t)) = \exp_{\tilde{x}} tI(\dot{\gamma}_j(0+)) = \tilde{\gamma}_j(t),$$

for $0 \le t \le t_1$. Proceeding by induction on $i = 1, \ldots, k$ we now have

$$\begin{aligned}
\varphi(\gamma_j(t)) &= \varphi(\exp_{\gamma_j(t_i)}(t - t_i)\dot{\gamma}_j(t_i+)) \\
&= \exp_{\tilde{\gamma}_j(t_i)}(t - t_i)(d\varphi_{\gamma_j(t_i)}\dot{\gamma}_j(t_i+)) \\
&= \exp_{\tilde{\gamma}_j(t_i)}(t - t_i)(I_{{}_i\gamma_j}\dot{\gamma}_j(t_i+)) \\
&= \tilde{\gamma}_j(t),
\end{aligned}$$

for $t_i \le t \le t_{i+1}$; this implies $\varphi({}_i\theta_j) = {}_i\tilde{\theta}_j$ and then

$$\begin{aligned}
d\varphi_{\gamma_j(t_{i+1})} &= P_{{}_i\tilde{\theta}_j} \circ d\varphi_{\gamma_j(t_i)} \circ P^{-1}_{{}_i\theta_j} \\
&= P_{{}_i\tilde{\theta}_j} \circ I_{{}_i\gamma_j} \circ P^{-1}_{{}_i\theta_j} \\
&= I_{{}_{i+1}\gamma_j},
\end{aligned}$$

and the induction step is complete. This proves that $\varphi \circ \gamma_j = \tilde{\gamma}_j$ and hence $\tilde{\gamma}_0(1) = \varphi(\gamma_0(1)) = \varphi(\gamma_1(1)) = \tilde{\gamma}_1(1)$.

The second case we want to consider is when for all $i = 0, \ldots, k-1$ there is a normal coordinate neighborhood of $\gamma_0(t_i)$ (resp. $\tilde{\gamma}_0(t_i)$) containing $\gamma_0(t_{i+1})$, $\gamma_1(t_{i+1})$ and $\gamma_1(t_{i+2})$ (resp. $\tilde{\gamma}_0(t_{i+1})$, $\tilde{\gamma}_1(t_{i+1})$ and $\tilde{\gamma}_1(t_{i+2})$) and the minimal geodesic segments between them. If $k = 1$, then γ_0 and γ_1 are contained in a normal coordinate neighborhood around x and the result follows from the first case. We proceed by induction on k. Suppose that γ_0 and γ_1 have $k \ge 2$ breaks. We introduce the auxiliary minimal geodesic $\tau : [t_{k-1}, t_k] \to M$ from $\gamma_0(t_{k-1})$ to $\gamma_1(t_k)$. By the induction hypothesis we have

$$(4.2.2) \qquad (\widetilde{{}_{k-1}\gamma_0 \cup \tau})(t_k) = {}_k\tilde{\gamma}_1(t_k) \quad \text{and} \quad I_{{}_{k-1}\gamma_0 \cup \tau} = I_{{}_k\gamma_1}.$$

Notice that the isometry $I_{{}_{k-1}\gamma_0} : T_{\gamma_0(t_{k-1})}M \to T_{\tilde{\gamma}_0(t_{k-1})}\tilde{M}$ induces a correspondence $\eta \mapsto \hat{\eta}$ between geodesics in M starting at $\gamma_0(t_{k-1})$ and geodesics in $\tilde{M}$ starting at $\tilde{\gamma}_0(t_{k-1})$. Set

$$\eta_0 = \gamma_0|[t_{k-1}, 1] \quad \text{and} \quad \eta_1 = \tau \cup \gamma_1|[t_k, 1] = \tau \cup {}_k\theta_1.$$

By the first case we have

$$\hat{\eta}_0(1) = \hat{\eta}_1(1) \quad \text{and} \quad I_{\eta_0} = I_{\eta_1},$$

which is clearly equivalent to

$$\tilde{\gamma}_0(1) = (\widetilde{{}_{k-1}\gamma_0 \cup \eta_1})(1) \quad \text{and} \quad I_{\gamma_0} = I_{{}_{k-1}\gamma_0 \cup \eta_1},$$

and by (4.2.2) we have that

$$(\widetilde{{}_{k-1}\gamma_0 \cup \eta_1})(1) = (\widetilde{{}_{k-1}\gamma_0 \cup \tau} \cup {}_k\theta_1)(1) = \tilde{\gamma}_1(1).$$

It follows that $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$, as desired.

Finally assume that γ_0 and γ_1 are arbitrary broken geodesics in M starting at x such that $\gamma_0(1) = \gamma_1(1)$. Since M is simply connected, γ_0 and γ_1 are homotopic to each other, so by Lemma 4.1.1 a homotopy Γ between γ_0 and γ_1 can be chosen such that the $\gamma_s(t) = \Gamma(s,t)$ for $s \in [0,1]$ are broken geodesics, with common break points at $t_1, \ldots, t_k$. By refining the partition $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = 1$ we may assume that $k \geq 1$ and, for $s \in [0,1]$ and $i = 0, \ldots, k-1$, that $\gamma_s(t_{i+1})$ and $\gamma_s(t_{i+2})$ belong to a normal coordinate neighborhood of $\gamma_s(t_i)$.

Now if s_0, $s_1 \in [0,1]$ are sufficiently close, then γ_{s_0} and γ_{s_1} are seen to satisfy the conditions of the second case. It follows that $\tilde{\gamma}_{s_0}(1) = \tilde{\gamma}_{s_1}(1)$. This shows that $\gamma_s(1)$ is locally constant with respect to $s \in [0,1]$, which obviously implies that $\tilde{\gamma}_s(1)$ is constant with respect to $s \in [0,1]$. In particular, $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$. This completes the proof of the claim.

