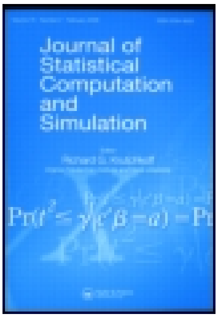

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The beta exponentiated Weibull distribution

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The Weibull distribution is one of the most important distributions in reliability. For the first time, we introduce the beta exponentiated Weibull distribution which extends recent models by Lee *et al.* [*Beta-Weibull distribution: some properties and applications to censored data*, J. Mod. Appl. Statist. Meth. 6 (2007), pp. 173–186] and Barreto-Souza *et al.* [*The beta generalized exponential distribution*, J. Statist. Comput. Simul. 80 (2010), pp. 159–172]. The new distribution is an important competitive model to the Weibull, exponentiated exponential, exponentiated Weibull, beta exponential and beta Weibull distributions since it contains all these models as special cases. We demonstrate that the density of the new distribution can be expressed as a linear combination of Weibull densities. We provide the moments and two closed-form expressions for the moment-generating function. Explicit expressions are derived for the mean deviations, Bonferroni and Lorenz curves, reliability and entropies. The density of the order statistics can also be expressed as a linear combination of Weibull densities. We obtain the moments of the order statistics. The expected information matrix is derived. We define a log-beta exponentiated Weibull regression model to analyse censored data. The estimation of the parameters is approached by the method of maximum likelihood. The usefulness of the new distribution to analyse positive data is illustrated in two real data sets.

Keywords: beta exponentiated Weibull distribution; beta Weibull distribution; exponentiated Weibull distribution; information matrix; maximum likelihood; moment-generating function; Weibull distribution

1. Introduction

The Weibull distribution is a very popular model, and has been extensively used over the past decades for modelling data in reliability, engineering and biological studies. In this article, we introduce and study several mathematical properties of a new model refereed to as the beta exponentiated Weibull (BEW) distribution. The Weibull distribution represents only a special case of the new model. We provide a comprehensive description of some mathematical properties of the BEW distribution with the hope that it will attract wider applications in reliability, engineering and in other areas of research.

The exponentiated distribution is constructed by raising a baseline cumulative distribution function (cdf) $G(x)$ to an arbitrary power $\alpha > 0$, and then a new cdf $F(x) = G(x)^{\alpha}$ emerges with one additional parameter. In this construction, $F(x)$ may be referred to as the exponentiated G

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distribution. The relation between the corresponding density functions is $f(x) = \alpha G(x)^{\alpha-1} g(x)$. We note that for $\alpha > 1$ and $\alpha < 1$ and for larger values of x, the multiplicative factor $\alpha G(x)^{\alpha-1}$ is greater and smaller than 1, respectively. The reverse assertion is also true for smaller values of x. The latter immediately implies that the ordinary moments associated with the density $f(x)$ are strictly larger (smaller) than those associated with the density $g(x)$ when $\alpha > 1$ ($\alpha < 1$).

Based on this idea, Gupta and Kundu [1] introduced the exponentiated exponential (EE) distribution as a generalization of the exponential distribution, and Nadarajah and Kotz [2] proposed four more exponentiated-type distributions to extend the gamma, Weibull, Gumbel and Fréchet distributions in the same way as the EE distribution extends the exponential distribution. They also provided some mathematical properties for each exponentiated distribution. In the same way, Mudholkar and Srivastava [3] generalized the Weibull distribution by introducing the exponentiated Weibull (EW) cumulative distribution defined as

$$G_{\lambda,\alpha,c}(x) = \{1 - \exp[-(\lambda x)^c]\}^\alpha, \quad x > 0. \tag{1}$$

The two parameters $\alpha > 0$ and $c > 0$ in Equation (1) represent shape parameters and $\lambda > 0$ is a scale parameter. Clearly, the exponential distribution is a particular case of the EW distribution when $\alpha = c = 1$. The EW distribution (which extends the EE distribution) was studied by Mudholkar *et al.* [4], Mudholkar and Hutson [5] and Nassar and Eissa [6]. The EW distribution is a special case of the beta Weibull (BW) distribution proposed by Lee *et al.* [7].

Generalized Weibull distributions (with additional shape parameters) are usually developed in order to introduce skewness and to vary tail weights and to improve the fit of the model in the non-central probability regions. More recently, Gusmão *et al.* [8] introduced the generalized inverse Weibull distribution. In this article, we consider a generalization of the Weibull distribution by introducing three extra shape parameters which provide greater flexibility in the form of the new distribution and consequently in modelling observed positive data. If G denotes the baseline cdf of a random variable, then the beta-G distribution is defined as

$$F(x) = I_{G(x)}(a, b) = \frac{1}{B(a, b)} \int_0^{G(x)} w^{a-1}(1 - w)^{b-1}\, \mathrm{d}w, \tag{2}$$

for $a > 0$ and $b > 0$. Here, $I_y(a, b) = B_y(a, b)/B(a, b)$ is the incomplete beta function ratio, $B_y(a, b) = \int_0^y w^{a-1}(1 - w)^{b-1}\, \mathrm{d}w$ is the incomplete beta function and $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ is the beta function, where $\Gamma(\cdot)$ is the gamma function. The class of generalized distributions (2) has been receiving considerable attention over the last years, in particular, after the studies by Eugene *et al.* [9] and Jones [10].

Eugene *et al.* [9], Nadarajah and Kotz [11], Nadarajah and Gupta [12], Nadarajah and Kotz [2] and Barreto-Souza *et al.* [13] proposed the beta normal, beta Gumbel, beta Fréchet, beta exponential (BE) and beta exponentiated exponential (BEE) distributions by taking $G(x)$ in Equation (2) to be the cdf of the normal, Gumbel, Fréchet, exponential and EE distributions, respectively. Another distribution that happens to belong to Equation (2) is the beta logistic distribution, which has been around for over 20 years [13], even if it did not originate directly from this equation.

The properties of $F(x)$ for any beta-G distribution defined from a parent $G(x)$ in Equation (2) could, in principle, follow from the properties of the hypergeometric function which are well established in the literature; see, for example, Section 9.1 of Gradshteyn and Ryzhik [14].

The probability density function (pdf) corresponding to Equation (2) has the form

$$f(x) = \frac{1}{B(a, b)}\, G(x)^{a-1}\{1 - G(x)\}^{b-1} g(x). \tag{3}$$

Here, $f(x)$ will be most tractable when the cdf $G(x)$ and the pdf $g(x) = \mathrm{d}G(x)/\mathrm{d}x$ have simple analytic expressions. Except for some special choices for $G(x)$ in Equation (3), as is the case when $G(x)$ is given by Equation (1), it seems that the pdf $f(x)$ will be difficult to deal with in generality.

