Geometry of isometric actions

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Basic setting

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- (G, M): isometric action

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- Given (G, M), relate the geometry of the orbits with algebraic invariants of the action.

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An isometric action of a compact Lie group on a complete Riemannian manifold is variationally complete if it produces enough Jacobi fields along geodesics to determine the multiplicities of focal points to the orbits.

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Theorem (Hermann, PAMS 1960)

Let G/K, G/H be compact symmetric spaces. Then the H action on G/K is variationally complete.

K-transversal domains

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Theorem (Conlon, JDG 1971) A hyperpolar action of a compact Lie group on a complete Riemannian manifold is variationally complete.

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- Dadok (TAMS 1985) classified the polar representations.
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- Di Scala and Olmos (PAMS 2000) proved that a variationally complete representation of a compact Lie group is polar.

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- Case of rank greater than one: so far no example known of nonhyperpolar, polar action.

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$$\tau(f) = \frac{1}{\mathrm{vol}(S^{n-1})} \int_{\nu^1(M)} |G| \, d\mathrm{vol}_{\nu^1(M)},$$

where $\eta^* \, d\mathrm{vol}_{S^{n-1}} = G \, d\mathrm{vol}_{\nu^1(M)}$

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- Chern and Lashof also proved that if $\tau(f) = 2$, *then* M *is a convex hypersurface in an affine subspace.*

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- Equivalently: an immersion $f : M \to \mathbf{R}^m$ is tight if every Morse height function $h_\xi(x) = \langle f(x), \xi \rangle$, $x \in M$, has the property that its number of critical points is equal to $\beta(M; \mathbf{F})$. (i.e. h_ξ is F*-perfect*)

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- Since $\tau(f) \geq \gamma(M) \geq \beta(M; \mathbf{F})$, a tight immersion of a compact manifold has minimum total absolute curvature.
- For example: the standard embeddings of the projective spaces are tight.

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- A spherical tight immersion is taut.

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- Generalized flag manifolds are homogeneous examples of *isoparametric submanifolds*.

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- Palais and Terng (TAMS 1987) showed that the only compact homogeneous isoparametric submanifolds of Euclidean space are the principal orbits of polar representations.

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- For instance, the Lens spaces distinct from the real projective space cannot be tautly embedded in Euclidean space.

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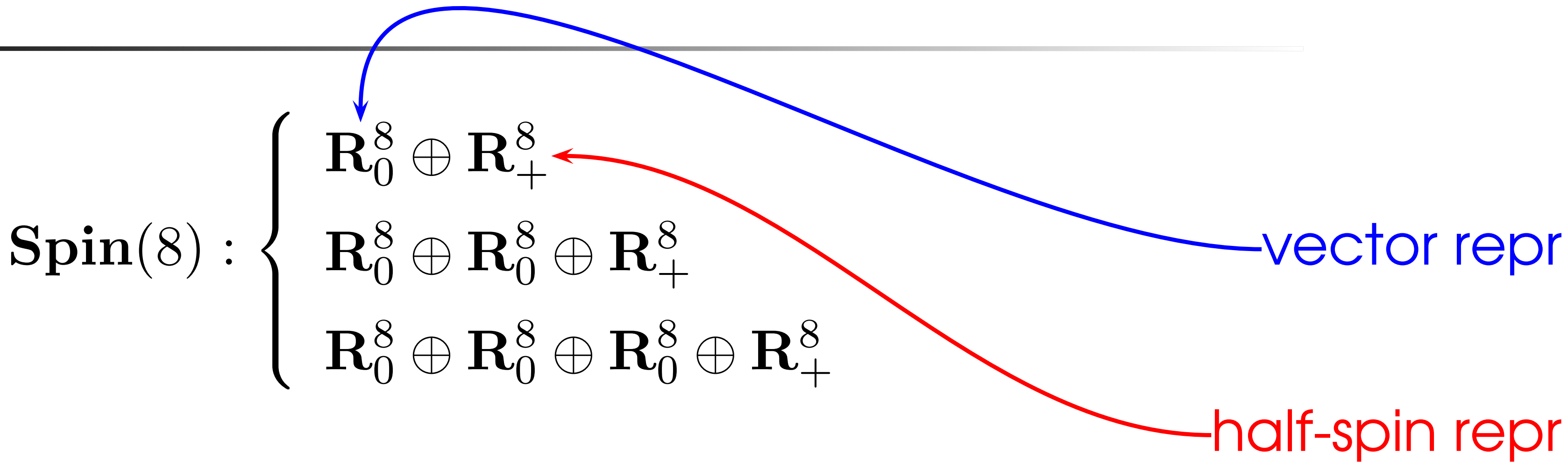

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vector repr

half-spin repr

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But these results are still far from the complete classification of taut homogeneous submanifolds of Euclidean space...

The exceptional examples

Let us recall...

Taut irreducible representations

- G. and Thorbergsson (Crelle 2003) classified the irreducible representations of compact Lie groups all of whose orbits are taut submanifolds.
- Besides the linear isotropy representations of symmetric spaces, there are three exceptional families ($n \geq 2$):

$\mathbf{SO}(2) \times \mathbf{Spin}(9)$	(standard) $\otimes_{\mathbf{R}}$ (spin)
$\mathbf{U}(2) \times \mathbf{Sp}(n)$	(standard) $\otimes_{\mathbf{C}}$ (standard)
$\mathbf{SU}(2) \times \mathbf{Sp}(n)$	(standard)^3 $\otimes_{\mathbf{H}}$ (standard)

- These are precisely the irreducible representations of cohomogeneity three.

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- The principal orbits of the exceptional taut irreducible representations fail to be isoparametric along one direction.
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- From another perspective: there is a one-dimensional subspace of the normal space which rotates in the direction of the tangent space when we move along a along a normal geodesic.
- Question: *is it possible to understand the geometry of the exceptional representations and find other, similar examples?*

Copolarity of (G, M)

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- G., Olmos and Tojeiro (TAMS 2004): a minimal K-section through a regular point of the action is the smallest connected, complete, totally geodesic submanifold of M through that point which intersects all the orbits and such that, at any intersection point with a principal orbit, its tangent space contains the normal space of that orbit with codimension k.

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- This is a good definition and uniquely specifies an integer k which we call the copolarity of (G, M).
- The case $k = 0$ case precisely corresponds to the polar actions.
- For most actions, the minimal k-action coincides with the ambient space. In this case, k equals the dimension of a principal orbit. We say that such isometric actions have trivial copolarity.

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- *What is the meaning of the integer* k*?*

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- An irreducible representation of a compact Lie group is taut if and only if $k = 0$ or $k = 1$.
- The codimension of a nontrivial minimal k-section of a nonpolar irreducible representation is at least 3.

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Let (G, V) be an orthogonal representation with copolarity $k = 1$ and let N be a principal orbit. Then the submanifold N of V splits extrinsically as $N = N_0 \times N_1$, where N_0 is either a homogeneous isoparametric submanifold or a point, and N_1 is one of the following:

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The main tool in the proof of this theorem is the concept of normal holonomy of Olmos.

Final remarks

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An open problem in the area is to similarly characterize the principal orbits of more general orthogonal representations in terms of their submanifold geometry and topology.

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Recall that the principal orbits of polar representations can be characterized as being the only compact homogeneous isoparametric submanifolds of Euclidean space (Palais and Terng, TAMS 1987).

An open problem in the area is to similarly characterize the principal orbits of more general orthogonal representations in terms of their submanifold geometry and topology.

We believe that orthogonal representations of low copolarity may serve as testing cases for this problem.

Thank you!