General Results for the Beta Modified Weibull Distribution

Saralees Nadarajah
School of Mathematics, University of Manchester,
Manchester M13 9PL, UK
e-mail:*saralees.nadarajah@manchester.ac.uk*
Gauss M. Cordeiro
Departamento de Estatística e Informática,
Universidade Federal Rural of Pernambuco,
52171-900, Recife, PE, Brazil
e-mail:*gausscordeiro@uol.com.br*
Edwin M. M. Ortega
Departamento de Ciências Exatas
Universidade de São Paulo,
13418-900, Piracicaba, SP Brazil.
e-mail:*edwin@esalq.usp.br*

Abstract

We study in detail the so-called beta modified Weibull distribution, motivated by the wide use of the Weibull distribution in practice, and also for the fact that the generalization provides a continuous crossover towards cases with different shapes. The new distribution is important since it contains as special sub-models some widely-known distributions, such as the generalized modified Weibull, beta Weibull, exponentiated Weibull, beta exponential, modified Weibull and Weibull distributions, among several others. It also provides more flexibility to analyze complex real data. Various mathematical properties of this distribution are derived, including its moments and moment generating function. We examine the asymptotic distributions of the extreme values. Explicit expressions are also derived for the characteristic function, mean deviations, Bonferroni and Lorenz curves, reliability and entropies. The estimation of parameters is approached by two methods: moments and maximum likelihood. We compare by simulation the performances of the estimates from these methods. We obtain the expected information matrix. Two applications are presented to illustrate the proposed distribution.

Keywords: Beta distribution; Exponentiated exponential; Exponentiated Weibull; Fisher information matrix; Generalized Modified Weibull; Maximum likelihood; Modified Weibull; Weibull distribution.

1 Introduction

The Weibull distribution, having exponential and Rayleigh as special sub-models, is a very popular distribution for modeling lifetime data and for modeling phenomenon with monotone failure rates. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, the Weibull distribution does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates such as the bathtub shaped and the unimodal failure rates which are common in reliability and biological studies. Such bathtub hazard curves have nearly flat middle portions and the corresponding densities have a positive anti-mode. An example of bathtub shaped failure rate is the human mortality experience with a high infant mortality rate which reduces rapidly to reach a low level. It then remains at that level for quite a few years before picking up again. Unimodal failure rates can be observed in course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

The models that present bathtub shaped failure rate are very useful in survival analysis. But, according to Nelson (1982), the distributions presented in shape literature with this type of data, such as the distributions proposed by Hjorth (1980), are sufficiently complex and, therefore, difficult to be modeled. Later, other works had introduced new distributions for modeling bathtub shaped failure rate. For example, Rajarshi and Rajarshi (1988) presented a revision of these distributions and Haupt and Schabe (1992) considered a lifetime model with bathtub failure rates. But, these models do not present much practicability to be used. However, in the last few years, new classes of distributions were proposed based on modifications of the Weibull distribution to cope with bathtub shaped failure rate. A good review of some of these models is presented in Pham and Lai (2007). Between these, the exponentiated Weibull (EW) (Mudholkar *et al.*, 1995, 1996), the additive Weibull (Xie and Lai, 1995), the extended Weibull (Xie *et al.*, 2002), the modified Weibull (MW) (Lai *et al.*, 2003), the beta exponential (BE) (Nadarajah and Kotz, 2006) and the extended flexible Weibull (Bebbington *et al.*, 2007) distributions. More recently, extensions are the generalized modified Weibull (GMW) by Carrasco *et al.* (2008) and the beta Weibull (BW) (Cordeiro and Nadarajah, 2010) distributions.

In this paper, we introduce a new distribution with five parameters, referred to as the beta modified Weibull (BMW) distribution, with the hope it will attract wider application in reliability, biology and other areas of research. This generalization contains as special sub-models several distributions such as the EW (Mudholkar *et al.*, 1995, 1996), exponentiated exponential (EE) (Gupta and Kundu, 1999, 2001), MW (Lai *et al.*, 2003), generalized Rayleigh (GR) (Kundu and Rakab, 2005) and GMW distributions, among several others. The new distribution due to its flexibility in accommodating all the forms of the risk function seems to be an important distribution that can be used in a variety of problems in modeling survival data. The BMW distribution is not only convenient for modeling comfortable bathtub-shaped failure rate data but it is also suitable for testing goodness-of-fit of some special sub-models such as the EW, BW, MW and GMW distributions.

The rest of the paper is organized as follows. In Section 2, we define the BMW distribution, present some special sub-models and provide expansions for its cdf and pdf. Two methods for simulating BMW variates and an expansion for the quantile function are provided in Section 3. General expansions for the moments are given in Section 4. Expansions for the moment generating function (mgf) and characteristic function (chf) are presented in Section 5. Section 6 is devoted to mean deviations about the mean and the median. Bonferroni and Lorenz curves are given in Section 7. The asymptotic distributions of the extreme values is discussed in Section 8. Estimation methods of moments and maximum likelihood, including the case of censoring, and the Fisher information matrix are presented in Section 9. The performances of the two estimation methods (moments and maximum likelihood) are also compared in this section. Section 10 provides two applications to real data. Section 11 ends with some conclusions. The paper also contains three appendices giving technical details.

2 Model Definition

The BMW distribution stems from the following general class: if $G(x)$ denotes the cumulative distribution function (cdf) of a random variable, then a generalized class of distributions can be defined for $a > 0$ and $b > 0$ by

$$F(x) = I_{G(x)}(a, b) = \frac{1}{B(a, b)} \int_0^{G(x)} w^{a-1}(1 - w)^{b-1} dw. \tag{1}$$

This class of generalized distributions has been receiving increased attention over the last years, in particular after the works of Eugene *et al.* (2002) and Jones (2004). The beta normal distribution obtained by taking $G(x)$ in (1) to be the cdf of the normal distribution was studied by Gupta and Nadarajah (2004) and Nadarajah and Kotz (2004). Nadarajah and Kotz (2004) and Barreto-Souza *et al.* (2009) provided closed form expressions for the moments and discussed maximum likelihood estimation for the beta Gumbel and beta Fréchet distributions, respectively. Consider the cdf of the modified Weibull distribution

$$G_{\alpha,\gamma,\lambda}(x) = 1 - \exp\left\{-\alpha x^{\gamma} \exp(\lambda x)\right\}, \tag{2}$$

due to Lai *et al.* (2003). Putting (2) into equation (1) yields the cdf of the BMW distribution (with five positive parameters and $x > 0$)

$$F(x) = \frac{1}{B(a, b)} \int_0^{1-\exp\{-\alpha x^{\gamma} \exp(\lambda x)\}} w^{a-1}(1 - w)^{b-1} dw. \tag{3}$$

The probability density function (pdf) and the hazard rate function (hrf) associated with (3) since $I_x(a, b) = I_{1-x}(b, a)$ are

$$f(x) = \frac{\alpha x^{\gamma-1}(\gamma + \lambda x) \exp(\lambda x)}{B(a, b)} \left[1 - \exp\left\{-\alpha x^{\gamma} \exp(\lambda x)\right\}\right]^{a-1} \exp\left\{-b\alpha x^{\gamma} \exp(\lambda x)\right\}, \tag{4}$$

and

$$h(x) = \frac{\alpha x^{\gamma-1}(\gamma + \lambda x) \exp(\lambda x)}{B(a, b) I_{\exp\{-\alpha x^{\gamma} \exp(\lambda x)\}}(b, a)} \left[1 - \exp\left\{-\alpha x^{\gamma} \exp(\lambda x)\right\}\right]^{a-1} \exp\left\{-b\alpha x^{\gamma} \exp(\lambda x)\right\}, \tag{5}$$

respectively.

If X is a random variable with pdf (4), we write $X \sim \text{BMW}(a, b, \alpha, \gamma, \lambda)$. Plots of the BMW pdf (4) are shown in Figures 1a, 1b and 1c. Figure 2 illustrates some of the possible shapes of the hazard function (5).

[Figures 1 and 2 about here.]

The BMW pdf is important since it includes as special sub-models several well-known distributions (Silva *et al.*, 2010). For $\lambda = 0$, it reduces to the BW distribution. If $\gamma = 1$ in addition to $\lambda = 0$, it simplifies further to the beta exponential (BE) distribution. The GMW distribution is also a special case when $b = 1$. If $a = 1$ in addition to $b = 1$, it yields the MW distribution. For $b = 1$ and $\lambda = 0$, the BMW distribution reduces to the EW distribution. If $\gamma = 1$ in addition to $b = 1$ and $\lambda = 0$, the BMW distribution becomes the EE distribution. For $\gamma = 2$, $\lambda = 0$ and $b = 1$, the BMW distribution reduces to the GR distribution. The Weibull distribution is clearly the simple special case for $a = b = 1$ and $\lambda = 0$. Other special sub-models of the BMW distribution are: the beta modified Rayleigh (BMR), beta modified exponential (BME), generalized modified Rayleigh (GMR), generalized modified exponential (GME), beta Rayleigh (BR), modified Rayleigh (MR) and modified exponential (ME), all sub-models reported in Silva *et al.* (2010).

The asymptotes of (3), (4) and (5) as $x \to 0, \infty$ are given by

$$F(x) \sim \frac{\alpha^a}{aB(a,b)} x^{\gamma a}$$

as $x \to 0$,

$$F(x) \sim 1 - \frac{1}{bB(a,b)} \exp\left\{-b\alpha x^{\gamma} \exp(\lambda x)\right\}$$

as $x \to \infty$,

$$f(x) \sim \frac{\gamma \alpha^a}{B(a,b)} x^{\gamma a - 1}$$

as $x \to 0$,

$$f(x) \sim \frac{\alpha}{B(a,b)} x^{\gamma-1} (\gamma + \lambda x) \exp\left\{\lambda x - b\alpha x^{\gamma} \exp(\lambda x)\right\}$$

as $x \to \infty$,

$$\tau(x) \sim \frac{\gamma \alpha^a}{B(a,b)} x^{\gamma a - 1}$$

as $x \to 0$, and

$$\tau(x) \sim \alpha b x^{\gamma-1} (\gamma + \lambda x) \exp(\lambda x)$$

as $x \to \infty$. Note that the lower tails of the pdf are polynomials. The hazard rate always increases as $x \to \infty$. The initial hazard rate can be increasing or decreasing depending on whether $\gamma a > 1$ or $\gamma a < 1$.