We define the five-parameter BEW distribution by taking $G(x)$ in Equation (2) to be the cdf (1). The BEW cumulative distribution then becomes

$$F(x) = I_{[1-\mathrm{e}^{-(\lambda x)^c}]^\alpha}(a,b) = \frac{1}{B(a,b)} \int_0^{[1-\mathrm{e}^{-(\lambda x)^c}]^\alpha} \omega^{a-1}(1-\omega)^{b-1}\,\mathrm{d}\omega, \quad x > 0, \tag{4}$$

for $\alpha > 0$, $\lambda > 0$, $a > 0$, $b > 0$ and $c > 0$. The pdf and the hazard rate function corresponding to Equation (4) are

$$f(x) = \frac{\alpha c \lambda^c}{B(a,b)} x^{c-1} \mathrm{e}^{-(\lambda x)^c} (1-\mathrm{e}^{-(\lambda x)^c})^{\alpha a-1} \{1-(1-\mathrm{e}^{-(\lambda x)^c})^\alpha\}^{b-1}, \quad x > 0, \tag{5}$$

and

$$h(x) = \frac{\alpha c \lambda^c x^{c-1} \mathrm{e}^{-(\lambda x)^c} (1-\mathrm{e}^{-(\lambda x)^c})^{\alpha a-1} \{1-(1-\mathrm{e}^{-(\lambda x)^c})^\alpha\}^{b-1}}{B(a,b) I_{1-(1-\mathrm{e}^{-(\lambda x)^c})^\alpha}(b,a)}, \quad x > 0, \tag{6}$$

respectively. If X is a random variable with density (5), then we write $X \sim \mathrm{BEW}(\alpha, \lambda, a, b, c)$.

Plots of the density (5) and failure rate function (6) for selected values of α, λ, a, b and c are given in Figures 1 and 2, respectively. The BEW failure rate function can be bathtub-shaped, monotonically decreasing or increasing and upside-down bathtub depending on the values of its parameters.

The rest of the article is organized as follows. In Section 2, we present some special submodels. In Section 3, we demonstrate that the BEW density function can be expressed as a linear combination of Weibull densities. This result is important to provide mathematical properties of the BEW model directly from those properties of the Weibull distribution. A range of mathematical properties is considered in Sections 4–6. These include quantile function, simulation, moment-generating and characteristic functions, mean deviations and Bonferroni and Lorenz curves. In Section 7, the density function of the BEW order statistics is expressed as a linear combination of Weibull densities. Explicit formulae for the moments of BEW order statistics and L-moments are derived in Section 8. The reliability and the Rényi and Shannon entropies are calculated in Sections 9 and 10, respectively. Maximum likelihood estimation is investigated in Section 11. In

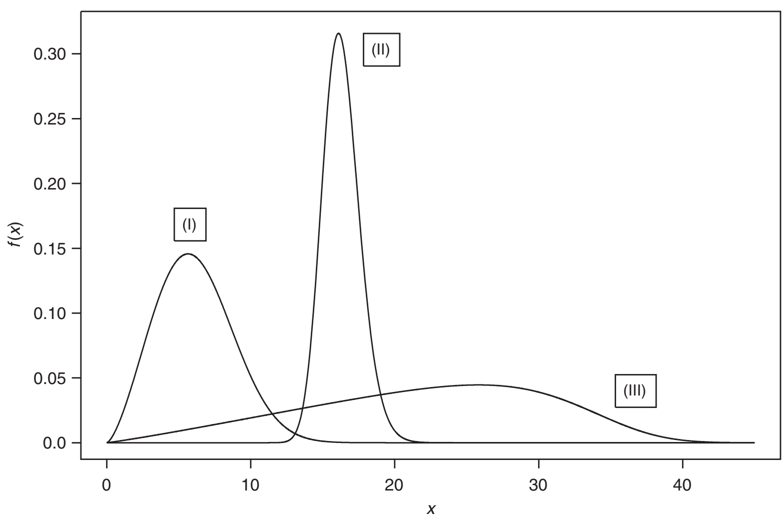

Figure 1. Plots of the BEW density function. (I): $\alpha = 1.00$, $\lambda = 0.10$, $a = 1.00$, $b = 2.50$, $c = 2.50$. (II): $\alpha = 2.00$, $\lambda = 0.10$, $a = 30.00$, $b = 2.50$, $c = 2.50$. (III): $\alpha = 3.35$, $\lambda = 0.02$, $a = 0.20$, $b = 54.00$, $c = 3.00$.

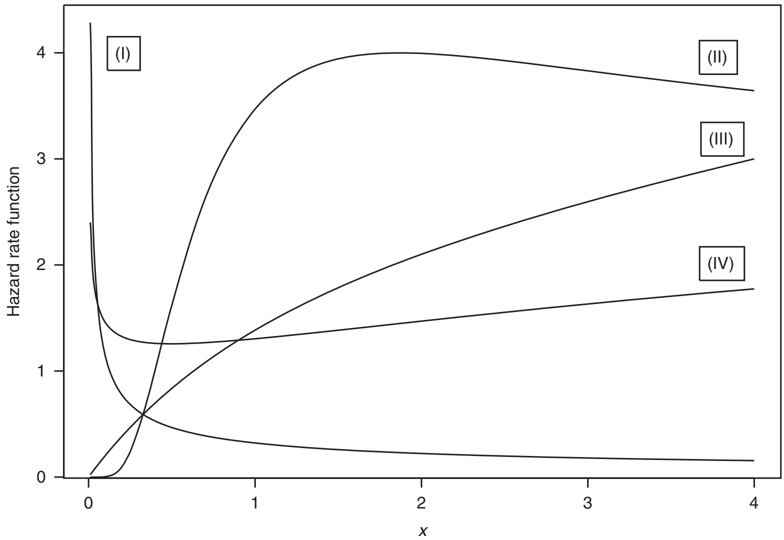

Figure 2. Plots of the BEW hazard rate functions. (I): $\alpha = 1.0$, $\lambda = 1.5$, $a = 0.8$, $b = 0.5$, $c = 0.5$. (II): $\alpha = 3.0$, $\lambda = 3.0$, $a = 3.5$, $b = 2.5$, $c = 0.8$. (III): $\alpha = 1.2$, $\lambda = 1.0$, $a = 1.1$, $b = 1.0$, $c = 1.5$. (IV): $\alpha = 0.7$, $\lambda = 1.0$, $a = 0.8$, $b = 0.9$, $c = 1.3$.

