3.107 $T \in \mathcal{L}(V,W)$ $\dim V, \dim W < \infty$

$\Rightarrow$ (a) $\ker T' \overset{?}{=} (\operatorname{im} T)^0$

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ & & \cup \\ & & \operatorname{im} T \end{array} \Rightarrow (\operatorname{im} T)^0 \subset W'$$

$$V' \xleftarrow{T'} W' \supset \ker T'$$

3.108 $\dim V, \dim W < \infty$, $T \in \mathcal{L}(V,W)$

$\Rightarrow$ (T sobrej $\Leftrightarrow$ T' inj)

3.109 $\dim V, \dim W < \infty$ $T \in \mathcal{L}(V,W)$

$\Rightarrow$ (a) $\dim \operatorname{im} T' = \dim \operatorname{im} T$

(b) $\operatorname{im} T' = (\ker T)^0$

3.110 $\dim V, \dim W < \infty$ $T \in \mathcal{L}(V,W)$

$\Rightarrow$ (T inj $\Leftrightarrow$ T' sobrej)

<u>Dem de 3.109</u> $T' : W' \to V'$

(a) $\dim \operatorname{im} T' = \dim W' - \dim \ker T'$ (T.F.A.L.)

$= \dim W - \dim (\operatorname{im} T)^0$ (3.107)

$= \dim \operatorname{im} T$ (3.106)

(b) Suponhamos que $\varphi \in \underline{\operatorname{im} T'}$. Então $\varphi = T'\psi$ para alguma $\psi \in W'$. Seja $v \in \ker T$. Então

$$\varphi(v) = (T'\psi)v = (\psi \circ T)v = \psi(\underbrace{Tv}_{=0}) = \psi(0) = 0$$

$\therefore \varphi \in (\ker T)^0$

$\therefore \operatorname{im} T' < (\ker T)^0 \quad (1)$

$$\dim \operatorname{im} T' \overset{(a)}{=} \dim \operatorname{im} T$$
$$= \dim V - \dim \ker T \quad (\text{T.F.A.L.})$$
$$= \dim (\ker T)^0 \quad (3.106) \quad (2)$$

(1) e (2) $\Rightarrow$ $\operatorname{im} T' = (\ker T)^0$ //.

—//—

<u>Matriz</u> <u>transposta</u> de $A \in M(m \times n, \mathbb{F})$, $A = (a_{ij})$, é $A^t = (a_{ji}) \in M(n \times m, \mathbb{F})$

<u>Transposição</u> <u>de</u> <u>matrizes</u> $A \mapsto A^t$:

$$\left.\begin{aligned} (A+B)^t &= A^t + B^t \\ (\lambda A)^t &= \lambda A^t \end{aligned}\right\} \quad \underline{\text{linear}}$$
$$(AB)^t = B^t A^t$$
$$(A^t)^t = A \qquad \underline{\text{idempotente}}$$

3.114 Suponhamos que $\dim V, \dim W < \infty$, sejam

$B_V: v_1, \ldots, v_n$ bases de V, e sejam
$B_W: w_1, \ldots, w_m$ $\quad W$

$B_{V'}: \varphi_1, \ldots, \varphi_n$ as bases duais de V'
$B_{W'}: \psi_1, \ldots, \psi_m$ $\quad W'$.

Seja também $T \in \mathcal{L}(V,W)$, e consideremos $T' \in \mathcal{L}(W',V')$. Então

$$[T']^{B_{W'}}_{B_{V'}} = \left([T]^{B_V}_{B_W}\right)^t$$

Dem. Sejam $A = [T]^{B_V}_{B_W} = (a_{ij})$, $B = [T']^{B_{W'}}_{B_{V'}} = (b_{ij})$.

$$\underset{v_k}{V} \xrightarrow{T} \underset{w_1,\dots,w_m}{W}$$

$$\underset{\varphi_1,\dots,\varphi_n}{V'} \xleftarrow{T'} \underset{\psi_1,\dots,\psi_m}{W'}$$

$$T'(\psi_j) = \sum_{i=1}^{n} b_{ij}\,\varphi_i \qquad (1)$$

$$\begin{aligned} T'(\psi_j)(v_k) &= \psi_j(Tv_k) \\ &= \psi_j\left(\sum_{l=1}^{m} a_{lk} w_l\right) \end{aligned}$$

$$= \sum_{l=1}^{m} a_{lk} \underbrace{\psi_j(w_l)}_{=\begin{cases} 1, & l=j \\ 0, & l\neq j \end{cases}} = a_{jk} \qquad (2)$$