The proof so far implies that $\Phi(y) = \tilde{\gamma}(1)$ and $d\Phi_y = I_\gamma$, where γ is any broken geodesic defined on $[0,1]$ and joining x to y. Let V be a normal coordinate neighborhood of y in M, and let $\tilde{y} = \Phi(y)$. Now it is clear that Φ restricted to V coincides with the map

$$\exp_{\tilde{y}} \circ I_\gamma \circ \exp_y^{-1}|_V,$$

which is an isometry by Proposition 1.3.3. Therefore Φ is a local isometry. It is well known that a local isometry from M to $\tilde{M}$ is a covering map if M is complete. Hence we have proved that Φ is a covering map which finishes the proof. □

5 Appendix: A review of semisimple and compact Lie algebras

5.1 Invariant integration

In 1897 Adolf Hurwitz introduced the idea “invariant integration” on Lie groups and in 1933 Alfred Haar considered the more general idea of a “left invariant Haar measure” on locally compact topological groups.

On a compact Lie group, the bi-invariant Haar integral has a description in terms of volume forms. Let G be a Lie group. A differential form ω on G is called *left-invariant* if $L_g^*\omega = \omega$ for all $g \in G$. Similarly, and defines right-invariant differential forms. Since a left-invariant form is determined by its value at 1, the space of left-invariant n-forms on G is one-dimensional for $n = \dim G$. For a nonzero left invariant n-form ω, consider the associated orientation on G. For each compactly supported continuous function f on G,

$$f \mapsto \int_G f\omega \tag{5.1.1}$$

defines a positive continuous linear functional and hence yields a regular Borel measure on G. Since $L_g : G \to G$ is a diffeomorphism that preserves the orientation of G, we have $\int_G f\omega = \int_G L_g^*(f\omega) = \int_G (f \circ L_g)\omega$ for all $g \in G$, and then (5.1.1) is called a *left Haar integral* on G. In case G is compact, there is a unique left invariant n-form ω with $\int_G \omega = 1$, up to sign. Henceforth we will identify this n-form with the associated measure on G and denote them by dg.

What about right-invariance of (5.1.1)? For each $h \in G$, $R_h^* dg$ is a left-invariant form on G and thus we can write $R_h^* dg = \tilde{\lambda}(h) dg$ for a homomorphism $\tilde{\lambda} : G \to \mathbf{R}^\times$. Now $R_h : G \to G$ is a diffeomorphism which preserves (resp. reverses) the orientation if $\tilde{\lambda}(h) > 0$ (resp. if $\tilde{\lambda}(h) < 0$), so

$$\begin{aligned}\int_G f\,dg &= (\operatorname{sgn} \tilde{\lambda}(h)) \int_G R_h^*(f\,dg) \\ &= (\operatorname{sgn} \tilde{\lambda}(h)) \int_G (f \circ R_h)\tilde{\lambda}(h) dg \\ &= \int_G (f \circ R_h)\lambda(h) dg,\end{aligned}$$

where the homomorphism $\lambda = |\tilde{\lambda}| : G \to (0, +\infty)$ is called the *modular function* on G. In case G is a compact Lie group, $\lambda(G)$ is a compact subgroup of $(0, +\infty)$ and thus trivial. This shows that the left Haar integral is also right-invariant, and hence we have a two-sided Haar integral on G.

Denote the Lie algebra of G by $\mathfrak{g}$. Fix a basis $X_1 \ldots, X_n$ of $\mathfrak{g}$. Let $\theta_1, \ldots, \theta_n$ be the dual basis of $\mathfrak{g}^*$. Then an explicit left-invariant n-form on G is given by $\omega = \theta_1 \wedge \ldots \wedge \theta_n$. In case G is compact and endowed a bi-invariant Riemannian metric (cf. subsection 5.6), and we take $X_1 \ldots, X_n$ to be orthonormal, dg coincides with the *Riemannian volume form*, and the Haar measure coincides with the Riemannian measure.

5.1.2 Proposition *Let $\rho : G \to \mathbf{GL}(V)$ be a real (resp. complex) representation of a compact Lie group G. Then there exists a positive definite Euclidean (resp. Hermitian) inner product $\langle , \rangle$ on $\mathfrak{g}$ such that each $\rho(g)$, with $g \in G$, is an orthogonal (resp. unitary) transformation. In particular, there exists an Euclidean inner product on the Lie algebra $\mathfrak{g}$ of G such that each* Ad_g, *with $g \in G$, is an orthogonal transformation.*

Proof. Let $\langle \cdot, \cdot \rangle_0$ be an arbitrary inner (resp. Hermitian) product on V. For u, $v \in V$, set

$$\langle u, v \rangle = \int_G \underbrace{\langle \rho(g)u, \rho(g)v \rangle_0}_{=:f(g)} \, dg,$$

where dg is a bi-invariant Haar measure. Then $\langle\cdot,\cdot\rangle$ is an inner (resp. Hermitian) product on V and, for $h\in G$,

$$\begin{aligned}
\langle\rho(h)u,\rho(h)v\rangle &= \int_G \langle\rho(g)(\rho(h)u),\rho(g)(\rho(h)v)\rangle_0\,dg \\
&= \int_G \langle\rho(gh)u,\rho(gh)v\rangle_0\,dg \\
&= \int_G f(gh)\,dg \\
&= \int_G f(g)\,dg \\
&= \langle u,v\rangle,
\end{aligned}$$

which completes the proof. □

5.1.3 Proposition *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Then: an inner product $\langle,\rangle$ on $\mathfrak{g}$ is* Ad*-invariant if and only if*

$$\langle \mathrm{ad}_X Y, Z\rangle + \langle Y, \mathrm{ad}_X Z\rangle = 0$$

for all X, Y, $Z\in\mathfrak{g}$; we say it is ad-invariant.