Section 12, we propose a log-beta exponentiated Weibull (LBEW) regression model, which can be useful for lifetime analysis. In Section 13, we fit the LBEW model to two real data sets to illustrate its usefulness. Finally, concluding remarks are addressed in Section 14.

2. Special sub-models

The BEW density (5) allows for greater flexibility of its tails and can widely be applied in many areas of engineering and biology. We study mathematical properties of this distribution because it extends several distributions previously considered in the literature. In fact, the Weibull

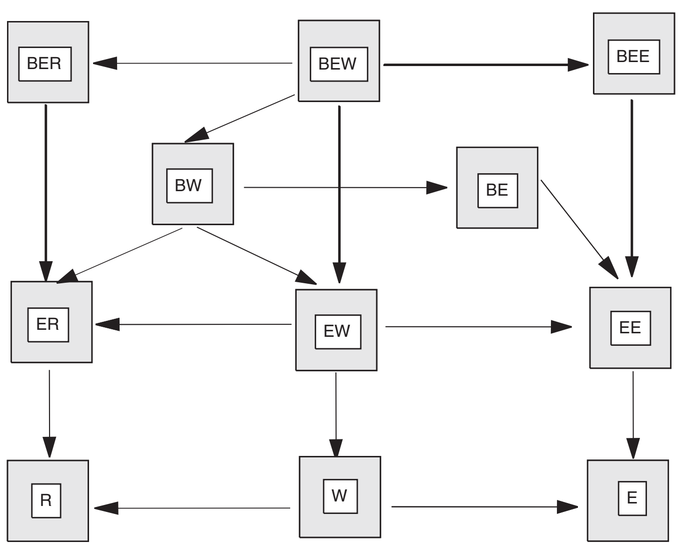

Figure 3. The BEW distribution and its sub-models as listed in Table 1.

Table 1. Exponentiated and beta exponentiated type distributions.

Distribution	BEW a	b	c	λ	α	Main reference
BEE	a	b	1	λ	α	[13]
BW	a	b	c	λ	1	[7]
BE	a	b	1	λ	1	[2]
ER	1	1	2	λ	1	[17,18]
EW($\alpha a, \lambda, c$)	a	1	c	λ	α	[4]
EW(α, λ, c)	1	1	c	λ	α	[4]
EE($\alpha a, \lambda$)	a	1	1	λ	α	[1]
R(s)	1	1	2	$1/\sqrt{2}s$	1	[19]
W($\lambda b^{1/c}, c$)	1	b	c	λ	1	[20]
E	1	1	1	λ	1	[21]

Note: B, Beta; EE, exponentiated exponential; E, exponential; W, Weibull and R, Rayleigh.

model (with parameters c and λ) is clearly a special case for $\alpha = a = b = 1$, with a continuous crossover towards models with different shapes (e.g. a particular combination of skewness and kurtosis). The BEW distribution also contains as sub-models the EW [4–6,15], EE [16], BW [7] and BEE [13] distributions for $\alpha = b = 1$, $\alpha = b = c = 1$, $\alpha = 1$ and $c = 1$, respectively. When $\alpha = a = 1$, Equation (5) yields the Weibull distribution with parameters $\lambda b^{1/c}$ and c. The BE distribution [2] is also a sub-model for $\alpha = c = 1$. Moreover, while the transformation (2) is not analytically tractable in the general case, the formulas related with the BEW distribution turn out manageable (as shown in the rest of this article), and with the use of modern computer resources with analytic and numerical capabilities, may turn into adequate tools comprising the arsenal of applied statisticians. Figure 3 and Table 1 summarize some sub-models of the BEW distribution. We hope that the general results in the paper will make the BEW model attract even more applications in reliability, engineering, biology and statistics.

3. Expansion for the density function

Here and henceforth, let X be a random variable having the BEW density function (5). Equations (4) and (5) are straightforward to compute using any software with algebraic facilities. However, we can obtain expansions for $F(x)$ and $f(x)$ in terms of infinite (or finite) weighted sums of cdf's and pdf's of Weibull distributions, respectively. First, for $b > 0$ real non-integer, we replace $(1-w)^{b-1}$ under the integral by the power series and integrate to obtain

$$\int_0^x w^{a-1}(1-w)^{b-1}\,\mathrm{d}w = \sum_{j=0}^{\infty} \frac{(-1)^j \binom{b-1}{j}}{(a+j)} x^{a+j},$$

where the binomial term $\binom{b-1}{j} = \Gamma(b)/\Gamma(b-j)j!$ is defined for any real b. From Equation (4), we have

$$F(x) = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \binom{b-1}{j}}{(a+j)} \{1 - \exp[-(\lambda x)^c]\}^{\alpha(a+j)}.$$

Again using the binomial expansion and (1), we can write

$$F(x) = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k} \binom{b-1}{j}\binom{\alpha(a+j)}{k}}{(a+j)B(a,b)} [1 - G_{\lambda_k,1,c}(x)], \tag{7}$$

where $G_{\lambda_k,1,c}(x)$ is the Weibull cdf with scale parameter $\lambda_k = k^{1/c}\lambda$ and shape parameter c. Note that $\lambda_0 = 0$ implies the degenerate case for which $G_{\lambda_0,1,c}(x) = 0$. Differentiating Equation (7) yields a useful expansion for the BEW density function

$$f(x) = \sum_{k=1}^{\infty} w_{+,k}\, g_{\lambda_k,1,c}(x), \tag{8}$$

where $g_{\lambda_k,1,c}(x) = \mathrm{d}G_{\lambda_k,1,c}(x)/\mathrm{d}x$ denotes the Weibull density with scale parameter λ_k and shape parameter c, $w_{+,k} = \sum_{j=0}^{\infty} w_{j,k}$ and the quantities $w_{j,k}$ are given by

$$w_{j,k} = \frac{(-1)^{j+k+1}\binom{b-1}{j}\binom{\alpha(a+j)}{k}}{(a+j)B(a,b)}. \tag{9}$$

Clearly, $\sum_{k=1}^{\infty} w_{+,k} = 1$. The linear combination form (8) is a useful representation for the BEW distribution and holds for any parameter values. If $b > 0$ is an integer, the index j in the sum stops at $b-1$, and if both α and a are integers, then the index k in the sum stops at $\alpha(a+j)$. The ordinary, incomplete, inverse and factorial moments, generating function and mean deviations of the BEW distribution can be expressed as functions of those quantities for Weibull distributions. For example, the sth moment of the Weibull distribution with parameters λ and c, say $\mu'_s = \Gamma(s/c+1)\lambda^{-s}$, and Equation (8) yield the sth moment of X (for both a and α real non-integers)

$$E(X^s) = \Gamma\left(\frac{s}{c}+1\right)\sum_{k=1}^{\infty} w_{+,k}\lambda_k^{-s}. \tag{10}$$

For $a = b = \alpha = 1$, Equation (10) yields precisely the sth moment of the Weibull distribution. Figures 4–6 show great flexibility in the values of the skewness and kurtosis of the BEW distribution.