Throughout this paper we use the following representations for (3) and (4) due to Silva *et al.* (2010):

$$F(x) = 1 - \sum_{j=0}^{\infty} w_j \left\{1 - G_{\alpha(b+j),\gamma,\lambda}(x)\right\},$$

and

$$f(x) = \sum_{j=0}^{\infty} w_j g_{\alpha(b+j),\gamma,\lambda}(x), \tag{6}$$

where $g_{\alpha(b+j),\gamma,\lambda}(x) = dG_{\alpha(b+j),\gamma,\lambda}(x)/dx$ and

$$w_j = \frac{(-1)^j \Gamma(a)}{B(a,b)\Gamma(a-j)(b+j)j!}. \tag{7}$$

Clearly, expansion (6) reveals that the BMW pdf is a mixture of MW densities (holding for any parameter values). It is very useful to derive the ordinary, central, inverse and factorial moments of the BMW distribution from a weighted infinite (or finite if a is an integer) linear combination of those quantities for MW distributions.

We shall also use the following result due to Carrasco *et al.* (2008, Section 4, equations (5)-(7)):

$$\int_A \kappa(x) dG_{\alpha,\gamma,\lambda}(x) = \alpha \int_A \kappa\left(\sum_{j=1}^{\infty} a_j x^{j/\gamma}\right) \exp(-\alpha x) dx, \tag{8}$$

for an integrable function $\kappa(\cdot)$ and for an integrable set A, where

$$a_j = \frac{(-1)^{j+1} j^{j-2} \lambda^{j-1}}{(j-1)! \gamma^{j-1}}. \tag{9}$$

In fact, using the Lambert $W(\cdot)$ function, which is exactly equal to the $F(\cdot)$ function given in Carrasco *et al.* (2008, Section 4), we can rewrite (8) as

$$\int_A \kappa(x) dG_{\alpha,\gamma,\lambda}(x) = \alpha \int_A \kappa\left(\frac{\gamma}{\lambda} W\left(\frac{\lambda x^{1/\gamma}}{\gamma}\right)\right) \exp(-\alpha x) dx. \tag{10}$$

The shape of the pdf (4) can be described analytically. The critical points of the pdf are the solutions of the equation:

$$\lambda + \frac{\gamma - 1}{x} + \frac{\lambda}{\gamma + \lambda x} = \alpha x^{\gamma - 1}(\gamma + \lambda x)\exp(\lambda x)\left[b - \frac{a-1}{\exp\left\{\alpha x^{\gamma}\exp(\lambda x)\right\} - 1}\right]. \tag{11}$$

There may be more than one solution to (11). If $x = x_0$ is a root of (11) then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\lambda(x_0) < 0$, $\lambda(x_0) > 0$ or $\lambda(x_0) = 0$, where

$$\begin{aligned}
\lambda(x) \;&=\; \lambda + \frac{1-\gamma}{x^2} - \frac{\lambda^2}{(\gamma + \lambda x)^2} \\
&\quad + (a-1)\alpha x^{\gamma-2}\exp(\lambda x)\left[\frac{(\lambda x + \gamma - 1)(\gamma + \lambda x) + \lambda x}{\exp\left\{\alpha x^{\gamma}\exp(\lambda x)\right\} - 1} - \frac{\alpha x^{\gamma}(\gamma + \lambda x)^2 \exp(\lambda x)}{\left\{\exp\left\{\alpha x^{\gamma}\exp(\lambda x)\right\} - 1\right\}^2}\right] \\
&\quad - b\alpha x^{\gamma-2}\exp(\lambda x)\left\{(\gamma + \lambda x - 1)(\gamma + \lambda x) + \lambda x\right\}.
\end{aligned}$$

3 Simulation and Quantile Function

We present two methods for simulation from the BMW distribution in (3). The first uses the inversion method. Let U be a uniform variate on the unit interval $[0, 1]$. Setting

$$I_{1-\exp\{-\alpha X^{\gamma}\exp(\lambda X)\}}(a, b) = U$$

and solving, we see that BMW variates X can be obtained as roots of the equation

$$\log X + \lambda X + \log\alpha - \log\left[-\log\left\{1 - I_U^{-1}(a, b)\right\}\right] = 0,$$

where $I_u^{-1}(a,b)$ denotes the inverse of the incomplete beta function ratio.

Our second method for simulation from the BMW distribution is based on the rejection method. Take h to be the pdf of a gamma random variable with shape parameter γ and scale parameter λ. Define a constant M by

$$M = \frac{\alpha\Gamma(\gamma)}{\lambda^\gamma}\exp\left(M^*\right),$$

where

$$M^* = \sup_{x>0}\left\{\log(\gamma+\lambda x) + 2\lambda x - \alpha x^\gamma \exp(\lambda x)\right\}.$$

Then, the following scheme holds for simulating BMW variates:

1. simulate $X = x$ from the pdf h;
2. simulate $Y = VMg(x)$, where V is a beta random variable with shape parameters a and b;
3. accept $X = x$ as a BMW variate if $Y < f(x)$. If $Y \geq f(x)$ return to step 2.

Note that routines are widely available for simulating from the gamma distribution.

We now give an expansion for the quantile function $q = F^{-1}(p)$. First, we have $p = F(q) = I_s(a,b)$, where $s = G_{\alpha,\gamma,\lambda}(q) = 1 - \exp\{-\alpha q^\gamma \exp(\lambda q)\}$. From the $W(\cdot)$ function defined in Appendix A, we can express q in terms of s

$$q = \frac{\gamma}{\lambda} W\left(\frac{\lambda[-\alpha^{-1}\log(1+s)]^{1/\gamma}}{\gamma}\right). \tag{12}$$

The $W(\cdot)$ function can be calculated easily using Mathematica, for example,

$$W(z) = z - z^2 + \frac{3z^3}{2} - \frac{8z^4}{3} + \frac{125z^5}{24} - \frac{54z^6}{5} + \frac{16807z^7}{720} - \frac{16384z^8}{315} + \frac{531441z^9}{4480} - \frac{156250z^{10}}{567} + O\left(z^{11}\right).$$

Further, it is possible to obtain s as function of p from some expansions for the inverse of the beta incomplete function $s = I_p^{-1}(a,b)$. One of them can be found in Wolfram website[1] as

$$\begin{aligned} s = I_p^{-1}(a,b) &= w + \frac{b-1}{a+1}w^2 + \frac{(b-1)(a^2+3ba-a+5b-4)}{2(a+1)^2(a+2)}w^3 \\ &+ \frac{(b-1)[a^4+(6b-1)a^3+(b+2)(8b-5)a^2+(33b^2-30b+4)a+b(31b-47)+18]}{3(a+1)^3(a+2)(a+3)}w^4 + O(p^{5/a}), \end{aligned}$$

where $w = [apB(a,b)]^{1/a}$ for $a > 0$. Inserting the last expansion in equation (12), q is expressed in terms of p.

4 Moments

Let $X \sim \text{BMW}(a,b,c,\alpha,\gamma,\lambda)$. Combining (6) and (8), the kth moment of X follows as

$$E(X^k) = \sum_{j=0}^{\infty} w_j I_1(j,k), \tag{13}$$

[1] http://functions.wolfram.com/06.23.06.0004.01

where

$$
\begin{aligned}
I_1(j,k) &= \alpha(b+j)\int_0^\infty \left(\sum_{m=1}^\infty a_m x^{m/\gamma}\right)^k \exp\{-\alpha(b+j)x\}\,dx \\
&= \alpha(b+j)\sum_{m_1=1}^\infty \cdots \sum_{m_k=1}^\infty a_{m_1}\cdots a_{m_k} \\
&\qquad \times \int_0^\infty x^{(m_1+\cdots+m_k)/\gamma} \exp\{-\alpha(b+j)x\}\,dx,
\end{aligned}
$$

and thus

$$
\begin{aligned}
I_1(j,k) &= \sum_{m_1=1}^\infty \cdots \sum_{m_k=1}^\infty a_{m_1}\cdots a_{m_k} \\
&\qquad \times \{\alpha(b+j)\}^{-(m_1+\cdots+m_k)/\gamma}\,\Gamma\left(\frac{m_1+\cdots+m_k}{\gamma}+1\right), \qquad (14)
\end{aligned}
$$

where w_j and a_j are defined by (7) and (9), respectively.

A much simpler representation for the kth moment, using (10) and equation (11) in Corless *et al.* (1996), can be obtained as

$$
\begin{aligned}
I_1(j,k) &= \alpha(b+j)\int_0^\infty \left(\gamma W\left(\frac{x^{1/\gamma}}{\gamma}\right)\right)^k \exp\{-\alpha(b+j)x\}\,dx \\
&= \alpha(b+j)k(-\gamma)^k \sum_{n=1}^\infty \frac{(-1)^n n^{n-k}(n-1)\cdots(n-k+1)}{n!\gamma^n}\int_0^\infty x^{n/\gamma}\exp\{-\alpha(b+j)x\}\,dx \\
&= k(-\gamma)^k \sum_{n=1}^\infty \frac{(-1)^n n^{n-k}(n-1)\cdots(n-k+1)}{n!\gamma^n\{\alpha(b+j)\}^{n/\gamma}}\Gamma\left(\frac{n}{\gamma}+1\right). \qquad (15)
\end{aligned}
$$

Equation (15) gives a representation for $E(X^k)$ involving only a doubly infinite series.

For lifetime models, it is also of interest to know what $E(X^k \mid X > x)$ is. Using (6) and (8), one can show that

$$
E\left(X^k \mid X > x\right) = \frac{1}{I_{\exp\{-\alpha x^\gamma \exp(\lambda x)\}}(b,a)}\sum_{j=0}^\infty w_j I_2(j,k),
$$

where

$$
\begin{aligned}
I_2(j,k) &= \sum_{m_1=1}^\infty \cdots \sum_{m_k=1}^\infty a_{m_1}\cdots a_{m_k} \\
&\qquad \times \{\alpha(b+j)\}^{-(m_1+\cdots+m_k)/\gamma}\,\Gamma\left(\frac{m_1+\cdots+m_k}{\gamma}+1, \alpha(b+j)x\right),
\end{aligned}
$$

where w_j and a_j are defined by (7) and (9), respectively.

A much simpler representation for $E(X^k \mid X > x)$, using (10) and equation (11) in Corless *et al.* (1996), can be obtained as

$$
I_2(j,k) = k(-\gamma)^k\lambda^{-k}\sum_{n=1}^\infty \frac{(-\lambda)^n n^{n-k}(n-1)\cdots(n-k+1)}{n!\gamma^n\{\alpha(b+j)\}^{n/\gamma}}\Gamma\left(\frac{n}{\gamma}+1, \alpha(b+j)x\right). \qquad (16)
$$

Again equation (16) gives a representation for $E(X^k \mid X > x)$ involving only a doubly infinite series. The mean residual lifetime function is $E(X \mid X > x) - x$.

[Figures 3 and 4 about here.]