Proof. Let $\mathbf{O}(\mathfrak{g})$ denote the orthogonal group of a given inner product $\langle,\rangle$ on $\mathfrak{g}$, and let $\mathbf{SO}(\mathfrak{g})$ denote its identity component; these are Lie group and we let $\mathfrak{so}(\mathfrak{g})$ denote their common Lie algebra. For each $X\in\mathfrak{g}$, $\mathrm{ad}_X\in\mathfrak{so}(\mathfrak{g})$ if and only if $\mathrm{Ad}_{\exp(tX)}\in\mathbf{SO}(\mathfrak{g})$ for all $t\in\mathbf{R}$, and this is equivalent to the statement of the proposition, if we use the connectedness of G to have that $\exp[\mathfrak{g}]$ generates G. □

5.2 Adjoint group

Let $\mathfrak{g}$ be a Lie algebra. Then $\mathbf{GL}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathfrak{gl}(\mathfrak{g})$ consisting of all endomorphisms of the vector space underlying $\mathfrak{g}$. The group of automorphisms of $\mathfrak{g}$, denoted by $\mathrm{Aut}(\mathfrak{g})$, is clearly a closed subgroup of $\mathbf{GL}(\mathfrak{g})$. Recall that a closed subgroup of a Lie group is a Lie subgroup with the subspace topology. Hence $\mathrm{Aut}(\mathfrak{g})$ is a Lie subgroup of $\mathbf{GL}(\mathfrak{g})$. Its Lie algebra consists of the endomorphisms $D\in\mathfrak{gl}(\mathfrak{g})$ such that

$$\exp(tD)\cdot[X,Y] = [\exp(tD)\cdot X, \exp(tD)\cdot Y]$$

for X, $Y\in\mathfrak{g}$, $t\in\mathbf{R}$. Differentiating this equation at $t=0$, we obtain

$$D[X,Y] = [DX,Y] + [X,DY] \tag{5.2.1}$$

for X, $Y\in\mathfrak{g}$. The endomorphisms D satisfying equation (5.2.1) are called *derivations* of $\mathfrak{g}$. Conversely, if D is a derivation of $\mathfrak{g}$, then it is easily checked by induction that

$$D^m[X,Y] = \sum_{i+j=m}\frac{m!}{i!j!}[D^iX, D^jX]$$

for all $m \geq 1$. It follows that

$$e^D[X,Y] = \sum_{m\geq 0} \frac{1}{m!} D^m[X,Y] = [e^D X, e^D Y],$$

that is, e^D is an automorphism. Therefore the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is the space $\mathrm{Der}(\mathfrak{g})$ of all derivations of $\mathfrak{g}$.

In particular, the Jacobi identity shows that $\mathrm{ad}_X \in \mathrm{Der}(\mathfrak{g})$ for all $X \in \mathfrak{g}$, so that the image of the adjoint representation $\mathrm{ad}[\mathfrak{g}]$ is a subalgebra of $\mathrm{Der}(\mathfrak{g})$, again by Jacobi. The derivation property says that $[D, \mathrm{ad}_X] = \mathrm{ad}_{DX}$ for $D \in \mathrm{Der}(\mathfrak{g})$ and $X \in \mathfrak{g}$, so $\mathrm{ad}[\mathfrak{g}]$ is indeed an *ideal* of $\mathrm{Der}(\mathfrak{g})$.

The elements of $\mathrm{ad}[\mathfrak{g}]$ are called *inner derivations*. Let $\mathrm{Inn}(\mathfrak{g})$ be the connected subgroup of $\mathrm{Aut}(\mathfrak{g})$ defined by $\mathrm{ad}[\mathfrak{g}]$. As $\mathrm{ad}[\mathfrak{g}]$ is an ideal of $\mathrm{Der}(\mathfrak{g})$, $\mathrm{Inn}(\mathfrak{g})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{g})$, and in accordance with the next proposition, that group is called the *adjoint group* of $\mathfrak{g}$ and its elements are called *inner automorphisms* of $\mathfrak{g}$.

5.2.2 Proposition *The adjoint group* $\mathrm{Inn}(\mathfrak{g})$ *is canonically isomorphic to* $G/Z(G)$*, where* G *is any connected Lie group with Lie algebra* $\mathfrak{g}$ *and* $Z(G)$ *denotes the center of* G*.*

Proof. The image of the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is contained in $\mathrm{Inn}(\mathfrak{g})$, because $\mathrm{Ad}_{\exp X} = e^{\mathrm{ad}_X}$ for $X \in \mathfrak{g}$, and the image of exp generates G. Since $d(\mathrm{Ad}) = \mathrm{ad}$, the Lie algebra of the image of Ad is $\mathrm{ad}[\mathfrak{g}]$, thus we get equality $\mathrm{Ad}(G) = \mathrm{Inn}(\mathfrak{g})$. Finally, note that the kernel of Ad is $Z(G)$. □

Fix a real Lie algebra $\mathfrak{g}$. Regarding *all* the Lie groups that have Lie algebras isomorphic to $\mathfrak{g}$, now the following picture emerges. If $G_1 \to G_2$ is a covering homomorphism, then G_1 and G_2 have isomorphic Lie algebras, but the converse statement does not hold, namely, there exist Lie groups with isomorphic Lie algebras such that neither one of them covers the other one (find examples!). However, there exists a simply-connected Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$, and $\tilde{G}$ does cover any other Lie group with Lie algebra $\mathfrak{g}$. Moreover, if $\mathfrak{g}$ is centerless, the adjoint group $\bar{G} := \mathrm{Inn}(\mathfrak{g})$ has Lie algebra $\mathfrak{g}$ and it is covered by any Lie group G with Lie algebra $\mathfrak{g}$, since $\bar{G} \cong G/Z(G)$. Hence $\tilde{G}$ sits at the top of the hierarchy and $\bar{G}$ sits at its bottom.

5.3 Killing form

Let $\mathfrak{g}$ be a Lie algebra. The *Killing form* of $\mathfrak{g}$ is the symmetric bilinear form

$$\beta(X,Y) = \mathrm{tr}\,\mathrm{ad}X\mathrm{ad}Y \qquad \text{(trace)}$$

where X, $Y \in \mathfrak{g}$.