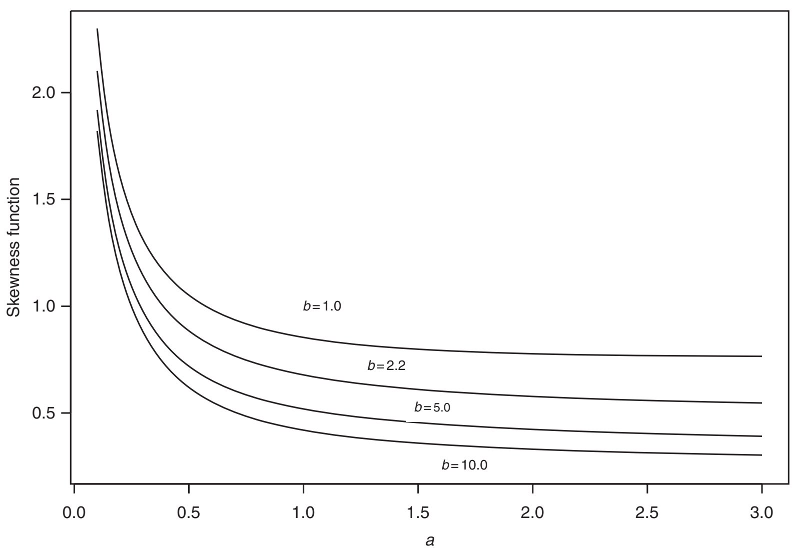

Figure 4. Skewness of the BEW distribution as function of a for some values of b and $\alpha = 2.1$, $\lambda = 1.7$ and $c = 1.5$.

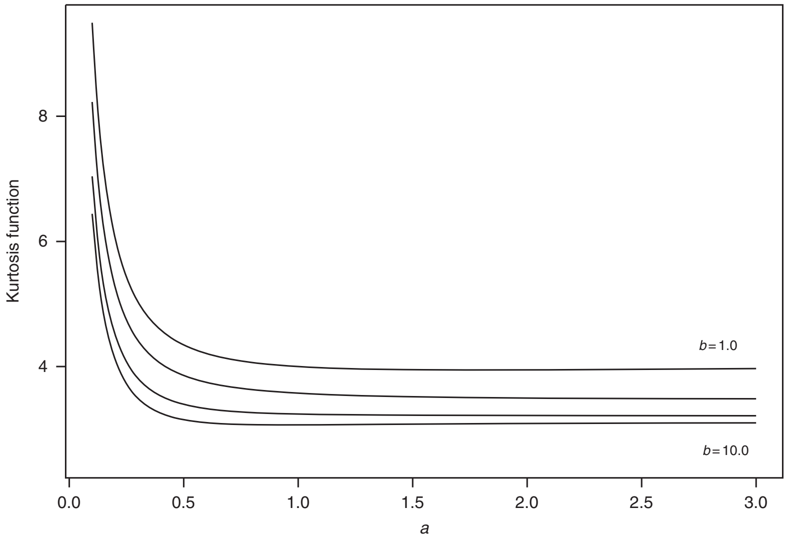

Figure 5. Kurtosis of the BEW distribution as function of a for some values of b and $\alpha = 2.1$, $\lambda = 1.7$ and $c = 1.5$.

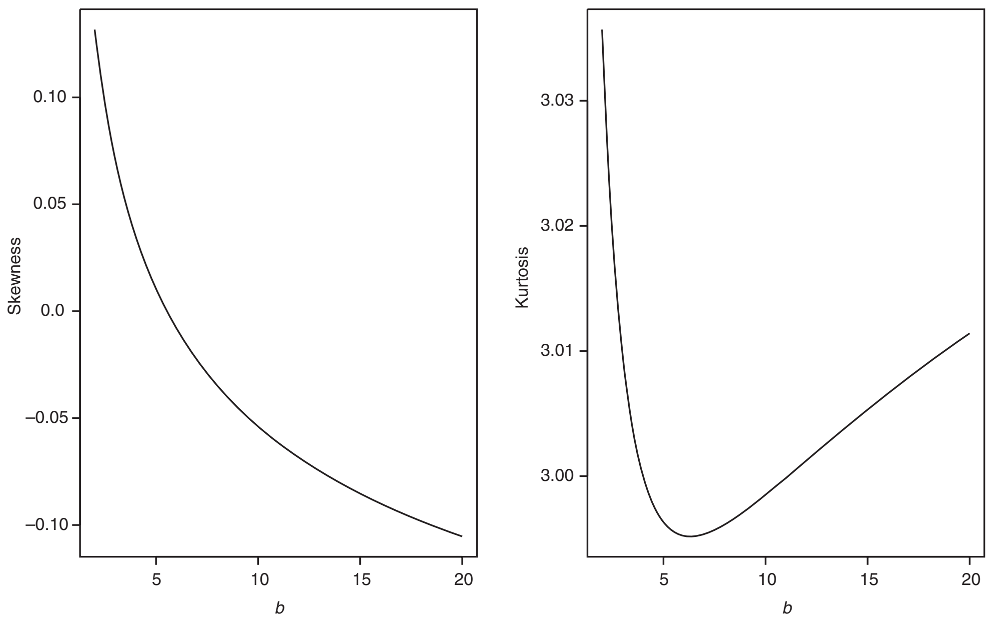

Figure 6. Skewness and kurtosis of the BEW distribution as function of b for fixed values $\alpha = 2.0$, $\lambda = 0.8$, $a = 3.0$ and $c = 3.0$.

4. Quantile function and simulation

The quantile function corresponding to Equation (4) is

$$x = Q(u) = F^{-1}(u) = \frac{1}{\lambda}(-\log\{1 - [I_u^{-1}(a,b)]^{1/\alpha}\})^{1/c}, \tag{11}$$

where $I_u^{-1}(a,b)$ denotes the inverse of the incomplete beta function with parameters a and b. The following expansion for the inverse of the beta incomplete function $I_u^{-1}(a,b)$ can be found on the Wolfram website[1]

$$\begin{aligned}I_u^{-1}(a,b) = w &+ \frac{b-1}{a+1}w^2 + \frac{(b-1)(a^2+3ab-a+5b-4)}{2(a+1)^2(a+2)}w^3 \\ &+ \frac{(b-1)[a^4+(6b-1)a^3+(b+2)(8b-5)a^2]}{3(a+1)^3(a+2)(a+3)}w^4 \\ &+ \frac{(b-1)[(33b^2-30b+4)a+b(31a-47)+18]}{3(a+1)^3(a+2)(a+3)}w^4 \\ &+ O(p^{5/a}),\end{aligned}$$

where $w = [a\,B(a,b)\,u]^{1/a}$ for $a > 0$.