The skewness and kurtosis measures can now be calculated from the ordinary moments using well-known relationships. Plots of the skewness and kurtosis for some choices of the parameter b as function of the parameter a, and for some choices of the parameter a as function of the parameter b, for $\alpha = 0.7$, $\gamma = 0.8$ and $\lambda = 0.2$, are shown in Figures 3 and 4, respectively. These figures immediately reveal that the skewness and kurtosis curves, respectively, as function of a and b first decrease and then increase, whereas as functions of b and a they always decrease, in all cases the other parameter being fixed.

5 Moment Generating Function and Characteristic Function

Let $X \sim \mathrm{BMW}(a,b,c,\alpha,\gamma,\lambda)$. The moment generating function of X, $M(t) = E[\exp(tX)]$, and the characteristic function, $\phi(t) = E[\exp(\mathrm{i}tX)]$, where $\mathrm{i} = \sqrt{-1}$, are expressed as

$$M(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k) \ \text{ and } \ \phi(t) = \sum_{k=0}^{\infty} \frac{(\mathrm{i}t)^k}{k!} E(X^k),$$

where $E(X^k)$ is given by (13). We now give another representation for $M(t)$ which can be expressed from equation (6) as an infinite weighted sum

$$M(t) = \sum_{j=0}^{\infty} w_j M_j(t), \tag{17}$$

where $M_j(t)$ is the mgf of the $\mathrm{MW}(\alpha(b+j),\gamma,\lambda)$ distribution and w_j is defined by (7). By combining (10) and equation (16) in Corless *et al.* (1996), a simple representation for $M_j(t)$ can be written as

$$\begin{aligned} M_j(t) &= \alpha(b+j) \int_0^{\infty} \exp\left\{ \frac{t\gamma}{\lambda} W\left(\frac{\lambda x^{1/\gamma}}{\gamma} \right) - \alpha(b+j)x \right\} dx \\ &= -\alpha(b+j)t\gamma \sum_{n=0}^{\infty} \frac{(-1)^n (n\lambda - t\gamma)^{n-1}}{n!\gamma^n} \int_0^{\infty} x^{n/\gamma} \exp\{-\alpha(b+j)x\}\, dx, \end{aligned}$$

and then

$$M_j(t) = -t\gamma \sum_{n=0}^{\infty} \frac{(-1)^n (n\lambda - t\gamma)^{n-1}}{n!\gamma^n \{\alpha(b+j)\}^{n/\gamma}} \Gamma\left(\frac{n}{\gamma} + 1 \right).$$

The corresponding chf is

$$\phi(t) = \sum_{j=0}^{\infty} w_j \phi_j(t), \tag{18}$$

where

$$\phi_j(t) = -\mathrm{i}t\gamma \sum_{n=0}^{\infty} \frac{(-1)^n (n\lambda - \mathrm{i}t\gamma)^{n-1}}{n!\gamma^n \{\alpha(b+j)\}^{n/\gamma}} \Gamma\left(\frac{n}{\gamma} + 1 \right).$$

Equations (17) and (18) are representations for $M(t)$ and $\phi(t)$, respectively, involving only doubly infinite series.

6 Mean Deviations

Let $X \sim \text{BMW}(a, b, c, \alpha, \gamma, \lambda)$. The amount of scatter in X is evidently measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median – defined by

$$\delta_1(X) = \int_0^\infty |x - \mu| \, f(x) dx \text{ and } \delta_2(X) = \int_0^\infty |x - M| \, f(x) dx,$$

respectively, where $\mu = E(X)$ and $M = Median(X)$ denotes the median. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the relationships

$$\begin{aligned}\delta_1(X) &= \int_0^\mu (\mu - x) f(x) dx + \int_\mu^\infty (x - \mu) f(x) dx \\ &= 2\mu F(\mu) - 2\mu + 2 \int_\mu^\infty x f(x) dx,\end{aligned}$$

and

$$\begin{aligned}\delta_2(X) &= \int_0^M (M - x) f(x) dx + \int_M^\infty (x - M) f(x) dx \\ &= 2 \int_M^\infty x f(x) dx - \mu.\end{aligned}$$

Using (6) and (8), one can show that

$$\int_\mu^\infty x f(x) dx = \sum_{j=0}^\infty w_j I_3(j) \text{ and } \int_M^\infty x f(x) dx = \sum_{j=0}^\infty w_j I_4(j).$$

Here,

$$I_3(j) = \sum_{m=1}^\infty a_m \left\{\alpha(b + j)\right\}^{-m/\gamma} \Gamma\left(\frac{m}{\gamma} + 1, \alpha(b + j)\mu\right),$$

and

$$I_4(j) = \sum_{m=1}^\infty a_m \left\{\alpha(b + j)\right\}^{-m/\gamma} \Gamma\left(\frac{m}{\gamma} + 1, \alpha(b + j)M\right),$$

where w_j and a_j are defined by (7) and (9), respectively. So, it follows that

$$\delta_1(X) = 2\mu F(\mu) - 2\mu + 2\sum_{j=0}^\infty w_j I_3(j),$$

and

$$\delta_2(X) = 2\sum_{j=0}^\infty w_j I_4(j) - \mu.$$

7 Bonferroni and Lorenz Curves

Bonferroni and Lorenz curves have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. For $X \sim \text{BMW}(a, b, c, \alpha, \gamma, \lambda)$, they are defined by

$$B(p) = \frac{1}{p\mu} \int_0^q x f(x) dx \text{ and } L(p) = \frac{1}{\mu} \int_0^q x f(x) dx, \tag{19}$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$ is calculated by (12). Using (6) and (8), we can show that

$$\int_0^q x f(x) dx = \sum_{j=0}^{\infty} w_j I_5(j),$$

where

$$I_5(j) = \sum_{m=1}^{\infty} a_m \left\{ \alpha(b+j) \right\}^{-m/\gamma} \gamma \left(\frac{m}{\gamma} + 1, \alpha(b+j) q \right),$$

and the constants w_j and a_j are defined by equations (7) and (9), respectively. So, we can reduce the curves in (19) to

$$B(p) = \frac{1}{p\mu} \sum_{j=0}^{\infty} w_j I_5(j) \text{ and } L(p) = \frac{1}{\mu} \sum_{j=0}^{\infty} w_j I_5(j),$$

respectively.

8 Extreme Values

If $\overline{X} = (X_1 + \cdots + X_n)/n$ denotes the sample mean then by the usual central limit theorem $\sqrt{n}(\overline{X} - E(X))/\sqrt{Var(X)}$ approaches the standard normal distribution as $n \to \infty$ under suitable conditions. Sometimes one would be interested in the asymptotics of the extreme values $M_n = \max(X_1, \ldots, X_n)$ and $m_n = \min(X_1, \ldots, X_n)$.

Let $g(t) = t^{-\gamma} \exp(-\lambda t)/(\lambda \alpha b)$, a strictly positive function. Take the cdf and the pdf as specified by (3) and (4), respectively. It can be seen that

$$\begin{aligned}
\frac{1 - F(t + x g(t))}{1 - F(t)} &= \exp\left[\alpha b t^{\gamma} \exp(\lambda t) - \alpha b \left\{ t + x g(t) \right\} \exp\left[\lambda \left\{ t + x g(t) \right\}\right]\right] \\
&= \exp\left[\alpha b t^{\gamma} \exp(\lambda t) \left\{ 1 - \left(1 + \frac{x g(t)}{t} \right)^{\gamma} \exp\left(\lambda x g(t)\right) \right\}\right] \\
&= \exp\left[\alpha b t^{\gamma} \exp(\lambda t) \left\{ 1 - \left(1 + \frac{\gamma x g(t)}{t} + \cdots \right) \left(1 + \lambda x g(t) + \cdots \right) \right\}\right] \\
&= \exp\left[-\lambda \alpha b x t^{\gamma} \exp(\lambda t) g(t) + o(1)\right] \\
&= \exp\left\{-x + o(1)\right\},
\end{aligned}$$

as $t \to \infty$. It can also be seen using L'Hospital's rule that

$$\lim_{t \to 0} \frac{F(tx)}{F(t)} = \lim_{t \to 0} \frac{x f(tx)}{f(t)} = x^{\gamma a}.$$

Hence, it follows from Theorem 1.6.2 in Leadbetter *et al.* (1987) that there must be norming constants $a_n, b_n, c_n > 0$ and d_n such that

$$\Pr\left\{a_n \left(M_n - b_n\right) \le x\right\} \to \exp\left\{-\exp(-x)\right\},$$

and

$$\Pr\left\{c_n \left(m_n - d_n\right) \le x\right\} \to 1 - \exp\left(-x^{\gamma a}\right),$$

as $n \to \infty$. The form of the norming constants can also be determined. For instance, using Corollary 1.6.3 in Leadbetter *et al.* (1987), one can see that $b_n = F^{-1}(1 - 1/n)$ and $a_n = 1/g(b_n)$, where $F^{-1}(\cdot)$ denotes the inverse function of $F(\cdot)$.

9 Estimation

Here, we consider estimation by the methods of moments and maximum likelihood and provide expressions for the associated Fisher information matrix. We also consider estimation issues for censored data.

Suppose $x_1, \ldots, x_n$ is a random sample from the BMW distribution (4). For the moment estimation, let $m_k = (1/n) \sum_{j=1}^{n} x_j^k$ for $k = 1, \cdots, 5$. By equating the theoretical moments of (4) with the sample moments, one obtains the equations:

$$\sum_{j=0}^{\infty} w_j I_1(j,k) = m_k, \tag{20}$$

for $k = 1, \cdots, 5$, where w_j and $I_1(j,k)$ are given by (7) and (14), respectively. The method of moment estimators (MMEs) are the simultaneous solutions of the equations: (20) for $k = 1, \cdots, 5$.