5.3.1 Proposition *a. If* $\mathfrak{a} \subset \mathfrak{g}$ *is an ideal, then the Killing form of* $\mathfrak{a}$ *is the restriction of* β *to* $\mathfrak{a} \times \mathfrak{a}$*.*

b. *If* $s \in \mathrm{Aut}(\mathfrak{g})$*, then* $\beta(sX, sY) = \beta(X, Y)$ *for* X*,* $Y \in \mathfrak{g}$.

c. $\beta(\mathrm{ad}_X Y, Z) + \beta(Y, \mathrm{ad}_X Z) = 0$ *for* X*,* Y*,* $Z \in \mathfrak{g}$ *(*β *is* ad*-invariant).*

Proof. (a) If X, $Y \in \mathfrak{a}$ then $\mathrm{ad}_X \mathrm{ad}_Y$ maps $\mathfrak{g}$ into $\mathfrak{a}$. (b) If $s \in \mathrm{Aut}(\mathfrak{g})$ then $\mathrm{ad}_{sX} = s \circ \mathrm{ad}_X \circ s^{-1}$. (c) It follows from Jacobi. □

5.4 Semisimplicity

In order to avoid unnecessary technicalities, we adopt the following nonstandard, but completely equivalent definition. We call a Lie algebra $\mathfrak{g}$ *semisimple* if β is nondegenerate, and it is called *simple* if it is semisimple and has no nontrivial ideals.[10] Note that by the ad-invariance of β, its kernel is always an ideal of the underlying Lie algebra.

A Lie group G is called *semisimple* (resp. simple) if its Lie algebra is semisimple (resp. simple). Note that the definition of simple Lie group is different from the definition of simple abstract group, in that a simple Lie group is allowed to contain non-trivial discrete normal subgroups; for instance, $\mathbf{SU}(n)$ is considered a simple Lie group but it has a finite center.

5.4.1 Proposition *Let* $\mathfrak{g}$ *be a semisimple Lie algebra, let* $\mathfrak{a} \subset \mathfrak{g}$ *be an ideal and*

$$\mathfrak{a}^\perp = \{X \in \mathfrak{g} : \beta(X, \mathfrak{a}) = 0\}.$$

Then $\mathfrak{a}^\perp$ *is an ideal,* $\mathfrak{a}$ *and* $\mathfrak{a}^\perp$ *are semisimple and* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ *(direct sum of ideals).*

Proof. The ad-invariance of β implies that $\mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$. Then $\mathfrak{a} \cap \mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$, and again by ad-invariance of β, $\mathfrak{a} \cap \mathfrak{a}^\perp$ is Abelian; in fact, for every $Z \in \mathfrak{g}$ and X, $Y \in \mathfrak{a} \cap \mathfrak{a}^\perp$,

$$\beta(Z, [X, Y]) = \beta([Z, X], Y) = 0,$$

so $[X, Y] = 0$ by nondegeneracy of β. Fix now a complementary subspace $\mathfrak{b}$ of $\mathfrak{a} \cap \mathfrak{a}^\perp$ in $\mathfrak{g}$. Then, for $X \in \mathfrak{a} \cap \mathfrak{a}^\perp$ and $Y \in \mathfrak{g}$, the linear map $\mathrm{ad}_X \mathrm{ad}_Y$ maps $\mathfrak{a} \cap \mathfrak{a}^\perp$ to zero and $\mathfrak{b}$ to $\mathfrak{a} \cap \mathfrak{a}^\perp$, so it has no diagonal elements and thus $\beta(X, Y) = 0$, yielding $X = 0$ by nondegeneracy of β. We have shown that $\mathfrak{a} \cap \mathfrak{a}^\perp = 0$. The nondegeneracy of β also implies that $\dim \mathfrak{a} + \dim \mathfrak{a}^\perp = \dim \mathfrak{g}$, whence $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. That $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are semisimple is a consequence of Proposition 5.3.1(a) and $\beta(\mathfrak{a}, \mathfrak{a}^\perp) = 0$. □

5.4.2 Corollary *A semisimple Lie algebra* $\mathfrak{g}$ *is centerless and decomposes into a direct sum of simple ideals* $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$*. In particular,* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

[10] The more usual definitions are that a Lie algebra is *semisimple* if it has no nontrivial solvable ideals, and it is *simple* if it is non-Abelian and has no nontrivial ideals; in particular, a simple Lie algebra is semisimple. The equivalence with the definitions given above follows from Cartan's criteria for semisimplicity and solvability in terms of the Killing form.

Proof. Clearly the center of $\mathfrak{g}$ is contained in the kernel of the Killing form, hence it is zero. If $\mathfrak{a}$ is a proper ideal of $\mathfrak{g}$, then the proposition says that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ and the decomposition result follows by induction on the dimension of $\mathfrak{g}$. Finally,

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1, \mathfrak{g}_1] \oplus \cdots \oplus [\mathfrak{g}_r, \mathfrak{g}_r]$$

and each $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$ since $[\mathfrak{g}_i, \mathfrak{g}_i]$ is a nonzero ideal of $\mathfrak{g}_i$. □

5.4.3 Proposition *If $\mathfrak{g}$ is semisimple then $\mathrm{ad}[\mathfrak{g}] = \mathrm{Der}(\mathfrak{g})$, i.e. every derivation is inner.*

Proof. Since $\mathfrak{g}$ is centerless, $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is injective, so $\mathrm{ad}[\mathfrak{g}]$ is semisimple. Denote by β, β' the Killing forms of $\mathrm{ad}[\mathfrak{g}]$ and $\mathrm{Der}(\mathfrak{g})$. $\mathrm{ad}[\mathfrak{g}]$ is an ideal of $\mathrm{Der}(\mathfrak{g})$. Let $\mathfrak{a}$ denote its β'-orthogonal complement in $\mathrm{Der}(\mathfrak{g})$. Then also $\mathfrak{a}$ is an ideal of $\mathrm{Der}(\mathfrak{g})$. We have, using Proposition 5.3.1,

$$\beta(\mathrm{ad}[\mathfrak{g}], \mathrm{ad}[\mathfrak{g}] \cap \mathfrak{a}) = \beta'(\mathrm{ad}[\mathfrak{g}], \mathrm{ad}[\mathfrak{g}] \cap \mathfrak{a}) = 0$$

so $\mathrm{ad}[\mathfrak{g}] \cap \mathfrak{a} = 0$. Therefore $\mathrm{ad}_{DX} = [D, \mathrm{ad}_X] \in \mathrm{ad}[\mathfrak{g}] \cap \mathfrak{a} = 0$ for $D \in \mathfrak{a}$ and $X \in \mathfrak{g}$. Since $\ker \mathrm{ad} = 0$, this implies $\mathfrak{a} = 0$, as desired. □

5.4.4 Corollary *If $\mathfrak{g}$ is semisimple then $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})^0$.*

Proof. $\mathrm{Inn}(\mathfrak{g})$ is connected and both hand sides have the same Lie algebra. □

5.5 Compact Lie algebras

A Lie algebra $\mathfrak{g}$ is called *compact* if there exists a compact Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$.