Simulation of X is straightforward from Equation (11) by

$$X = \frac{1}{\lambda}\{-\log(1 - V^{1/\alpha})\}^{1/c},$$

where V is a beta variate with shape parameters a and b.

5. Moment-generating function

We derive two closed-form expressions for the moment-generating function (mgf), $M(t) = E[\exp(tX)]$, of X. Setting $\xi_k = \lambda(k+1)^{1/c}$, we obtain

$$\begin{aligned}M(t) &= \frac{\alpha c\lambda^c}{B(a,b)}\int_0^\infty \mathrm{e}^{tx}x^{c-1}\mathrm{e}^{-(\lambda x)^c}[1-\exp\{-(\lambda x)^c\}]^{\alpha a-1}\{1-[1-\exp\{-(\lambda x)^c\}]^\alpha\}^{b-1}\,\mathrm{d}x \\ &= \frac{\alpha c\lambda^c}{B(a,b)}\sum_{j=0}^\infty\binom{b-1}{j}(-1)^j\int_0^\infty \mathrm{e}^{tx}x^{c-1}\exp\{-(\lambda x)^c\}[1-\exp\{-(\lambda x)^c\}]^{\alpha(a+j)-1}\,\mathrm{d}x \\ &= \frac{\alpha c\lambda^c}{B(a,b)}\sum_{k=0}^\infty f_k\int_0^\infty \mathrm{e}^{tx}x^{c-1}\exp\{-(k+1)(\lambda x)^c\}\,\mathrm{d}x\end{aligned}$$

and then

$$M(t) = \frac{\alpha\, c\, \lambda^c}{B(a,b)}\sum_{k=0}^\infty f_k\sum_{m=0}^\infty\frac{t^m}{m!}\int_0^\infty x^{m+c-1}\,\exp\{-(\xi_k x)^c\}\,\mathrm{d}x, \tag{12}$$

where

$$f_k = \sum_{j=0}^\infty(-1)^{j+k}\binom{b-1}{j}\binom{\alpha(a+j)-1}{k}.$$

If $b \geq 1$ is an integer, then the index j in the last sum stops at $b-1$, and if both α and a are integers, the index k in Equation (12) stops at $\alpha(a+j)-1$. A simple representation for the integral in Equation (12) can be obtained using the Wright-generalized hypergeometric function

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p)\\(\beta_1, B_1), \ldots, (\beta_q, B_q)\end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q}\Gamma(\beta_j + B_j n)}\frac{x^n}{n!}. $$

We assert that

$$ \begin{aligned} I &= \sum_{m=0}^{\infty}\frac{t^m}{m!}\int_0^{\infty} x^{m+c-1}\exp\{-(\xi_k x)^c\}\,\mathrm{d}x = \frac{1}{c\xi_k^c}\sum_{m=0}^{\infty}\frac{(t/\xi_k)^m}{m!}\Gamma\left(\frac{m}{c}+1\right) \\ &= \frac{1}{c\xi_k^c}\,{}_1\Psi_0\left[\begin{matrix}(1, c^{-1})\\ -\end{matrix}; \frac{t}{\xi_k}\right] \end{aligned} \tag{13} $$

provided that $c > 1$. Combining Equations (12) and (13) yields the first representation for $M(t)$

$$ M(t) = \frac{\alpha\lambda^c}{B(a,b)}\sum_{k=0}^{\infty}\frac{f_k}{\xi_k^c}\,{}_1\Psi_0\left[\begin{matrix}(1, c^{-1})\\ -\end{matrix}; \frac{t}{\xi_k}\right]. \tag{14} $$

A second representation for $M(t)$ comes from the Meijer G-function defined as

$$ G_{p,q}^{m,n}\left(x\,\middle|\,\begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right) = \frac{1}{2\pi\mathrm{i}}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j + t)\prod_{j=1}^{n}\Gamma(1 - a_j - t)}{\prod_{j=n+1}^{p}\Gamma(a_j + t)\prod_{j=m+1}^{p}\Gamma(1 - b_j - t)}\,x^{-t}\,\mathrm{d}t, $$

where $\mathrm{i} = \sqrt{-1}$ is the complex unit and L denotes an integration path; see Section 9.3 in Gradshteyn and Ryzhik [14] for a description of this path. Using the result $\exp\{-g(x)\} = G_{0,1}^{1,0}\left(g(x) \mid \begin{matrix}-\\0\end{matrix}\right)$ for $g(\cdot)$ an arbitrary function, the integral in Equation (12) can be written as

$$ I = \int_0^{\infty} x^{c-1}\exp\{tx - (\xi_k x)^c\}\,\mathrm{d}x = \int_0^{\infty} x^{c-1}\exp(tx)G_{0,1}^{1,0}\left(\xi_k^c x^c\,\middle|\,\begin{matrix}-\\0\end{matrix}\right)\mathrm{d}x. $$

We now assume that $c = p/q$, where $p \geq 1$ and $q \geq 1$ are co-prime integers. By (2.24.1.1) in vol. 3 of Prudnikov *et al.* [22], I can be calculated as

$$ I = \frac{p^{c-1/2}(-t)^{-c}}{(2\pi)^{(p+q)/2-1}}G_{q,p}^{p,q}\left(\frac{(\xi_k c)^q p^p}{(-t)^p q^q}\,\middle|\,\begin{matrix}\frac{1-c}{p}, \frac{2-c}{p}, \ldots, \frac{p-c}{p}\\ 0, \frac{1}{q}, \ldots, \frac{q-1}{q}\end{matrix}\right). $$

From Equation (12) and the last two equations, we obtain

$$ M(t) = \frac{\alpha c\lambda^c p^{c-1/2}(-t)^{-c}}{(2\pi)^{(p+q)/2-1}B(a,b)}\sum_{k=0}^{\infty} f_k G_{q,p}^{p,q}\left(\frac{(\xi_k c)^q p^p}{(-t)^p\, q^q}\,\middle|\,\begin{matrix}\frac{1-c}{p}, \frac{2-c}{p}, \ldots, \frac{p-c}{p}\\ 0, \frac{1}{q}, \ldots, \frac{q-1}{q}\end{matrix}\right). \tag{15} $$

Note that the condition $c = p/q$ in Equation (15) is not restrictive since every real number can be approximated by a rational number.