Avaliando (1) em v_k:

$$T'(\psi_j)v_k = \sum_{i=1}^{n} b_{ij} \underbrace{\varphi_i(v_k)}_{=\begin{cases} 1, & i=k \\ 0, & i\neq k \end{cases}}$$

$$= b_{kj} \qquad (3)$$

Comparando (2) e (3), obtemos:

$$a_{jk} = b_{kj}$$

$$A = B^t \quad \text{ou} \quad A^t = B \quad //.$$

—n—

POSTO DE UMA MATRIZ

3.115 Def. Seja $A \in M(m \times n, \mathbb{F})$.

- O posto-linha de A é a dimensão do span de suas linhas (= nº máximo de linhas L.I.) em $\mathbb{F}^n$.
- O posto-coluna de A é a dimensão do span de suas colunas (= nº máx. de colunas L.I.) em $\mathbb{F}^m$.

3.116 Ex. $A = \begin{pmatrix} 4 & 7 & 1 & 8 \\ 3 & 5 & 2 & 9 \end{pmatrix} \in M(2 \times 4, \mathbb{R})$

$$\left.\begin{aligned} \text{posto-linha}(A) &= \dim \operatorname{span}((4,7,1,8), (3,5,2,9)) = 2 \\ \text{posto-coluna}(A) &= \dim \underbrace{\operatorname{span}\Big(\underbrace{\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}}_{\text{L.I.}}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 9 \end{pmatrix}\Big)}_{\subset \mathbb{R}^2} = 2 \end{aligned}\right] =$$

3.11.7 Suponhamos que $\dim V, \dim W < \infty$ e $T \in \mathcal{L}(V, W)$. Então $\dim \operatorname{im} T =$ posto-coluna $[T]^{B_V}_{B_W}$, onde B_V, B_W são qq. bases de V, W, resp.

<u>Dem.</u> Escrevamos $B_V: v_1, \dots, v_n$ e $B_W: w_1, \dots, w_m$.

Se $w \in \operatorname{im} T$, então $w = Tv$ para algum $v \in V$.

Temos $v = a_1 v_1 + \dots + a_n v_n$ para únicos $a_1, \dots, a_n \in \mathbb{F}$.

Agora

$$w = Tv = a_1 Tv_1 + \dots + a_n Tv_n$$

$$\Rightarrow \boxed{\operatorname{im} T \subset \operatorname{span}(Tv_1, \dots, Tv_n)} \quad (1)$$

Reciprocamente,

$$Tv_1, \dots, Tv_n \in \underbrace{\operatorname{im} T}_{\text{é um subespaço de } W}$$

$$\Rightarrow \boxed{\operatorname{span}(Tv_1, \dots, Tv_n) \subset \operatorname{im} T} \quad (2)$$

$$(1) \text{ e } (2) \Rightarrow \boxed{\operatorname{im} T = \operatorname{span}(Tv_1, \dots, Tv_n)}$$

$$\begin{tikzcd} V \arrow[r, "T"] & W \supset \operatorname{im} T \arrow[d, "\Phi\ (\text{isomorfismo})", "\cong"'] \\ & \mathbb{F}^m \supset \Phi(\operatorname{im} T) \end{tikzcd} \qquad \begin{tikzcd} w \arrow[d, mapsto, "\Phi"] \\ {[w]_{B_W}} \end{tikzcd}$$

$$[T]^{B_V}_{B_W} = \begin{array}{c} \\ w_1 \\ \vdots \\ w_m \end{array} \begin{array}{c} \begin{array}{ccc} Tv_1 & \cdots & Tv_n \end{array} \\ \left(\begin{array}{ccc} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{array} \right) \\ \begin{array}{ccc} \uparrow & & \uparrow \\ \Phi(Tv_1) & & \Phi(Tv_n) \end{array} \end{array}$$

$$\begin{aligned} \dim \operatorname{im} T &= \dim \operatorname{span}(Tv_1, \ldots, Tv_n) \\ &= \dim \operatorname{span}(\Phi(Tv_1), \ldots, \Phi(Tv_n)) \\ &= \text{posto-coluna } [T]^{B_V}_{B_W} \quad // \end{aligned}$$

3.113 Seja $A \in M(m \times n, \mathbb{F})$. Então

$$\boxed{\text{posto-linha}(A) = \text{posto-coluna}(A)}$$

Dem. Seja $T : M(n \times 1, \mathbb{F}) \to M(m \times 1, \mathbb{F})$ definida por $\underbrace{Tx}_{m\times 1} = \underbrace{A}_{m\times n}\underbrace{x}_{n\times 1}$. Então T é linear e