5.5.1 Theorem *Let $\mathfrak{g}$ be a Lie algebra. The following assertions are equivalent:*

a. *$\mathfrak{g}$ is a compact Lie algebra.*

b. $\mathrm{Inn}(\mathfrak{g})$ *is compact.*

c. *$\mathfrak{g}$ admits an* ad*-invariant positive definite inner product.*

d. *$\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$ is semisimple with negative definite Killing form.*

Proof. (c) implies (d). Let $\langle , \rangle$ be an ad-invariant positive definite inner product on $\mathfrak{g}$,

$$\langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle = 0$$

for X, Y, $Z \in \mathfrak{g}$. The center $\mathfrak{z}$ is ad-invariant, so also its $\langle , \rangle$-orthogonal complement $\mathfrak{z}^\perp$ is ad-invariant. Now $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}^\perp$, direct sum of ideals. The Killing form of $\mathfrak{z}^\perp$ is the restriction of the Killing form β of $\mathfrak{g}$. Owing to the ad-invariance of $\langle , \rangle$, ad_X is skew-symmetric with respect to $\langle , \rangle$ for $X \in \mathfrak{g}$, thus it has purely imaginary eigenvalues. Therefore $\beta(X,X) = \operatorname{tr}\operatorname{ad}_X^2 \leq 0$ and equality holds if and only if $\operatorname{ad}_X = 0$ if and only if $X \in \mathfrak{z}$. This proves that $B|_{\mathfrak{z}^\perp\times\mathfrak{z}^\perp}$ is negative definite and hence $\mathfrak{z}^\perp$ is semisimple and $[\mathfrak{g},\mathfrak{g}] = [\mathfrak{z}^\perp,\mathfrak{z}^\perp] = \mathfrak{z}^\perp$.

(d) implies (b). We have $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{z}) \times \mathrm{Inn}([\mathfrak{g},\mathfrak{g}]) = \mathrm{Inn}([\mathfrak{g},\mathfrak{g}])$ since $\mathfrak{z}$ is Abelian. Without loss of generality, we may thus assume $\mathfrak{g}$ is semisimple with negative definite Killing form. Let $\mathbf{O}(\mathfrak{g}) \subset \mathbf{GL}(\mathfrak{g})$ the compact subgroup of β-preserving transformations. Clearly $\mathrm{Aut}(\mathfrak{g})$ is contained in $\mathbf{O}(\mathfrak{g})$ as a closed, thus compact subgroup. Now Corollary 5.4.4 yields the result.

(b) implies (c). Since the group of inner automorphisms $\mathrm{Inn}(\mathfrak{g})$ is compact, by Proposition 5.1.2 there exists an Ad-invariant positive definite inner product on $\mathfrak{g}$. It is also ad-invariant.

Now (b), (c) and (d) are equivalent. We next show that (b) and (d) imply (a). $\mathrm{Inn}([\mathfrak{g},\mathfrak{g}]) = \mathrm{Inn}(\mathfrak{g})$ is compact and $\mathrm{Inn}([\mathfrak{g},\mathfrak{g}])$ has Lie algebra $\mathrm{ad}([\mathfrak{g},\mathfrak{g}]) \cong [\mathfrak{g},\mathfrak{g}]$ since $[\mathfrak{g},\mathfrak{g}]$ is semisimple. Now $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g},\mathfrak{g}]$ is the Lie algebra of $S^1 \times \cdots \times S^1 \times \mathrm{Inn}([\mathfrak{g},\mathfrak{g}])$.

Finally (a) implies (b). Since $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, $\mathrm{Inn}(\mathfrak{g}) \cong G/Z(G)$ is compact. $\square$

5.5.2 Corollary *A semisimple Lie algebra is compact if and only if its Killing form is negative definite.*

5.6 Geometry of compact Lie groups with bi-invariant metrics

Recall that a Riemannian metric on a smooth manifold M is simply a smoothly varying assignment of an inner product $\langle,\rangle_p$ on the tangent space T_pM for each $p \in M$; here the smoothness of $\langle,\rangle$ refers to the fact that $p \mapsto \langle X_p, Y_p\rangle_p$ defines a smooth function on M for all smooth vector fields X, Y on M.

As an important application of Proposition 5.1.2, we show that a compact Lie group G admits a *bi-invariant Riemannian metric*, that is, a Riemannian metric with respect to which left translations and right translations are isometries.

Indeed there is a bijective correspondence between positive-definite inner products on $\mathfrak{g}$ and left-invariant Riemannian metrics on G: every inner product on $\mathfrak{g} = T_1G$ gives rise to a left-invariant metric on G by declaring the left-translations to be isometries, namely, $dL_g : T_1G \to T_gG$ is a linear isometry for all $g \in G$; conversely, every left-invariant metric on G is completely determined by its value at 1. Now, when is a left-invariant metric $\langle,\rangle$ on G also right-invariant? Note that differentiation of the obvious formula $R_g = L_g \circ \mathrm{Inn}_{g^{-1}}$ at 1 yields

$$d(R_g)_1 = (dL_g)_1 \circ \mathrm{Ad}_{g^{-1}},$$

where $g \in G$. We deduce that g is right-invariant if and only if $\langle,\rangle_1$ is Ad-invariant. Thus the existence of a bi-invariant metric on G follows from Proposition 5.1.2.