Clearly, special formulas for the mgf of the Weibull, BW, BE, EW and EE distributions can be obtained immediately from Equations (14) and (15) by substitution of known parameters.

The characteristic function $\phi(t) = E[\exp(\mathrm{i}tX)]$ of X corresponding to Equation (14) is

$$\phi(t) = \frac{\alpha\lambda^c}{B(a,b)} \sum_{k=0}^{\infty} \frac{f_k}{\xi_k^c} {}_1\Psi_0 \left[\begin{matrix} (1, c^{-1}) \\ - \end{matrix} ; \frac{\mathrm{i}t}{\xi_k} \right]$$

provided that $c > 1$.

The characteristic function of X corresponding to Equation (15) is

$$\phi(t) = \frac{\alpha c \lambda^c p^{c-1/2}(-\mathrm{i}t)^{-c}}{(2\pi)^{(p+q)/2-1}B(a,b)} \sum_{k=0}^{\infty} f_k G_{q,p}^{p,q} \left(\frac{(\xi_k c)^q p^p}{(-it)^p q^q} \middle| \begin{matrix} \frac{1-c}{p}, \frac{2-c}{p}, \ldots, \frac{p-c}{p} \\ 0, \frac{1}{q}, \ldots, \frac{q-1}{q} \end{matrix} \right),$$

provided that $c = p/q$ and $p \geq 1$ and $q \geq 1$ are co-prime integers.

6. Mean deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and the median. If X has the BEW distribution, then we can derive the mean deviations about the mean $\mu = E(X)$ and about the median M from

$$\delta_1 = \int_0^{\infty} |\, x - \mu \,|\, f(x)\,\mathrm{d}x \quad \text{and} \quad \delta_2 = \int_0^{\infty} |\, x - M \,|\, f(x)\,\mathrm{d}x,$$

respectively. The mean μ is obtained from Equation (10) with $s = 1$ and the median M is the solution of the non-linear equation $I_{[1-\mathrm{e}^{-(\lambda M)^c}]^{\alpha}}(a,b) = \frac{1}{2}$.

These measures can be calculated using the following relationships:

$$\delta_1 = 2[\mu F(\mu) - J(\mu)] \quad \text{and} \quad \delta_2 = \mu - 2J(M), \tag{16}$$

where $J(a) = \int_0^a x f(x)\,\mathrm{d}x$ can be obtained from Equation (5). We have

$$J(a) = c \sum_{k=1}^{\infty} \lambda_k^c w_{+,k} \int_0^a x^c \exp\{-(\lambda_k x)^c\}\,\mathrm{d}x = \sum_{k=1}^{\infty} \frac{w_{+,k}}{\lambda_k} \gamma(c^{-1} + 1, (\lambda_k a)^c), \tag{17}$$

where $\gamma(\alpha, x) = \int_0^x w^{\alpha-1}\mathrm{e}^{-w}\,\mathrm{d}w$ (for $\alpha > 0$) is the incomplete gamma function. Hence, the measures in Equation (16) can be directly obtained from Equation (17).

The quantity $J(a)$ can also be used to determine Bonferroni and Lorenz curves which have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. They are given by

$$B(\pi) = \frac{J(q)}{\pi\mu} \quad \text{and} \quad L(\pi) = \frac{J(q)}{\mu},$$

respectively, where $q = Q(\pi)$ is calculated from Equation (11) for a given probability π.

7. Order statistics

The density function $f_{i:n}(x)$ of the ith order statistic for $i = 1, \ldots, n$ from data values $X_1, \ldots, X_n$ having the BEW distribution is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x)\, F(x)^{i-1}\, \{1 - F(x)\}^{n-i},$$

where $F(\cdot)$ is the cdf (4) and $f(\cdot)$ is the pdf (5). The binomial expansion yields

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i, n-i+1)} f(x) \sum_{l=0}^{n-i} (-1)^l \binom{n-i}{l} F(x)^{i-1+l} \\ &= \frac{cx^{c-1}}{B(i, n-i+1)} \left(\sum_{k=0}^{\infty} w_{+,k} \lambda_k^c \mathrm{e}^{-(\lambda_k x)^c} \right) \sum_{l=0}^{n-i} (-1)^{l+1} \binom{n-i}{l} \\ &\quad \times \left(\sum_{k=0}^{\infty} w_{+,k} \mathrm{e}^{-(\lambda_k x)^c} \right)^{i-1+l}. \end{aligned}$$

Setting $u = \exp\{-(\lambda x)^c\}$, we can write from Equations (7) and (8)

$$f_{i:n}(x) = \frac{c\lambda^c x^{c-1}}{B(i, n-i+1)} \left(\sum_{s=0}^{\infty} s w_{+,s}\, u^s \right) \sum_{l=0}^{n-i} (-1)^{l+1} \binom{n-i}{l} \left(\sum_{k=0}^{\infty} w_{+,k} u^k \right)^{i-1+l}.$$

We use an equation of Gradshteyn and Ryzhik [14, Section 0.314] for a power series raised to a positive integer r given by

$$\left(\sum_{k=0}^{\infty} a_k u^k \right)^r = \sum_{k=0}^{\infty} d_{r,k} u^k, \tag{18}$$

where the coefficients $d_{r,k}$ (for $k = 1, 2, \ldots$) can be determined from the recurrence equation

$$d_{r,k} = (ka_0)^{-1} \sum_{m=1}^{k} [m(r+1) - k] a_m d_{r,k-m} \tag{19}$$

and $d_{r,0} = a_0^r$. Hence, $d_{r,k}$ comes directly from $d_{r,0}, \ldots, d_{r,k-1}$ and, therefore, from $a_0, \ldots, a_k$. Using Equations (18) and (19), it follows that

$$f_{i:n}(x) = \frac{c\lambda^c x^{c-1}}{B(i, n-i+1)} \left(\sum_{s=0}^{\infty} s w_{+,s} u^s \right) \sum_{l=0}^{n-i} (-1)^{l+1} \binom{n-i}{l} \left(\sum_{k=0}^{\infty} c_{i-1+l,k} u^k \right),$$

where

$$c_{i-1+l,k} = (kw_{+,0})^{-1} \sum_{m=1}^{k} [m(i+l) - k] w_{+,m} c_{i-1+l,k-m}$$

and $c_{i-1+l,0} = w_{+,0}^{i-1+l} = (-1)^{i-1+l}$. Here, $c_{i-1+l,k}$ follows from $c_{i-1+l,0}, \ldots, c_{i-1+l,k-1}$ and, therefore, from $w_{+,0}, \ldots, w_{+,k-1}$. Combining terms, we obtain

$$f_{i:n}(x) = \frac{c\lambda^c x^{c-1}}{B(i, n-i+1)} \sum_{l=0}^{n-i} \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} (-1)^{l+1} \binom{n-i}{l} s c_{i-1+l,k} w_{+,s} u^{k+s}.$$