Now consider estimation by the method of maximum likelihood. The log likelihood (LL) function $\log L = \log L(a, b, \alpha, \lambda, \gamma)$ of the five parameters is:

$$\begin{aligned} \log L &= n \log \alpha - n \log B(a,b) + (\gamma - 1) \sum_{j=1}^{n} \log x_j + \sum_{j=1}^{n} \log(\gamma + \lambda x_j) + \lambda \sum_{j=1}^{n} x_j \\ &\quad + (a-1) \sum_{j=1}^{n} \log \left[1 - \exp \left\{ -\alpha x_j^\gamma \exp(\lambda x_j) \right\} \right] - b\alpha \sum_{j=1}^{n} x_j^\gamma \exp(\lambda x_j) . \end{aligned} \tag{21}$$

It follows that the maximum likelihood estimators (MLEs) are the simultaneous solutions of the equations:

$$\sum_{j=1}^{n} \log \left[1 - \exp \left\{ -\alpha x_j^\gamma \exp(\lambda x_j) \right\} \right] = n\psi(a) - n\psi(a+b),$$

$$\alpha \sum_{j=1}^{n} x_j^\gamma \exp(\lambda x_j) = n\psi(a+b) - n\psi(b),$$

$$\frac{n}{\alpha} + (a-1) \sum_{j=1}^{n} \frac{x_j^\gamma \exp(\lambda x_j)}{\exp\left\{ \alpha x_j^\gamma \exp(\lambda x_j) \right\} - 1} = b \sum_{j=1}^{n} x_j^\gamma \exp(\lambda x_j),$$

$$\sum_{j=1}^{n} \frac{x_k}{\gamma + \lambda x_k} + \alpha(a-1) \sum_{j=1}^{n} \frac{x_j^{\gamma+1} \exp(\lambda x_j)}{\exp\left\{ \alpha x_j^\gamma \exp(\lambda x_j) \right\} - 1} = b\alpha \sum_{j=1}^{n} x_j^{\gamma+1} \exp(\lambda x_j)$$

and

$$\sum_{j=1}^{n} \log x_j + \sum_{j=1}^{n} \frac{1}{\gamma + \lambda x_k} + \alpha(a-1) \sum_{j=1}^{n} \frac{\log x_j x_j^\gamma \exp(\lambda x_j)}{\exp\left\{ \alpha x_j^\gamma \exp(\lambda x_j) \right\} - 1} = b\alpha \sum_{j=1}^{n} \log x_j x_j^\gamma \exp(\lambda x_j) .$$

For interval estimation of $(a, b, \alpha, \lambda, \gamma)$ and tests of hypotheses, one requires the Fisher information matrix. The elements of this matrix for (21) are given in Appendix B.

Often with lifetime data, one encounters censored data. There are different forms of censoring: type I censoring, type II censoring, etc. Here, we consider the general case of multicensored data: there are n subjects of which

- n_0 are known to have failed at the times $x_1, \ldots, x_{n_0}$.
- n_1 are known to have failed in the interval $[s_{j-1}, s_j]$, $j = 1, \ldots, n_1$.
- n_2 survived to a time r_j, $j = 1, \ldots, n_2$ but not observed any longer.

Here, $n = n_0 + n_1 + n_2$. Note too that type I censoring and type II censoring are contained as particular cases of multi-censoring. The LL function $\log L = \log L(a, b, \alpha, \lambda, \gamma)$ of the five parameters for this multi-censoring data is:

$$
\begin{aligned}
\log L \;=\;& n_0 \log \alpha - n_0 \log B(a,b) + (\gamma - 1)\sum_{j=1}^{n_0} \log x_j + \sum_{j=1}^{n_0} \log(\gamma + \lambda x_j) + \lambda \sum_{j=1}^{n_0} x_j \\
&+ (a-1)\sum_{j=1}^{n_0} \log\left[1 - \exp\left\{-\alpha x_j^{\gamma} \exp(\lambda x_j)\right\}\right] - b\alpha \sum_{j=1}^{n_0} x_j^{\gamma} \exp(\lambda x_j) \\
&+ \sum_{j=1}^{n_1} \log\left[I_{1-\exp\left\{-\alpha s_j^{\gamma} \exp(\lambda s_j)\right\}}(a,b) - I_{1-\exp\left\{-\alpha s_{j-1}^{\gamma} \exp(\lambda s_{j-1})\right\}}(a,b)\right] \\
&+ \sum_{j=1}^{n_2} \log I_{\exp\left\{-\alpha r_j^{\gamma} \exp(\lambda r_j)\right\}}(b,a). \qquad (22)
\end{aligned}
$$

It follows that the MLEs are the simultaneous solutions of the five equations given in Appendix C. The Fisher information matrix corresponding to (22) is too complicated to be presented here. [Table 1 about here.]

We now compare the performances of the two estimation methods. For this purpose, we generated samples of size $n = 20$ from (4) for $a, b = 1, 2, \ldots, 6$ and α, γ, λ fixed as $\alpha = \lambda = 1$ and $\gamma = 1$. For each sample, we computed the MLEs and the MMEs, following the procedures described before. We repeated this process 100 times and computed the average of the estimates (AE) and the mean squared error (MSE). The results are reported in Table 1. It is clear that the MLE performs consistently better than the MME for all values of a, b and with respect to the AE and MSE. This is expected of course.

10 Applications

10.1 Voltage Data

Here, we compare the results of the fits of the BMW, BW, GMW, MW and EW distributions to the data set studied by Meeker and Escobar (1998, p. 383), which gives the times of failure and running times for a sample of devices from a field-tracking study of a larger system. At a certain point in time, 30 units were installed in normal service conditions. Two causes of failure were observed for each unit that failed: the failure caused by an accumulation of randomly occurring damage from power-line voltage spikes during electric storms and failure caused by normal product wear.
[Table 2 about here.]

In many applications, there is a qualitative information about the failure rate function shape, which can help in selecting a particular model. In this context, a device called the total time on test (TTT) plot (Aarset, 1987) is useful. The TTT plot is obtained by plotting $\mathsf{G}(r/n) = [(\sum_{i=1}^{r} T_{i:n}) + (n - r)T_{r:n}]/(\sum_{i=1}^{n} T_{i:n})$, where $r = 1, \ldots, n$ and $T_{i:n}$, $i = 1, \ldots, n$, are the order statistics of the sample, against r/n (Mudholkar *et al.*, 1996). Figure 5a shows that the TTT-plot for the data set has first a convex shape and then a concave shape. It indicates a bathtub-shaped hazard rate function. Hence, the BMW distribution could be an appropriate model for the fitting of these data. Table 2 gives the MLEs (and the corresponding standard errors in parentheses) of the parameters and the values of the following statistics for some models: AIC (Akaike Information Criterion) due to Akaike (1974), BIC (Bayesian Information Criterion) due to Schwarz (1978), and CAIC (Consistent Akaike Information Criterion) due to Bozdogan (1987). The computations were done using the NLMixed procedure in SAS. These results indicate that the BMW model has the lowest AIC, BIC and CAIC values among all fitted models, and hence it could be chosen as the best model.

In order to assess if the model is appropriate, Figure 5b gives the empirical and estimated survival functions of the BMW, BW, GMW, MW and EW distributions. Plots the histogram of the data and the fitted BMW, BW, GMW, MW and EW distributions are given in Figure 5c. We conclude that the BMW distribution provides a good fit for these data. In addition, the estimated hazard rate function in Figure 5d is a bathtub-shaped curve.
[Figures 5 and 6 about here.]

The conclusion based on the fitted pdfs, the histogram of the data and survival functions can also be verified by means of the probability plots given in Figures 6a-e. A probability plot (as recommended by Chambers *et al.* (1983)), consists of plots of the observed probabilities against the probabilities predicted by the fitted model. For example, for the BMW model,

$$F(x_{(j)}) = \frac{1}{B(\widehat{a},\widehat{b})}\int_0^{1-\exp\{-\widehat{\alpha}x_{(j)}^{\widehat{\gamma}}\exp(\widehat{\lambda}x_{(j)})\}} w^{\widehat{a}-1}(1-w)^{\widehat{b}-1}dw$$

was plotted versus $(j-0.375)/(n+0.25)$, $j=1,\ldots,n$, where $x_{(j)}$ are the sorted values of the observed fracture toughness. For each plot, we calculate the sum of squares

$$SS = \sum_{j=1}^{n}\left\{F(x_{(j)}) - \frac{(j-0.375)}{(n+0.25)}\right\}^2,$$

which is a measure of the closeness of the plot to the diagonal line. It is clear that the BMW model has the points closer to the diagonal line corresponding to the smallest SS.

10.2 Serum Reversal Data

The data set refers to the serum-reversal time (days) of 148 children contaminated with HIV from vertical transmission at the university hospital of the Ribeirão Preto School of Medicine (Hospital das Clínicas da Faculdade de Medicina de Ribeirão Preto) from 1986 to 2001 (Silva, 2004). More details, see, for example, in Carrasco *et al.* (2008). We assume that the lifetime are independently distributed, and also independent from the censoring mechanism. Considering right-censored lifetime data (censoring random). Figure 7a shows that the TTT-plot for the data set has first a convex shape and then a concave shape. It indicates a bathtub-shaped hazard rate function. Hence, the BMW distribution could be an appropriate model for the fitting of such data. Table 3 gives the MLEs (and the corresponding standard errors in parentheses) of the parameters and the values of the AIC, BIC and CAIC statistics. These results indicate that the BMW model has the lowest AIC, BIC and CAIC values among all fitted models, and hence it could be chosen as the best model.

[Table 3 and Figure 7 about here.]

In order to assess if the model is appropriate, plots of the empirical and estimated survival functions of the BMW, BW, GMW, MW and EW distributions are given in Figure 7b. We conclude that the BMW distribution provides a good fit for these data. Additionally, the estimated hazard rate function in Figure 7c is a bathtub-shaped curve.

11 Conclusions

In this paper, we study some mathematical properties of the beta modified Weibull (BMW) distribution which is quite flexible in analyzing positive data. It is an important alternative model to several models discussed in the literature since it contains the Weibull, exponentiated exponential, exponentiated Weibull, beta exponential, modified Weibull (MW), generalized modified Weibull and beta Weibull distributions, among others, as special sub-models. We demonstrate that the pdf of the BMW distribution can be expressed as a mixture of MW pdfs. We provide their moments and two closed form expressions for its moment generating function. We examine the asymptotic distributions of the extreme values. Explicit expressions are also derived for the characteristic function, mean deviations and Bonferroni and Lorenz curves. The pdf of the order statistics can also be expressed in terms of an infinite mixture of MW pdfs. We obtain a closed form expression for their moments and for the L moments. The estimation of parameters is approached by two methods: moments and maximum likelihood. We compare by simulation the performances of the estimates from these methods. The expected information matrix is derived. The usefulness of the BMW distribution is illustrated in two analyses of real data.

Appendix A

The calculations in this paper involve the following special functions: the gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty w^{\alpha-1} \exp(-w) dw,$$

the digamma function defined by

$$\psi(\alpha) = \frac{d \log \Gamma(\alpha)}{d\alpha},$$

the incomplete gamma function defined by

$$\gamma(\alpha, x) = \int_0^x w^{\alpha-1} \exp(-w) dw,$$

the complementary incomplete gamma function defined by

$$\Gamma(\alpha, x) = \int_x^\infty w^{\alpha-1} \exp(-w) dw,$$

the beta function defined by

$$B(a, b) = \int_0^1 w^{a-1}(1 - w)^{b-1} dw,$$

the incomplete beta function ratio defined by

$$I_y(a,b) = \frac{1}{B(a,b)} \int_0^y w^{a-1}(1-w)^{b-1} dw,$$

and the ${}_3F_2$ hypergeometric function defined by

$${}_3F_2(a,b,c;d,e;x) = \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(c)} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(b+j)\Gamma(c+j)}{\Gamma(d+j)\Gamma(e+j)} \frac{x^j}{j!}.$$

We shall also need the Lambert $W(z)$ function defined as the inverse of $z = x\exp(x)$, say $x = W(z)$. The properties of these special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000). The Lambert $W(z)$ function is the series expansion $F(z) = ProductLog[z]$ provided by the software Mathematica.