$[T]^{B_V}_{B_W} = A$, onde B_V, B_W são as bases canônicas de

$$M(n \times 1, \mathbb{F}) = \mathbb{F}^n$$
$$M(m \times 1, \mathbb{F}) = \mathbb{F}^m$$

$$[T] = \begin{array}{c} \\ e_1 \\ \vdots \\ e_m \end{array} \begin{array}{c} \begin{array}{ccc} Te_1 & \cdots & Te_n \end{array} \\ \left(\begin{array}{ccc} | & & | \\ Ae_1 & \cdots & Ae_m \\ | & & | \end{array} \right) \end{array}$$

Por ex:

$$Ae_1 = A\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix}a_{11}\\\vdots\\a_{m1}\end{pmatrix}$$

Agora:

$$\begin{aligned}
\text{posto-coluna}(A) &= \text{posto-coluna}\,[T]^{B_V}_{B_W} \\
&= \dim \operatorname{im} T && (3.117) \\
&= \dim \operatorname{im} T' && (3.109) \\
&= \text{posto-coluna}\,[T']^{B_V'}_{B_W'} && (3.117) \\
&= \text{posto-coluna}\left([T]^{B_V}_{B_W}\right)^t && (3.114) \\
&= \text{posto-linha}\,[T]^{B_V}_{B_W} \\
&= \text{posto-linha}(A) \quad //
\end{aligned}$$

bases duais de B_V, B_W

3.119 <u>Def</u>. [Posto de $A \in M(m\times n, \mathbb{F})$]

$$\text{posto}(A) := \text{posto-coluna}(A) = \text{posto-linha}(A)$$

<u>§3.C, Ex.3</u> $\dim V, \dim W < \infty$ $T \in \mathcal{L}(V,W)$

$\Rightarrow \exists$ bases B_V de V e B_W de W t.q.

$$[T]^{B_V}_{B_W} = \left(\begin{array}{c|c} \underbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}_{} & 0 \\ \hline \underbrace{0}_{\dim \operatorname{im} T} & 0\cdots 0 \end{array}\right)$$

Resolução: Seja $u_1, \dots, u_k$ uma base de $\ker T$. Completemos essa base a uma base

B_V: $\underbrace{v_1, \dots, v_n}, \underbrace{u_1, \dots, u_k}$

de V. Sejam

$$w_1 = Tv_1, \dots, w_n = Tv_n \qquad (1)$$

vetores em W. Mostremos que a lista (1) é L.I. Consideremos uma relação

$$a_1 w_1 + \dots + a_n w_n = 0. \qquad (2)$$

$\Rightarrow \quad a_1 Tv_1 + \dots + a_n Tv_n = 0$

$\Rightarrow \quad T(a_1 v_1 + \dots + a_n v_n) = 0$

$\Rightarrow a_1 v_1 + \dots + a_n v_n \in \ker T.$

$\Rightarrow a_1 v_1 + \dots + a_n v_n = b_1 u_1 + \dots + b_k u_k$

$\Rightarrow a_1 v_1 + \dots + a_n v_n - b_1 u_1 - \dots - b_k u_k = 0$

é uma relação lin entre os vetores da base B_V de V $\Rightarrow \underbrace{a_1 = \dots = a_n}_{=0} = b_1 = \dots = b_k = 0$

$\Rightarrow$ A relação (2) é trivial. Agora $w_1, \dots, w_n$ é LI.

Completemos $w_1, \dots, w_n$ a uma base

$$B_W: \underbrace{w_1, \ldots, w_n}_{\text{base de im}T}, w_{n+1}, \ldots, w_m$$

Finalmente:

$$\begin{array}{cc} & \begin{array}{ccc|ccc} Tv_1 & \cdots & Tv_n & \multicolumn{3}{c}{\overbrace{Tu_1 \quad \cdots \quad Tu_k}^{\ker T}} \end{array} \\ \begin{array}{c} \text{im}T\left\{\begin{array}{c} w_1 \\ \vdots \\ w_n \end{array}\right. \\ w_{n+1} \\ \vdots \\ w_m \end{array} & \left(\begin{array}{ccc|ccc} 1 & & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots & & \vdots \\ 0 & & 1 & 0 & \cdots & 0 \\ \hline & & & 0 & \cdots & 0 \\ & 0 & & \vdots & & \vdots \\ & & & 0 & \cdots & 0 \end{array}\right) \end{array}$$

//.