Substituting $\lambda_{k,s} = (k+s)^{1/c}\lambda$ into the above expression gives

$$f_{i:n}(x) = \frac{c}{B(i, n-i+1)} \sum_{l=0}^{n-i} \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} \frac{(-1)^{l+1}\binom{n-i}{1} s c_{i-1+l,k} w_{+,s} \lambda_{k,s}^{c}}{(k+s)} x^{c-1} \exp\{-(\lambda_{k,s}x)^c\},$$

and then

$$f_{i:n}(x) = \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} q(k,s) g_{\lambda_{k,s},1,c}(x), \tag{20}$$

where the coefficients $q(k,s)$ are given by

$$q(k,s) = \frac{s w_{+,s}}{(k+s)B(i, n-i+1)} \sum_{l=0}^{n-i} (-1)^{l+1} \binom{n-i}{l} c_{i-1+l,k}.$$

Equation (20) reveals that the density function of the BEW order statistics is expressed as a linear combination of Weibull densities. We can obtain some mathematical quantities of the BEW order statistics such as ordinary and incomplete moments, mgf, mean deviations, among others, directly from those quantities of the Weibull distribution. For example, when $c = p/q$ and $p \geq 1$ and $q \geq 1$ are co-prime integers, the mgf of the BEW order statistics immediately follow from Equations (15) and (20).

8. Moments of order statistics and L-moments

The moments of BEW order statistics can be written directly in terms of the moments of Weibull distributions from the mixture form (20). We have

$$E(X_{i:n}^r) = \Gamma\left(\frac{r}{c+1}\right) \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} q(k,s) \lambda_{k,s}^{-r}, \tag{21}$$

where $\lambda_{k,s}$ and $q(k,s)$ are defined in Section 7.

L-moments [23] are summary statistics for probability distributions and data samples but have several advantages over ordinary moments. For example, they apply for any distribution having a finite mean and no higher-order moments need be finite. The rth L-moment is computed from linear combinations of the ordered data values by

$$\lambda_r = \sum_{j=0}^{r-1} (-1)^{r-1-j} \binom{r-1}{j} \binom{r-1+j}{j} \beta_j, \tag{22}$$

where $\beta_j = E[XF(X)^j]$. In particular, $\lambda_1 = \beta_0$, $\lambda_2 = 2\beta_1 - \beta_0$, $\lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0$ and $\lambda_4 = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0$. In general, $\beta_k = (k+1)^{-1}E(X_{k+1:k+1})$, so it can be computed from Equation (21) with $i = n = k+1$ and $r = 1$.

9. Reliability

In the context of reliability, the stress–strength model describes the life of a component which has a random strength X_1 that is subjected to a random stress X_2. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X_1 > X_2$. Hence, $R = P(X_2 < X_1)$ is a measure of component reliability [24]. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and the aging of concrete pressure vessels.

We derive the reliability R when X_1 and X_2 have independent BEW$(\alpha_1, \lambda, a_1, b_1, c)$ and BEW$(\alpha_2, \lambda, a_2, b_2, c)$ distributions, respectively, with the same shape parameter c and scale parameter λ. The cdf F_1 of X_1 and pdf f_2 of X_2 are obtained from Equations (7) and (8), respectively. For $s = 1$ and 2, we define

$$w_{i,r}^{(s)} = \frac{(-1)^{i+r+1}\binom{b_s-1}{i}\binom{\alpha_s(a_s+i)}{r}}{(a_s+i)\,B(a_s,b_s)}.$$

Then,

$$\begin{aligned}
R = P(X_1 > X_2) &= \int_0^\infty \int_y^\infty f_1(x)f_2(y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty f_2(y)[1 - F_1(y)]\,\mathrm{d}y \\
&= 1 - \int_0^\infty f_2(y)F_1(y)\,\mathrm{d}y = 1 + \sum_{j,k=0}^\infty w_{j,k}^{(1)} \int_0^\infty f_2(y)[1 - G_{\lambda_k,1,c}(y)]\,\mathrm{d}y \\
&= 1 + \sum_{j,k=0}^\infty w_{j,k}^{(1)} \int_0^\infty f_2(y)\,\mathrm{e}^{-(\lambda_k y)^c}\,\mathrm{d}y.
\end{aligned}$$

The above integral K can be calculated as

$$\begin{aligned}
K &= E\{\exp[-(\lambda_k X_2)^c]\} = E\{\exp[-k(\lambda X_2)^c]\} \\
&= \frac{c\alpha_2\,\lambda^c}{B(a_2,b_2)} \int_0^\infty x^{c-1}\mathrm{e}^{-(k+1)(\lambda x)^c}[1-\mathrm{e}^{-(\lambda x)^c}]^{\alpha_2 a_2-1}\{1-[1-\mathrm{e}^{-(\lambda x)^c}]^{\alpha_2}\}^{b_2-1}\,\mathrm{d}x \\
&= \frac{c\alpha_2\lambda^c}{B(a_2,b_2)} \sum_{m=0}^\infty (-1)^m \binom{b_2-1}{m} \int_0^\infty x^{c-1}\mathrm{e}^{-(k+1)(\lambda x)^c}[1-\exp\{-(\lambda x)^c\}]^{\alpha_2(a_2+m)-1}\,\mathrm{d}x \\
&= \frac{\alpha_2}{(k+2)B(a_2,b_2)} \sum_{m,p=0}^\infty (-1)^{m+p} \binom{b_2-1}{m}\binom{\alpha_2(a_2+m)-1}{p} \\
&= -\frac{1}{(k+2)} \sum_{m,p=0}^\infty [\alpha_2(a_2+m)-p]w_{m,p}^{(2)}
\end{aligned}$$

and thus

$$R = 1 - \left(\sum_{j,k=0}^\infty \frac{w_{j,k}^{(1)}}{k+2}\right)\left(\sum_{m,p=0}^\infty [\alpha_2(a_2+m)-p]\,w_{m,p}^{(2)}\right).$$