Appendix B

The elements of the Fisher information matrix corresponding to the log likelihood function in (21) are:

$$E\left(-\frac{\partial^2 \log L}{\partial a^2}\right) = n\psi^{'}(a) - n\psi^{'}(a+b), \quad E\left(-\frac{\partial^2 \log L}{\partial a \partial b}\right) = -n\psi^{'}(a+b),$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial \alpha}\right) = -nT(\gamma,0,1,0,1), \quad E\left(-\frac{\partial^2 \log L}{\partial a \partial \lambda}\right) = -n\alpha T(\gamma+1,0,1,0,1),$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial \gamma}\right) = -n\alpha T(\gamma,1,1,0,1), \quad E\left(-\frac{\partial^2 \log L}{\partial b^2}\right) = n\psi^{'}(b) - n\psi^{'}(a+b),$$

$$E\left(-\frac{\partial^2 \log L}{\partial b \partial \alpha}\right) = nT(\gamma,0,1,0,0), \quad E\left(-\frac{\partial^2 \log L}{\partial b \partial \lambda}\right) = n\alpha T(\gamma+1,0,1,0,0),$$

$$E\left(-\frac{\partial^2 \log L}{\partial b \partial \gamma}\right) = n\alpha T(\gamma,1,1,0,0), \quad E\left(-\frac{\partial^2 \log L}{\partial \alpha^2}\right) = \frac{n}{\alpha^2} + n(a-1)T(2\gamma,0,2,1,2),$$

$$E\left(-\frac{\partial^2 \log L}{\partial \alpha \partial \lambda}\right) = nbT(\gamma+1,0,1,0,0) - n(a-1)\left\{T(\gamma+1,0,1,0,1) - \alpha T(2\gamma+1,0,2,1,2)\right\},$$

$$E\left(-\frac{\partial^2 \log L}{\partial \alpha \partial \gamma}\right) = nbT(\gamma,1,1,0,0) - n(a-1)\left\{T(\gamma,1,1,0,1) - \alpha T(2\gamma,1,2,1,2)\right\},$$

$$E\left(-\frac{\partial^2 \log L}{\partial \lambda^2}\right) = n\alpha bT(\gamma+2,0,1,0,0) - n\alpha(a-1)\left\{T(\gamma+2,0,1,0,1) - \alpha T(2\gamma+2,0,2,1,2)\right\},$$

$$\begin{aligned} E\left(-\frac{\partial^2 \log L}{\partial \lambda \partial \gamma}\right) &= nS(1,2) + n\alpha bT(\gamma+1,1,1,0,0) \\ &\quad - n\alpha(a-1)\left\{T(\gamma+1,1,1,0,1) - \alpha T(2\gamma+1,1,2,1,2)\right\} \end{aligned}$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial \gamma^2}\right) = nS(0,2) + n\alpha b T(\gamma,2,1,0,0) - n\alpha(a-1)\left\{T(\gamma,1,1,0,1) - \alpha T(2\gamma,1,2,1,2)\right\},$$

where

$$T(i,j,k,l,m) = E\left[\frac{X^i (\log X)^j \exp(k\lambda X)\exp\left\{l\alpha X^\gamma \exp\left(\lambda X\right)\right\}}{\left[\exp\left\{\alpha X^\gamma \exp\left(\lambda X\right)\right\} - 1\right]^m}\right]$$

and

$$S(i,j) = E\left[\frac{X^i}{(\gamma + \lambda X)^j}\right].$$

The expectations in $T(i,j,k,l,m)$ and $S(i,j)$ can be computed numerically.

Appendix C

The following five equations can be solved simultaneously to obtain the MLEs of the parameters of the log likelihood function given by (22):

$$\begin{aligned}
&\sum_{j=1}^{n_0} \log\left[1 - \exp\left\{-\alpha x_j^\gamma \exp\left(\lambda x_j\right)\right\}\right] \\
&+\sum_{j=1}^{n_1} \frac{\partial I_{1-\exp\left\{-\alpha s_j^\gamma \exp(\lambda s_j)\right\}}(a,b)/\partial a - \partial I_{1-\exp\left\{-\alpha s_{j-1}^\gamma \exp(\lambda s_{j-1})\right\}}(a,b)/\partial a}{I_{1-\exp\left\{-\alpha s_j^\gamma \exp(\lambda s_j)\right\}}(a,b) - I_{1-\exp\left\{-\alpha s_{j-1}^\gamma \exp(\lambda s_{j-1})\right\}}(a,b)} \\
&+\sum_{j=1}^{n_2} \frac{\partial I_{\exp\left\{-\alpha r_j^\gamma \exp(\lambda r_j)\right\}}(b,a)/\partial a}{I_{\exp\left\{-\alpha r_j^\gamma \exp(\lambda r_j)\right\}}(b,a)} = n_0\psi(a) - n_0\psi(a+b),
\end{aligned}$$

$$\begin{aligned}
&\alpha\sum_{j=1}^{n_0} x_j^\gamma \exp\left(\lambda x_j\right) - \sum_{j=1}^{n_1} \frac{\partial I_{1-\exp\left\{-\alpha s_j^\gamma \exp(\lambda s_j)\right\}}(a,b)/\partial b - \partial I_{1-\exp\left\{-\alpha s_{j-1}^\gamma \exp(\lambda s_{j-1})\right\}}(a,b)/\partial b}{I_{1-\exp\left\{-\alpha s_j^\gamma \exp(\lambda s_j)\right\}}(a,b) - I_{1-\exp\left\{-\alpha s_{j-1}^\gamma \exp(\lambda s_{j-1})\right\}}(a,b)} \\
&-\sum_{j=1}^{n_2} \frac{\partial I_{\exp\left\{-\alpha r_j^\gamma \exp(\lambda r_j)\right\}}(b,a)/\partial b}{I_{\exp\left\{-\alpha r_j^\gamma \exp(\lambda r_j)\right\}}(b,a)} = n_0\psi(a+b) - n_0\psi(b),
\end{aligned}$$

$$\begin{aligned}
&\frac{n_0}{\alpha} + (a-1)\sum_{j=1}^{n_0} \frac{x_j^\gamma \exp\left(\lambda x_j\right)}{\exp\left\{\alpha x_j^\gamma \exp\left(\lambda x_j\right)\right\} - 1} - b\sum_{j=1}^{n_0} x_j^\gamma \exp\left(\lambda x_j\right) \\
&+\frac{\alpha}{B(a,b)}\sum_{j=1}^{n_1} \frac{U\left(s_j\right) - U\left(s_{j-1}\right)}{I_{1-\exp\left\{-\alpha s_j^\gamma \exp(\lambda s_j)\right\}}(a,b) - I_{1-\exp\left\{-\alpha s_{j-1}^\gamma \exp(\lambda s_{j-1})\right\}}(a,b)} \\
= \quad &\frac{\alpha}{B(a,b)}\sum_{j=1}^{n_2} \frac{U\left(r_j\right)}{I_{\exp\left\{-\alpha r_j^\gamma \exp(\lambda r_j)\right\}}(b,a)},
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=1}^{n_0} \frac{x_k}{\gamma + \lambda x_k} + \alpha(a-1)\sum_{j=1}^{n_0} \frac{x_j^{\gamma+1} \exp\left(\lambda x_j\right)}{\exp\left\{\alpha x_j^\gamma \exp\left(\lambda x_j\right)\right\} - 1} - b\alpha\sum_{j=1}^{n_0} x_j^{\gamma+1} \exp\left(\lambda x_j\right) \\
&+\frac{1}{B(a,b)}\sum_{j=1}^{n_1} \frac{V\left(s_j\right) - V\left(s_{j-1}\right)}{I_{1-\exp\left\{-\alpha s_j^\gamma \exp(\lambda s_j)\right\}}(a,b) - I_{1-\exp\left\{-\alpha s_{j-1}^\gamma \exp(\lambda s_{j-1})\right\}}(a,b)} \\
= \quad &\frac{1}{B(a,b)}\sum_{j=1}^{n_2} \frac{V\left(r_j\right)}{I_{\exp\left\{-\alpha r_j^\gamma \exp(\lambda r_j)\right\}}(b,a)},
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{j=1}^{n_0} \log x_j + \sum_{j=1}^{n_0} \frac{1}{\gamma + \lambda x_k} + \alpha(a-1) \sum_{j=1}^{n_0} \frac{\log x_j x_j^\gamma \exp\left(\lambda x_j\right)}{\exp\left\{\alpha x_j^\gamma \exp\left(\lambda x_j\right)\right\} - 1} \\
& + \frac{\alpha}{B(a,b)} \sum_{j=1}^{n_1} \frac{Z\left(s_j\right) - Z\left(s_{j-1}\right)}{I_{1-\exp\left\{-\alpha s_j^\gamma \exp(\lambda s_j)\right\}}(a,b) - I_{1-\exp\left\{-\alpha s_{j-1}^\gamma \exp(\lambda s_{j-1})\right\}}(a,b)} \\
= \; & \frac{\alpha}{B(a,b)} \sum_{j=1}^{n_2} \frac{Z\left(r_j\right)}{I_{\exp\left\{-\alpha r_j^\gamma \exp(\lambda r_j)\right\}}(b,a)} + b\alpha \sum_{j=1}^{n_0} \log x_j x_j^\gamma \exp\left(\lambda x_j\right),
\end{aligned}$$

where $U(s) = s^{\gamma+1} \exp(\lambda s) \exp\{-\alpha b s^\gamma \exp(\lambda s)\}[1 - \exp\{-\alpha s^\gamma \exp(\lambda s)\}]^{a-1}$, $V(s) = s^\gamma \exp(\lambda s) \exp\{-\alpha b s^\gamma \exp(\lambda s)\}[1 - \exp\{-\alpha s^\gamma \exp(\lambda s)\}]^{a-1}$ and $Z(s) = s^\gamma \log s \exp(\lambda s) \exp\{-\alpha b s^\gamma \exp(\lambda s)\} [1 - \exp\{-\alpha s^\gamma \exp(\lambda s)\}]^{a-1}$. The partial derivatives of the incomplete beta function ratio can be computed using the facts

$$\frac{I_x(a,b)}{\partial a} = \{\log x - \psi(a) + \psi(a+b)\} I_x(a,b) - \frac{\Gamma(a)\Gamma(a+b)}{\Gamma(b)} x^a {}_3F_2\left(a, a, 1-b; a+1, a+1; x\right)$$

and

$$\begin{aligned}
\frac{I_x(a,b)}{\partial b} = \; & \frac{\Gamma(b)\Gamma(a+b)}{\Gamma(a)} (1-x)^b {}_3F_2\left(b, b, 1-a; b+1, b+1; 1-x\right) \\
& + \{\psi(b) - \psi(a+b) - \log(1-x)\} I_{1-x}(b,a),
\end{aligned}$$

see, for example, http://functions.wolfram.com/GammaBetaErf/BetaRegularized/20/01/02/0001/ and http://functions.wolfram.com/GammaBetaErf/BetaRegularized/20/01/03/0001/.