Let G be a compact connected Lie group endowed with a bi-invariant Riemannian metric and denote its Lie algebra by $\mathfrak{g}$. The Koszul formula for the

Levi-Civitâ connection ∇ on G is

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}\big(&X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ &+\langle [X,Y],Z\rangle + \langle [Z,X],Y\rangle + \langle [Z,Y],X\rangle\big),\end{aligned} \tag{5.6.1}$$

where X, Y, $Z \in \mathfrak{g}$. The inner product of left-invariant vector fields is constant, so the three terms in the first line of the right hand-side of (5.6.1) vanish; we apply ad-invariance of the inner product on $\mathfrak{g}$ to manipulate the remaining three terms and we arrive at

$$\nabla_X Y = \frac{1}{2}[X,Y].$$

In particular, $\nabla_X X = 0$ so every one-parameter subgroup of G is a geodesic through 1. Since there one-parameter groups going in all directions, they comprise all geodesics of G through 1 (of course, the geodesics through other points in G differ from one-parameter groups by a left translation).

5.6.2 Remark The statement about one-parameter groups coinciding with Riemannian geodesics through 1 is equivalent to saying that the exponential map of the Lie group coincides with the Riemannian exponential map $\mathrm{Exp}_1 : T_1G \to G$ that maps each $X \in T_1G$ to the value at time 1 of the geodesic through 1 with initial velocity X. Now geodesics through 1 are defined for all values of the parameter; in view of the Hopf-Rinow theorem, this means that G is complete as a Riemannian manifold, and any point in G can be joined by a geodesic to 1, or Exp_1 is surjective. We deduce that the (group) exponential map of a compact Lie group is a surjective map.

Now we compute the Riemannian sectional curvature of G. Let X, $Y \in \mathfrak{g}$ be an orthonormal pair. Then

$$\begin{aligned} K(X,Y) &= -\langle R(X,Y)X,Y\rangle \\ &= -\langle \nabla_X\nabla_Y X + \nabla_Y\nabla_X X - \nabla_{[X,Y]}X, Y\rangle \\ &= \frac{1}{4}||[X,Y]||^2, \end{aligned}$$

using again ad-invariance of the inner product.

Finally, let $\{X_i\}_{i=1}^n$ be an orthonormal basis of $\mathfrak{g}$. Then the Ricci curvature

$$\begin{aligned} Ric(X,X) &= \sum_{i=1}^n K(X,X_i) \\ &= \frac{1}{4}\sum_{i=1}^n ||[X,X_i]||^2. \end{aligned}$$

It follows that $Ric(X,X) \geq 0$ and $Ric(X,X) = 0$ if and only if X lies in the center of $\mathfrak{g}$.

There are several proofs of the following theorem. Here we use basic Riemannian geometry.

5.6.3 Theorem (Weyl) *Let G be a compact connected semisimple Lie group. Then the universal covering Lie group $\tilde{G}$ is also compact (equivalently, the fundamental group of G is finite).*

Proof. The universal covering $\tilde{G}$ has a structure of Lie group so that the projection $\tilde{G} \to G$ is a smooth homomorphism. Equip G with a bi-invariant Riemannian metric. Since $\mathfrak{g}$ is centerless, $\mathrm{Ric}(X, X) > 0$ for $X \neq 0$. By compactness of the unit sphere, $\mathrm{Ric}(X, X) \geq a\langle X, X\rangle$ for some $a > 0$. The Bonnet-Myers theorem yields that $\tilde{G}$ is compact. □

5.7 Cartan-Weyl structural theory

Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra. A *Cartan subalgebra* (CSA, for short) $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal Abelian subalgebra such that ad_H is a semisimple endomorphism of $\mathfrak{g}$ for all $H \in \mathfrak{h}$.

If $\mathfrak{g}$ is a compact Lie algebra, the CSA's of $\mathfrak{g}$ are exactly the Lie algebras of the maximal tori of the adjoint group of $\mathfrak{g}$. In this case, the existence of CSA is thus obvious, and their uniqueness, up to conjugation, is proved as in Proposition 2.5.5.

Consider the case $\mathfrak{g}$ is a complex semisimple Lie algebra. Usually the existence of a CSA is proved using Lie's and Engel's theorem. Then a compact real form of $\mathfrak{g}$ is obtained from a delicate construction of the so-called *Weyl basis* of $\mathfrak{g}$, with a specific form of the structure constants. Alternatively, Cartan noticed that if a basis $\{e_i\}$ of $\mathfrak{g}$ minimizes $\sum_{ijk} |c_{ij}^k|^2$ over all bases with $\beta(e_i, e_j) = -\delta_{ij}$ (Kronecker delta) for all i, j, where $[e_i, e_j] = \sum_k c_{ij}^k e_k$, then $\{e_i\}$ spans over $\mathbf{R}$ a compact real form of $\mathfrak{g}$. A proof of the existence of such a basis was accomplished by Richardson [39]. An immediate corollary is the existence of CSA of $\mathfrak{g}$, namely, the complexification of a maximal Abelian subalgebra of the compact real form.

Now fix a complex semisimple Lie algebra $\mathfrak{g}$ and a CSA $\mathfrak{h}$. Then $\{\mathrm{ad}_H : H \in \mathfrak{h}\}$ is a commuting family of semisimple endomorphisms of $\mathfrak{g}$. The common eigenspace decomposition is written

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}$ is the zero-eigenspace, Δ is a finite set of linear functionals on $\mathfrak{h}$, and

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \mathrm{ad}_H X = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}.$$

Each $\alpha \in \Delta$ is called a *root* and $\mathfrak{g}_\alpha$ is the associated *root space*. Since $\mathfrak{g}$ has a compact real form $\mathfrak{g}_u$ and $\mathfrak{h}$ is the complexification of a maximal Abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}_u$, $\alpha \in \Delta$ if and only $-\alpha \in \Delta$. The roots take pure imaginary values on $\mathfrak{t}$, and take real values on $\mathfrak{h}_{\mathbb{R}} = \sqrt{-1}\mathfrak{t}$. The Killing form is real-valued and positive-definite on $\mathfrak{h}_{\mathbb{R}}$, so it induces an Euclidean inner product there.

The set of roots Δ satisfies the following properties:

a. $\mathbf{R}\Delta = \mathfrak{h}_{\mathbf{R}}^*$

b. $a_{\alpha,\beta} := 2\frac{\langle\alpha,\beta\rangle}{||\alpha||^2} \in \mathbf{Z}$ for α, $\beta \in \Delta$.