Acknowledgments

The authors would like to thank the Editor, the Associate Editor and the referee for carefully reading the paper and for their comments which greatly improved the paper.

References

[1] Aarset, M. V. (1987). How to identify bathtub hazard rate. *IEEE Transactions Reliability*, 36, 106-108.

[2] Akaike, H. (1974). A new look at the statistical model identification. *IEEE Transactions on Automatic Control*, 19, 716-723.

[3] Barreto-Souza, W., Santos, A. H. S. and Cordeiro, G. M. (2009). The beta generalized exponential distribution. *Journal of Statistical Computation and Simulation*, doi: 10.1080/00949650802552402

[4] Bebbington, M., Lai, C. D. and Zitikis, R. (2007). A flexible Weibull extension. *Reliability Engineering and System Safety*, 92, 719-726.

[5] Bozdogan, H. (1987). Model selection and Akaike's information criterion (AIC): the general theory and its analytical extensions. *Psychometrika*, 52, 345-370.

[6] Carrasco, J. M. F., Ortega, E. M. M. and Cordeiro, G. M. (2008). A generalized modified Weibull distribution for lifetime modeling. *Computational Statistics and Data Analysis*, 53, 450-462.

[7] Chambers, J., Cleveland, W., Kleiner, B. and Tukey, P. (1983). *Graphical Methods for Data Analysis*. Chapman and Hall, London.

[8] Cordeiro, G. M. and Nadarajah, S. (2010). General results for the beta Weibull distribution. Submitted to *Computational Statistics and Data Analysis*.

[9] Corless, R., Gonnet, G., Hare, D., Jeffrey, D. and Knuth, D. (1996). On the Lambert W function. *Advances in Computational Mathematics*, 5, 329-359.

[10] Eugene, N., Lee, C. and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics – Theory and Methods*, 31, 497-512.

[11] Gradshteyn, I. S. and Ryzhik, I. M. (2000). *Table of Integrals, Series, and Products*, sixth edition. Academic Press, San Diego.

[12] Gupta, R. D. and Kundu, D. (1999). Generalized exponential distributions. *Australian and New Zealand Journal of Statistics*, 41, 173-188.

[13] Gupta, R. D. and Kundu, D. (2001). Exponentiated Exponential family: an alternative to gamma and Weibull distributions. *Biometrical Journal*, 43, 117-130.

[14] Gupta, A. K. and Nadarajah, S. (2004). On the moments of the beta normal distribution. *Communications in Statistics – Theory and Methods*, 33, 1-13.

[15] Haupt, E. and Schabe, H. (1992). A new model for a lifetime distribution with bathtub shaped failure rate. *Microelectronics and Reliability*, 32, 633-639.

[16] Hjorth, U. (1980). A realibility distribution with increasing, decreasing, constant and bathtub failure rates. *Technometrics*, 22, 99-107.

[17] Jones, M. C. (2004). Families of distributions arising from distributions of order statistics. *Test*, 13, 1-43.

[18] Kundu, D. and Rakab, M. Z. (2005). Generalized Rayleigh distribution: different methods of estimation. *Computational Statistics and Data Analysis*, 49, 187-200.

[19] Lai, C. D., Xie, M. and Murthy, D. N. P. (2003). A modified Weibull distribution. *Transactions on Reliability*, 52, 33-37.

[20] Leadbetter, M.R., Lindgren, G. and Rootzén, H. (1987). *Extremes and Related Properties of Random Sequences and Processes*. Springer Verlag, New York.

[21] Linhart, H. and Zucchini, W. (1986). *Model Selection*. John Wiley, New York.

[22] Meeker, W. Q. and Escobar, L. A. (1998). *Statistical Methods for Reliability Data*. John Wiley, New York.

[23] Mudholkar, G. S. and Hutson, A. D. (1996). The exponentiated Weibull family: some properties and a flood data application. *Communications in Statistics – Theory and Methods*, 25, 3059-3083.

[24] Mudholkar, G. S., Srivastava, D. K. and Freimer, M. (1995). The exponentiated Weibull family. *Technometrics*, 37, 436-45.

[25] Nadarajah, S. and Kotz, S. (2004). The beta Gumbel distribution. *Mathematical Problems in Engineering*, 10, 323-332.

[26] Nadarajah, S. and Kotz, S. (2006). The beta exponential distribution. *Reliability Engineering and System Safety*, 91, 689-697.

[27] Nelsen, W. (1982). *Applied Life Data Analysis*. Wiley, New York.

[28] Pham, H. and Lai, C. D. (2007). On recent generalizations of the Weibull distribution. *IEEE Transactions on Reliability*, 56, 454-458.

[29] Prudnikov, A. P., Brychkov, Y. A. and Marichev, O. I. (1986). *Integrals and Series*, volumes 1, 2 and 3. Gordon and Breach Science Publishers, Amsterdam.

[30] Rajarshi, S. and Rajarshi, M. B. (1988). Bathtub distributions: a review. *Communication in Statistics – Theory and Methods*, 17, 2521-2597.

[31] Schwarz, G. E. (1978). Estimating the dimension of a model. *Annals of Statistics*, 6, 461-464.

[32] Silva, A. N. F. (2004). Estudo evolutivo das crianças expostas ao HIV e notificadas pelo núcleo de vigilância epidemiológica do HCFMRP-USP. MSc Thesis, University of São Paulo, Brazil.

[33] Silva, G. O., Ortega, E. M. M. and Cordeiro, G. M. (2010). The beta modified Weibull distribution. *Lifetime and Data Analysis*, to appear.

[34] Xie, M. and Lai, C.D. (1995). Realibility analysis using additive Weibull model with bathtub-shaped failure rate function. *Realiability Engineering and System Safety*, 52, 87-93.

[35] Xie, M., Tang, Y. and Goh, T. N. (2002). A modified Weibull extension with bathtub failure rate function. *Reliability Engineering and System Safety*, 76, 279-285.

Table 1: Comparison of MLE versus MME.

a	b	MLE				MME			
		AE($\widehat{a}$)	AE($\widehat{b}$)	MSE($\widehat{a}$)	MSE($\widehat{b}$)	AE($\widehat{a}$)	AE($\widehat{b}$)	MSE($\widehat{a}$)	MSE($\widehat{b}$)
1	1	1.153	1.176	0.160	0.183	1.323	1.279	0.188	0.184
1	2	1.145	2.428	0.136	0.901	1.415	2.725	0.155	1.083
1	3	1.171	3.626	0.254	2.588	1.184	4.155	0.290	2.941
1	4	1.133	4.735	0.169	4.571	1.179	5.075	0.187	4.702
1	5	1.069	5.659	0.107	4.639	1.218	5.811	0.114	5.098
1	6	1.169	7.400	0.143	9.206	1.286	8.028	0.151	11.346
2	1	2.251	1.160	0.638	0.155	2.716	1.297	0.725	0.179
2	2	2.288	2.221	0.647	0.591	2.714	2.486	0.726	0.609
2	3	2.384	3.507	0.893	1.721	2.405	3.534	1.088	2.021
2	4	2.359	4.733	0.924	4.415	2.429	4.997	1.058	5.118
2	5	2.416	6.192	0.906	6.626	2.465	7.049	1.056	7.560
2	6	2.349	7.191	0.727	8.410	2.839	7.451	0.880	8.588
3	1	3.498	1.157	1.882	0.184	3.628	1.223	1.890	0.187
3	2	3.563	2.282	1.940	0.673	4.200	2.667	2.228	0.724
3	3	3.791	3.860	3.112	3.339	4.029	4.124	3.222	3.539
3	4	3.662	4.918	2.191	4.337	3.886	5.138	2.510	4.424
3	5	3.398	5.668	1.485	4.088	3.642	5.899	1.518	4.355
3	6	3.557	7.251	1.642	9.438	4.191	8.253	1.667	9.461
4	1	4.567	1.098	2.792	0.129	5.119	1.287	3.027	0.159
4	2	5.018	2.423	5.577	0.966	6.258	2.913	6.366	0.994
4	3	4.523	3.350	2.632	1.240	4.966	3.664	3.007	1.402
4	4	4.660	4.679	3.496	3.556	4.915	5.783	3.814	4.200
4	5	4.443	5.639	1.978	3.220	5.062	6.926	2.205	3.524
4	6	4.688	7.258	3.379	8.353	5.466	8.888	3.770	8.949
5	1	5.665	1.091	5.085	0.094	6.853	1.279	5.665	0.096
5	2	5.655	2.264	4.997	0.729	6.697	2.531	5.300	0.883
5	3	6.422	3.699	11.912	3.118	7.762	4.554	14.488	3.157
5	4	5.753	4.625	5.192	2.907	6.468	5.152	5.645	3.301
5	5	5.546	5.654	4.699	4.232	6.406	7.048	5.439	4.901
5	6	5.995	6.964	4.289	5.064	7.164	8.530	4.803	5.358
6	1	7.081	1.122	6.199	0.099	7.885	1.167	7.588	0.122
6	2	7.096	2.349	5.842	0.619	8.070	2.781	6.301	0.720
6	3	6.750	3.407	4.688	1.251	8.432	3.665	5.003	1.425
6	4	7.322	4.888	8.134	3.520	7.663	5.937	8.737	3.582
6	5	7.291	6.085	10.064	8.308	9.008	7.043	10.470	9.896
6	6	7.039	6.820	6.703	6.015	7.734	8.353	8.099	7.098

Table 2: MLEs of the model parameters for the voltage data, the corresponding SE (given in parentheses) and the measures AIC, BIC and CAIC.

Model	a	b	α	γ	λ	AIC	BIC	CAIC
Beta Modified	0.068	0.099	4.9e-17	4.266	0.0528	345.1	347.6	352.2
Weibull (BMW)	(0.016)	(0.049)	(0.000)	(0.011)	(0.002)			
Beta	0.203	0.083	8.9e-7	2.967	0	363.1	368.7	364.7
Weibull (BW)	(0)	(0)	(0)	(0)	-			
Generalized Modified	0.099	1	3.7e-16	3.597	0.048	353.0	358.6	354.6
Weibull (GMW)	(0.019)	-	(0.000)	(0.233)	(0.006)			
Modified	1	1	0.018	0.4536	0.007	362.1	366.3	363.1
Weibull (MW)	-	-	(0.018)	(0.220)	(0.002)			
Exponentiated	0.139	1	3.9e-17	6.540	0	360.5	364.7	361.4
Weibull (EW)	(0.025)	-	(0.000)	(0.0001)	-			

Table 3: MLEs of the model parameters for the serum-reversal data, the corresponding SE (given in parentheses) and the measures AIC, BIC and CAIC.