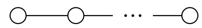
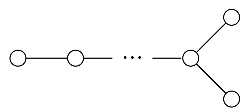
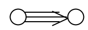
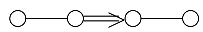
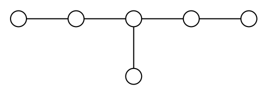
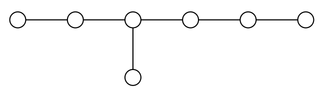
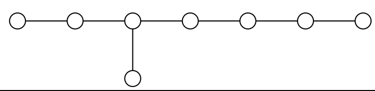
c. $s_\alpha(\Delta) = \Delta$, where s_α is the orthogonal reflection on $\alpha^\perp$.

A finite set Δ of nonzero linear functionals on an Euclidean space satisfying such properties is called an *abstract root system*. The second condition in the definition implies that if α, $\beta \in \Delta$, $\alpha = c\beta$ for some $c \in \mathbf{R}$ then $c = \pm 1$ or $\pm\frac{1}{2}$ or ± 2; in fact, $a_{c\beta,\beta} = 2/c$ and $a_{\beta,c\beta} = 2c$ must be integers. An abstract root system Σ is called *reduced* if it satisfies the additional condition that if α, $\beta \in \Delta$, $\alpha = c\beta$ for some $c \in \mathbf{R}$ then $c = \pm 1$. The root system Δ associated to a complex semisimple Lie algebra $\mathfrak{g}$ relative to a Cartan subalgebra $\mathfrak{h}$ is always reduced.

The group W generated by s_α for $\alpha \in \Delta$ is a finite reflection group, called the *Weyl group* of $\mathfrak{g}$ relative to $\mathfrak{h}$. The connected components of the complement of the union of the hyperplanes $\alpha = 0$ in $\mathfrak{h}_\mathbb{R}$ are called *Weyl chambers*. We fix a Weyl chamber $\mathcal{C}$. A root α is called *positive* if $\alpha > 0$ on $\mathcal{C}$. This gives a partition $\Delta = \Delta^+ \dot{\cup} (-\Delta^+)$, where Δ^+ is the set of positive roots. A root is *simple* if it is positive and cannot be written as the sum of two positive roots. The set Π of simple roots is a basis of $\mathfrak{h}^*$, and every $\alpha \in \Delta$ has an expression as an integral linear combination of simple roots, where the coefficients are all nonnegative in case $\alpha \in \Delta^+$.

Enumerate the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, where n is the rank of $\mathfrak{g}$. Put $a_{ij} = a_{\alpha_i,\alpha_j}$ (*Cartan integer*). The *Dynkin diagram* of $\mathfrak{g}$ is a kind of graph, where we take one vertex for each simple root, and we join vertices associated to α_i, α_j by $a_{ij}a_{ji} = 0$, 1, 2 or 3 lines. It turns out $||\alpha_i||^2 = ||\alpha_j||^2$ if $a_{ij}a_{ji} = 1$ and $||\alpha_i||^2 \neq ||\alpha_j||^2$ if $a_{ij}a_{ji} = 2$ or 3; in the latter case, we draw an arrow pointing to the shorter root on the double or triple lines. The Dynkin diagram is the disjoint union the Dynkin diagrams of the simple ideals of $\mathfrak{g}$, and it is connected if and only if $\mathfrak{g}$ is simple.

The classification of complex semisimple Lie algebras is thus reduced to a two-part problem: the classification of connected Dynkin diagrams, and the realization of each diagram as the diagram of a complex simple Lie algebra. The first part is a problem in Euclidean geometry, using properties of root systems. The list of admissible Dynkin diagrams is:

Cartan type	*Diagram*	*Condition*
$\mathbf{A}_n$		–
$\mathbf{B}_n$		$n \geq 2$
$\mathbf{C}_n$		$n \geq 3$
$\mathbf{D}_n$		$n \geq 4$
$\mathbf{G}_2$		–
$\mathbf{F}_4$		–
$\mathbf{E}_6$		–
$\mathbf{E}_7$		–
$\mathbf{E}_8$		–

Each diagram is realizable and the final list of complex simple Lie algebras, together with their compact real forms, is as follows (Killing-Cartan classification):

Type	$\mathfrak{g}$	$\mathfrak{g}_u$	dim	*Condition*
$\mathbf{A}_n$	$\mathfrak{sl}(n+1,\mathbf{C})$	$\mathfrak{su}(n+1)$	n^2+2n	$n \geq 1$
$\mathbf{B}_n$	$\mathfrak{so}(2n+1,\mathbf{C})$	$\mathfrak{so}(2n+1)$	$2n^2+n$	$n \geq 2$
$\mathbf{C}_n$	$\mathfrak{sp}(n,\mathbf{C})$	$\mathfrak{sp}(n)$	$2n^2+n$	$n \geq 3$
$\mathbf{D}_n$	$\mathfrak{so}(2n,\mathbf{C})$	$\mathfrak{so}(2n)$	$2n^2-n$	$n \geq 4$
$\mathbf{G}_2$	$\mathfrak{g}_2^{\mathbf{C}}$	$\mathfrak{g}_2$	14	–
$\mathbf{F}_4$	$\mathfrak{f}_4^{\mathbb{C}}$	$\mathfrak{f}_4$	52	–
$\mathbf{E}_6$	$\mathfrak{e}_6^{\mathbb{C}}$	$\mathfrak{e}_6$	78	–
$\mathbf{E}_7$	$\mathfrak{e}_7^{\mathbb{C}}$	$\mathfrak{e}_7$	133	–
$\mathbf{E}_8$	$\mathfrak{e}_8^{\mathbb{C}}$	$\mathfrak{e}_8$	248	–

5.7.1 Theorem *The simply-connected compact connected simple Lie groups are* $\mathbf{SU}(n)$, $\mathbf{Spin}(n)$, $\mathbf{Sp}(n)$, $\mathbf{G}_2$, $\mathbf{F}_4$, $\mathbf{E}_6$, $\mathbf{E}_7$ *and* $\mathbf{E}_8$.

6 References

[1] J. Berndt, *Polar actions on symmetric spaces*, Proceedings of the 15th International Workshop on Differential Geometry and the 4th KNUGRG-

OCAMI Differential Geometry Workshop [Volume 15], Natl. Inst. Math. Sci. (NIMS), Taejŏn, 2011, pp. 1–10.

[2] J. Berndt, S. Console, and C. Olmos, *Submanifolds and holonomy*, second ed., Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, Florida, USA, 2016.

[3] J. Berndt and C. Olmos, *On the index of symmetric spaces*, J. Reine Angew. Math. **737** (2018), 33–48.

[4] A. Borel, *Semisimple groups and Riemannian symmetric spaces*, Texts and Readings in Mathematics, vol. 16, Hindustan Book Agency, New Delhi, 1998.

[5] R. Bott, *An application of the Morse theory to the topology of Lie groups*, Bull. Soc. Math. France **84** (1956), 251–281.

[6] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958), 964–1029, Correction in Amer. J. Math. **83** (1961), 207–208.

[7] Claude C. Chevalley and R. D. Schafer, *The exceptional simple lie algebras* f_4 *and* e_6, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 137–141.

[8] É. Cartan, *La théorie des groupes et la géométrie*, Enseign. Math. **26** (1927), 200–225.

[9] ______, *Groupes simples clos et ouverts et géométrie riemannienne*, J. Math. Pures Appl. **8** (1929), 1–34.

[10] É Cartan, *La theorie des groupes finis et continus et l'analysis situs*, Mem. Sci. Math., vol. XLII, Gauthier-Villars, Paris, 1930.

[11] É. Cartan and J. A. Schouten, *On the geometry of the group-manifold of simple and semisimple groups*, Koninkl. Akad. Wetensch. Amsterdam Proc. **29** (1926), 803–815.

[12] J. Cheeger and D.G. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland Mathematical Library, no. 9, North-Holland Publishing Company, Amsterdam, Oxford, 1975.

[13] B.-Y. Chen and T. Nagano, *Totally geodesic submanifolds of symmetric spaces, II*, Duke Math. J. **45** (1978), no. 2, 405–425.

[14] ______, *A Riemannian geometric invariant and its applications to a problem of Borel and Serre*, Trans. Amer. Math. Soc. **308** (1988), no. 1, 273–297.

[15] C. Chevalley, *Theory of Lie groups I*, Princeton University Press, 1946.

[16] L. Conlon, *Variational completeness and K-transversal domains*, J. Differential Geom. **5** (1971), 135–147.

[17] R. J. Crittenden, *Minimum and conjugate points in symmetric spaces*, Canadian J. Math. **14** (1962), 320–328.

[18] J. Dadok, *Polar actions induced by actions of compact Lie groups*, Trans. Amer. Math. Soc. **288** (1985), 125–137.

[19] A. J. Di Scala and C. Olmos, *Variationally complete representations are polar*, Proc. Amer. Math. Soc. **129** (2001), 3445–3446.

[20] C. Gorodski and A. Kollross, *Some remarks on polar actions*, Ann. Global Anal. Geom. **49** (2016), 43–58.

[21] C. Gorodski and G. Thorbergsson, *Representations of compact Lie groups and the osculating spaces of their orbits*, Preprint, Univ. of Cologne, (also E-print math. DG/0203196), 2000.

[22] ———, *Variationally complete action on compact symmetric spaces*, J. Differential Geom. **62** (2002), 39–48.

[23] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001, corrected reprint of the 1978 original.

[24] R. Hermann, *Variational completeness for compact symmetric spaces*, Proc. Amer. Math. Soc. **11** (1960), 544–546.

[25] S. Klein, *Totally geodesic submanifolds of the exceptional Riemannian symmetric spaces of rank 2*, Osaka J. Math. **47** (2010), no. 4, 1077–1157.

[26] A. W. Knapp, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002.

[27] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Academic Press, 1963 and 1969, 2 volumes.

[28] A. Kollross, *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc. **354** (2002), 571–612.

[29] ———, *Hyperpolar actions on reducible symmetric spaces*, Transform. Groups **22** (2017), no. 1, 207–228.

[30] A. Kollross and A. Lytchak, *Polar actions on symmetric spaces of higher rank*, Bull. Lond. Math. Soc. **45** (2013), no. 2, 341–350.

[31] J.-L. Koszul, *Exposés sur les espaces homogènes symétriques*, Sociedade de Matemaática de São Paulo, São Paulo, 1959.

[32] O. Loos, *Symmetric spaces*, Benjamin Inc., 1969, 2 volumes.

[33] A. Lytchak and G. Thorbergsson, *Variationally complete actions on nonnegatively curved manifolds*, Illinois J. Math. **51** (2007), no. 2, 605–615.

[34] R. S. Palais and C.-L. Terng, *A general theory of canonical forms*, Trans. Amer. Math. Soc. **300** (1987), 771–789.

[35] ———, *Critical point theory and submanifold geometry*, Lect. Notes in Math., no. 1353, Springer-Verlag, 1988.

[36] F. Podestà and G. Thorbergsson, *Polar actions on rank one symmetric spaces*, J. Differential Geom. **53** (1999), 131–175.

[37] L. Pontrjagin, *Topological groups*, Princeton University Press, 1939, translated by E. Lehmer.

[38] J. C. Díaz Ramos, M. Domínguez Vázquez, and A. Kollross, *Polar actions on complex hyperbolic spaces*, Math. Z. **287** (2017), no. 3-4, 1183–1213.

[39] R. Richardson, *Compact real forms of a complex semi-simple Lie algebra*, J. Differential Geometry **2** (1968), 411–419.

[40] J. Szenthe, *Orthogonally transversal submanifolds and generalizations of the Weyl group*, Period. Math. Hungarica **15** (1984), 281–299.

[41] H. Weyl, *Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen. II.*, Math. Z. **24** (1926), no. 1, 328–376.

[42] J.A. Wolf, *Spaces of constant curvature*, sixth ed., AMS Chelsea Publishing, Providence, RI, 2011.