Model	a	b	α	γ	λ	AIC	BIC	CAIC
Beta Modified	0.147	0.184	1.8e-15	0.057	2.636	769.9	784.9	770.4
Weibull (BMW)	(0.020)	(0.072)	(0.000)	(0.001)	(0.014)			
Beta	0.508	0.117	9.8e-10	3.960	0	801.7	813.7	802.0
Weibull (BW)	(0.091)	(0.021)	(0.000)	(0.012)	-			
Generalized Modified	0.491	1	7.4e-06	0.649	0.023	779.8	795.7	795.8
Weibull (GMW)	(0.116)	-	(1.5-07)	(0.471)	(0.006)			
Modified	1	1	0.002	0.356	0.014	781.4	790.4	781.6
Weibull (MW)	-	-	(0.000)	(0.297)	(0.002)			
Exponentiated	0.385	1	5.5e-17	6.361	0	808.2	820.1	820.2
Weibull (EW)	(0.046)	-	(0.000)	(0.022)	-			

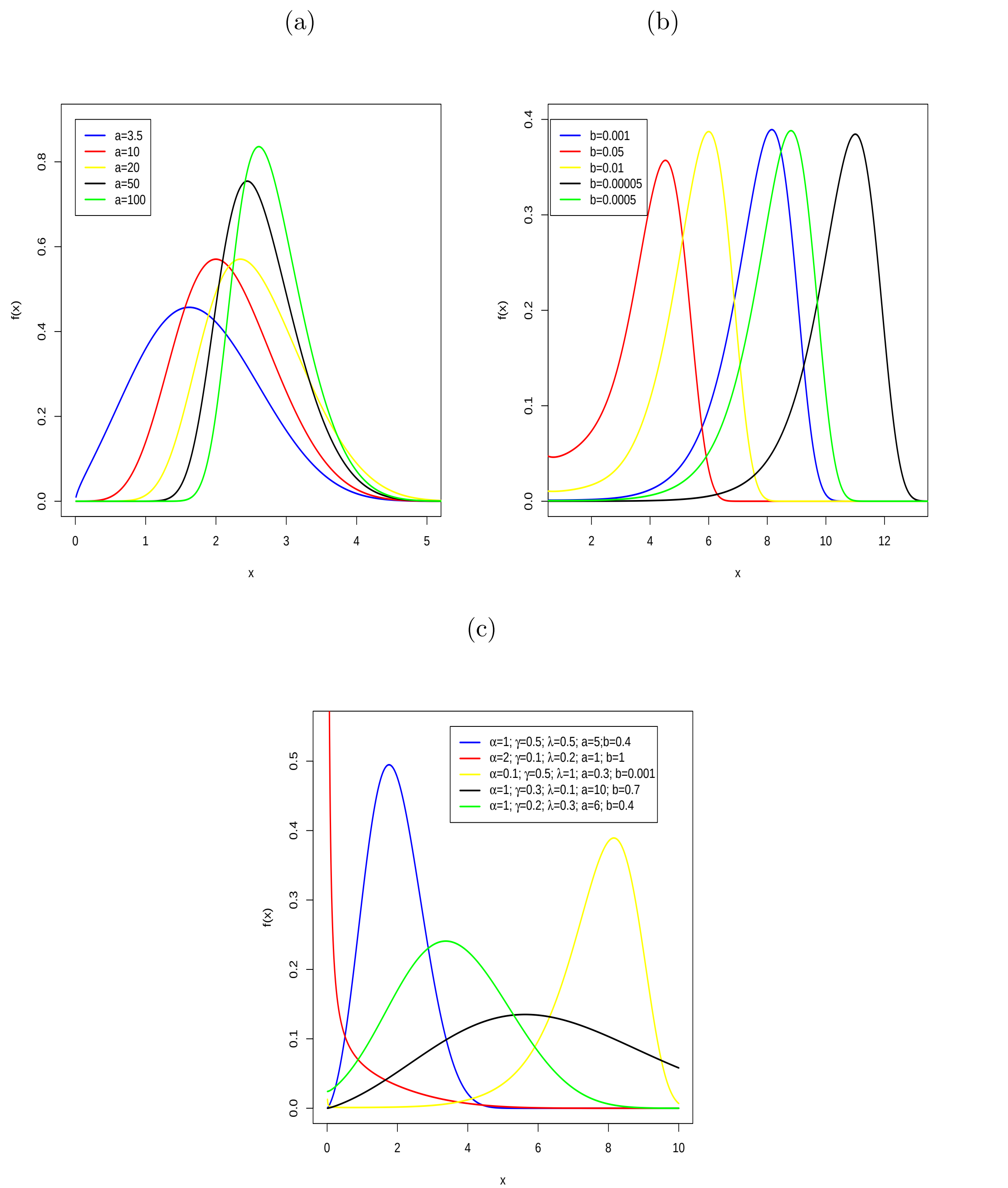


Figure 1: Plots of the BMW pdf for some parameter values. (a) Parameter values $\alpha = 1$, $\gamma = 0.5$ and $\lambda = 0.5$. (b) Parameter values $\alpha = 0.1$, $\gamma = 0.5$ and $\lambda = 1$.

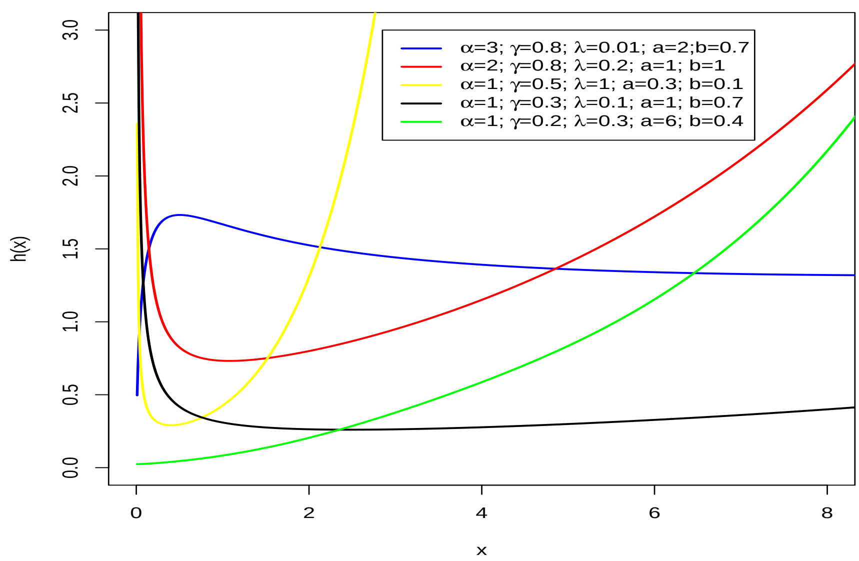


Figure 2: Plots of the hazard rate function (5) (increasing, decreasing, unimodal, bathtub shaped) for some parameter values.

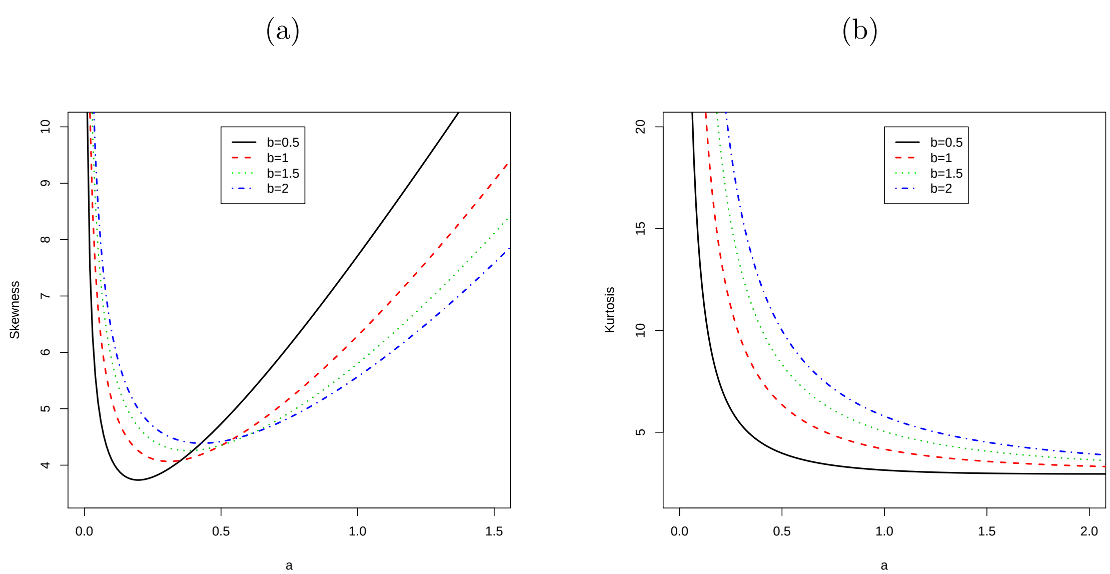


Figure 3: Skewness and kurtosis of the BMW distribution as a function of the parameter a for some values of b.

(a) (b)

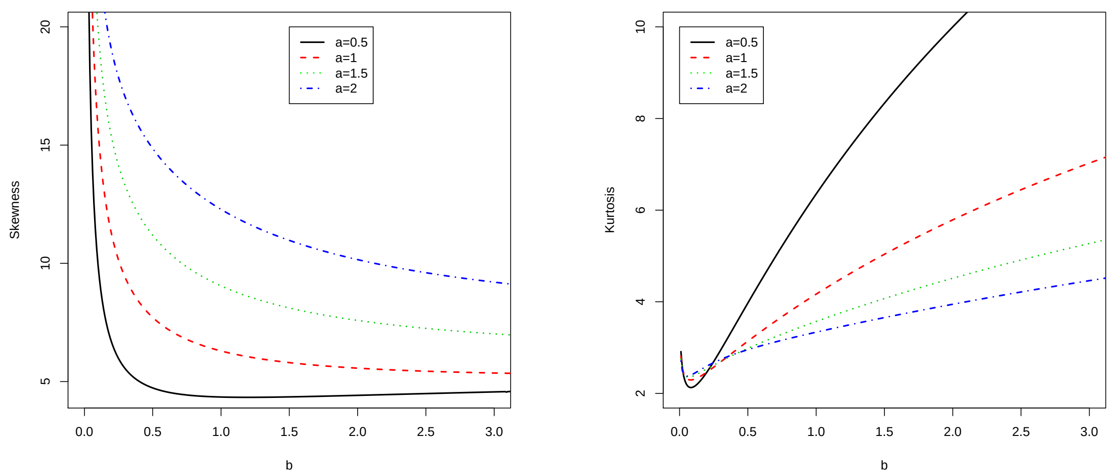


Figure 4: Skewness and kurtosis of the BMW distribution as a function of the parameter b for some values of a.

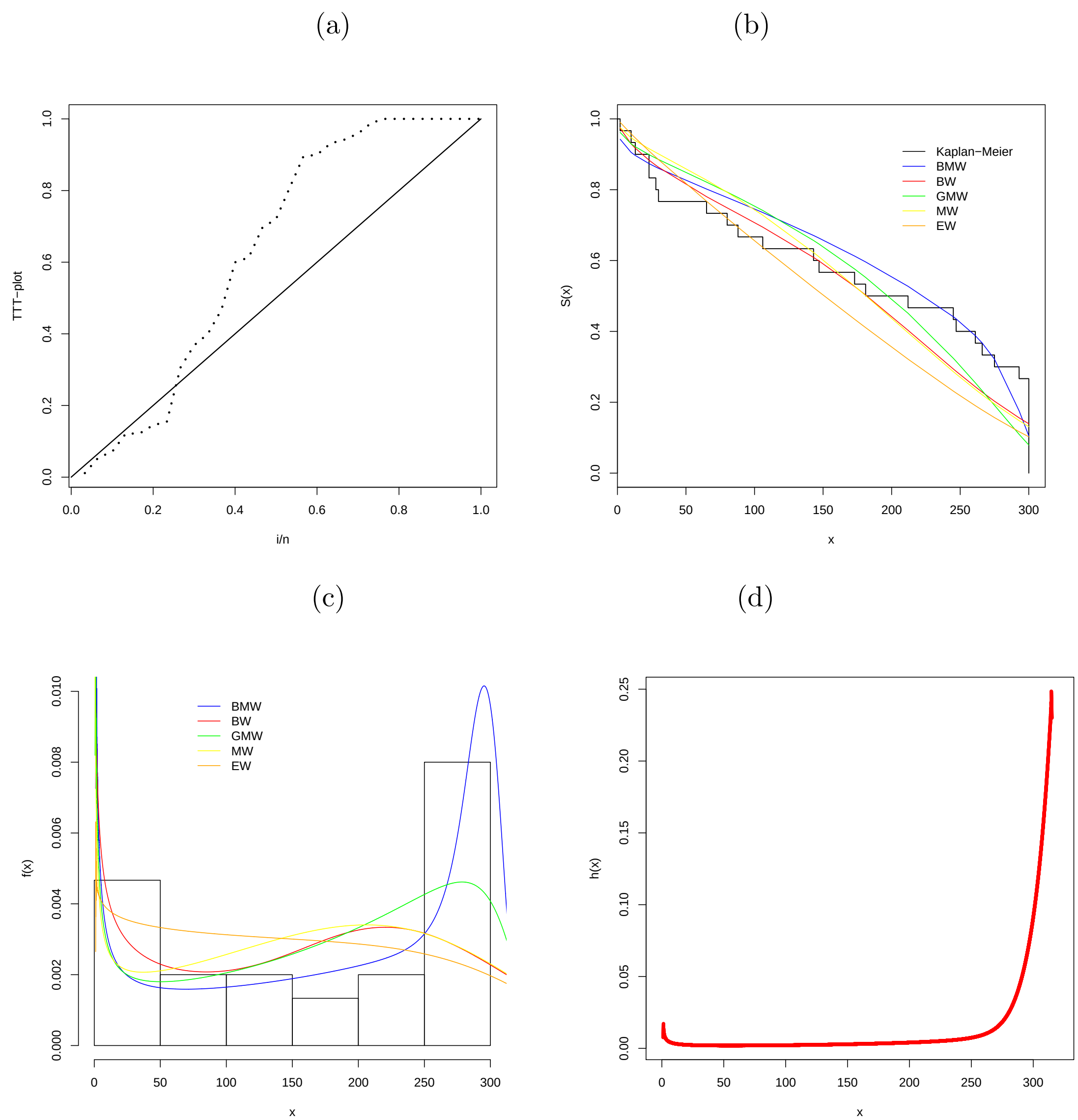
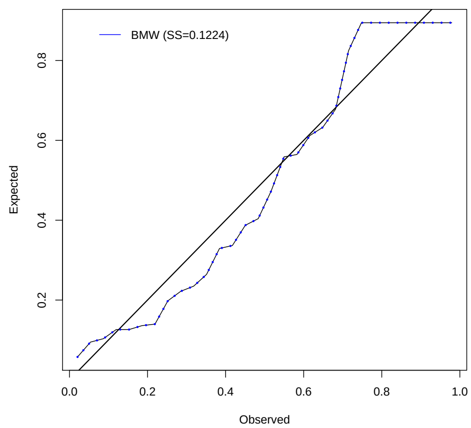
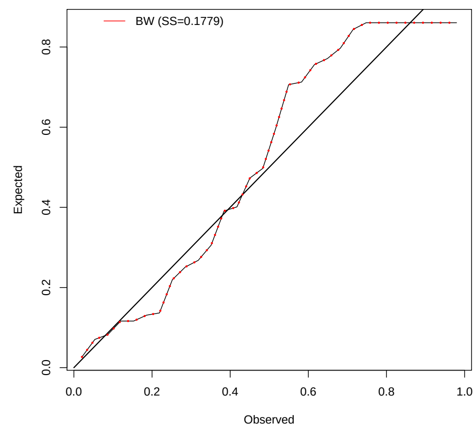


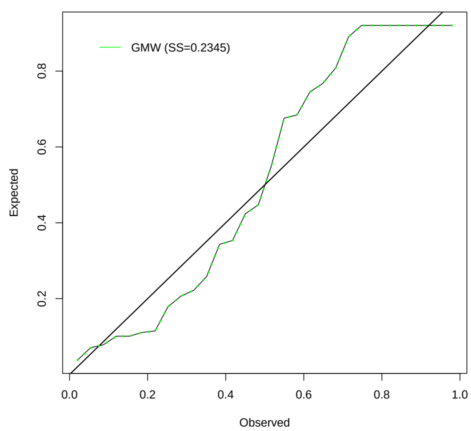
Figure 5: (a) TTT-plot on voltage data. (b) Estimated survival functions and the empirical survival for voltage data. (c) Estimated pdfs of the BMW, BW, GMW, MW and EW models for voltage data. (d) Estimated hazard rate function for the voltage data.

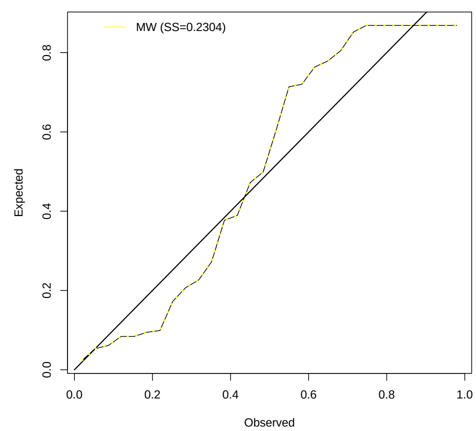
(a) (b)





(c) (d)





(e)

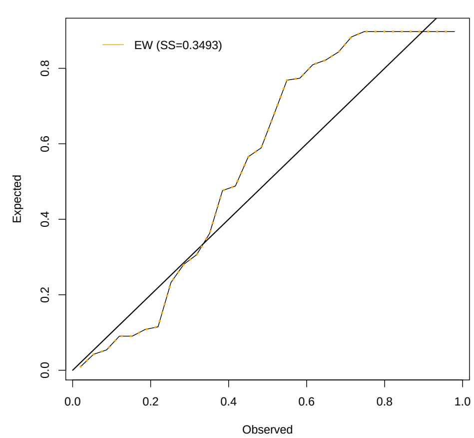
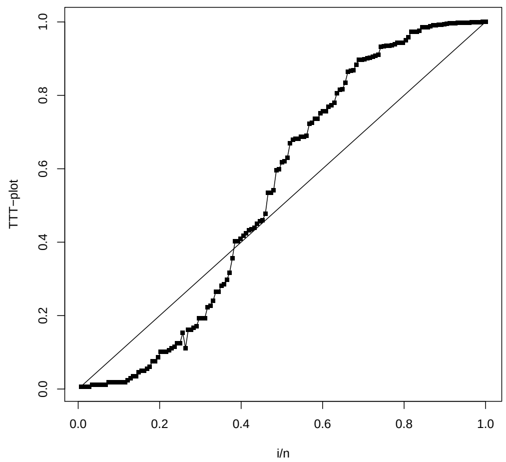
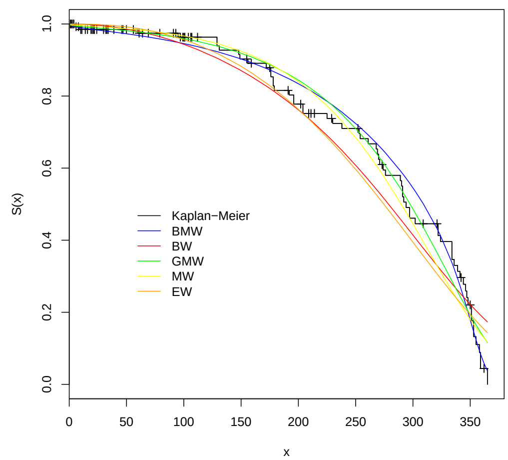


Figure 6: Probability plots of the fitted models to the voltage data. (a) BMW distribution. (b) BW distribution. (c) GMW distribution. (d) MW distribution. (e) EW distribution.

(a)



(b)



(c)

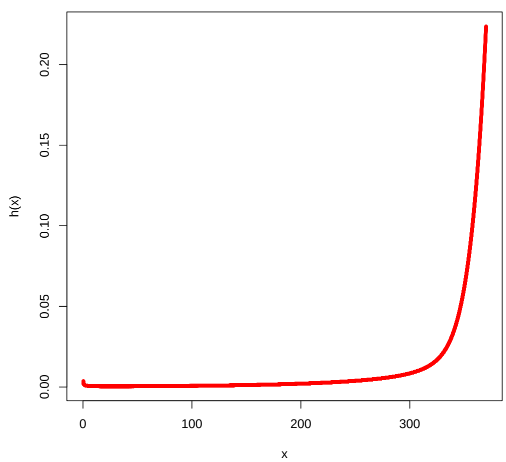


Figure 7: (a) TTT-plot on serum-reversal data. (b) Estimated survival function and the empirical survival for serum-reversal data. (c) Estimated hazard rate function for the serum-reversal data.