10. Entropy

The entropy of a random variable X with density $f(x)$ is a measure of variation of the uncertainty. A large value of entropy indicates the greater uncertainty in the data. The Rényi entropy is defined as

$$I_{\mathrm{R}}(\rho) = \frac{1}{1-\rho}\log\left\{\int f(x)^\rho\,\mathrm{d}x\right\},$$

where $\rho > 0$ and $\rho \neq 1$. For the BEW distribution, the integral in $I_{\mathrm{R}}(\rho)$ can be reduced to

$$\begin{aligned}\int_0^\infty f(x)^\rho\,\mathrm{d}x &= \left(\frac{c\alpha\lambda^c}{B(a,b)}\right)^\rho \int_0^\infty x^{\rho(c-1)}\mathrm{e}^{-\rho(\lambda x)^c}[1-\exp\{-(\lambda x)^c\}]^{\rho(\alpha a-1)}\\ &\quad\times\{1-[1-\exp\{-(\lambda x)^c\}]^\alpha\}^{\rho(b-1)}\,\mathrm{d}x\\ &= \left(\frac{c\alpha\lambda^c}{B(a,b)}\right)^\rho \sum_{j=0}^\infty (-1)^j \binom{\rho(b-1)}{j}\int_0^\infty x^{\rho(c-1)}\exp\{-\rho(\lambda x)^c\}\\ &\quad\times[1-\exp\{-(\lambda x)^c\}]^{\rho(\alpha a-1)+j\alpha}\,\mathrm{d}x\\ &= \left(\frac{c\alpha\lambda^c}{B(a,b)}\right)^\rho \sum_{j,k=0}^\infty (-1)^{j+k}\binom{\rho(b-1)}{j}\binom{\rho(\alpha a-1)+\alpha j}{k}\\ &\quad\times\int_0^\infty x^{\rho(c-1)}\exp\{-(k+\rho)(\lambda x)^c\}\,\mathrm{d}x.\end{aligned}$$

Setting $u = (k+\rho)(\lambda x)^c$, we obtain

$$\begin{aligned}I_{\mathrm{R}}(\rho) &= \frac{\rho}{1-\rho}\log\left\{\frac{\alpha}{\log B(a,b)}\right\} - \log(c\lambda)\\ &\quad+\frac{1}{1-\rho}\log\left\{\Gamma\left(\frac{\rho(c-1)+1}{c}\right)\right.\\ &\quad\left.\times\sum_{j,k=0}^\infty \frac{(-1)^{j+k}}{(k+\rho)^{\rho-(\rho-1)/c}}\binom{\rho(b-1)}{j}\binom{\rho(\alpha a-1)+j\alpha}{k}\right\}.\end{aligned}$$

The Shannon entropy is given by

$$\begin{aligned}E[-\log f(X)] &= -\log(c\alpha\lambda^c) + \log B(a+b) - (c-1)E[\log(X)] + E[(\lambda X)^c]\\ &\quad-(\alpha a-1)E\{\log[1-\mathrm{e}^{-(\lambda X)^c}]\} - (b-1)E\{\log[1-(1-\mathrm{e}^{-(\lambda X)^c})^\alpha]\}.\end{aligned}\tag{23}$$

The expectations in Equation (23) can easily be obtained. Setting $u = (\lambda_k x)^c$, we have

$$\begin{aligned}E[\log(X)] &= \int_0^\infty \log(x)f(x)\,\mathrm{d}x\\ &= \sum_{k=1}^\infty w_{+,k}\int_0^\infty \log(x)g_{\lambda_k,1,c}(x)\,\mathrm{d}x.\end{aligned}$$

The last integral, say T, is given by

$$\begin{aligned}T &= \frac{1}{c}\int_0^\infty \log(u)\mathrm{e}^{-u}\,\mathrm{d}u - \log(\lambda_k)\int_0^\infty \mathrm{e}^{-u}\,\mathrm{d}u\\ &= \frac{1}{c}\psi(1) - \log(\lambda_k) = -\left[\frac{\gamma}{c} + \log(\lambda_k)\right],\end{aligned}$$

where $\psi(z) = \mathrm{d}\log[\Gamma(z)]/\mathrm{d}z$ is the digamma function and γ is Euler's constant. Since $\lambda_k = k^{1/c}\lambda$, we obtain

$$E[\log(X)] = -\frac{\gamma}{c} - \frac{1}{c}\sum_{k=1}^\infty w_{+,k}\,\log(k) - \log(\lambda).$$

Further,

$$E[(\lambda X)^c] = \sum_{k=1}^{\infty} w_{+,k} \int_0^{\infty} (\lambda x)^c g_{\lambda_k,1,c}(x)\,\mathrm{d}x = \lambda^c \sum_{k=1}^{\infty} w_{+,k} \int_0^{\infty} x^c g_{\lambda_k,1,c}(x)\,\mathrm{d}x$$
$$= \sum_{k=1}^{\infty} k^{-1} w_{+,k}.$$

In the last expression in Equation (23) are obtained from Equations (24) and (25) given below. Hence,

$$E\{-\log[f(X)]\} = -\log(c\,\alpha\,\lambda^c) + \log[B(a+b)]$$
$$+ (c-1)\left[\frac{\gamma}{c} + \frac{1}{c}\sum_{k=1}^{\infty} w_{+,k}\log(k) + \log(\lambda)\right]$$
$$+ \sum_{k=1}^{\infty} k^{-1} w_{+,k} - \frac{(\alpha a - 1)}{\alpha}[\psi(a) - \psi(a+b)] - (b-1)[\psi(b) - \psi(a+b)].$$

11. Estimation

Let $\boldsymbol{\theta} = (\alpha, \lambda, a, b, c)^{\mathrm{T}}$ be the parameter vector of the BEW distribution (5). We consider the method of maximum likelihood to estimate $\boldsymbol{\theta}$. The log-likelihood function for the five parameters from a single observation $x > 0$, say $\ell(\alpha, \lambda, a, b, c)$, is

$$\ell(\alpha, \lambda, a, b, c) = \log(\alpha) + \log(c) + c\log(\lambda) - \log[B(a,b)] + (c-1)\log(x)$$
$$- (\lambda x)^c + (\alpha a - 1)\log\{1 - \exp[-(\lambda x)^c]\}$$
$$+ (b-1)\log\{1 - [1 - \exp\{-(\lambda x)^c\}]^{\alpha}\}.$$

The components of the unit score vector $U = (\partial\ell/\partial\alpha, \partial\ell/\partial\lambda, \partial\ell/\partial a, \partial\ell/\partial b, \partial\ell/\partial c)^{\mathrm{T}}$ are

$$\frac{\partial\ell}{\partial\alpha} = \frac{1}{\alpha} + a\,\log\{1 - \exp[-(\lambda x)^c]\} - (b-1)\frac{\{1 - \exp[-(\lambda x)^c]\}^{\alpha}\log\{1 - \exp[-(\lambda x)^c]\}}{1 - \{1 - \exp[-(\lambda x)^c]\}^{\alpha}},$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{c}{\lambda} - c\lambda^{c-1}x^c + (\alpha a - 1)\frac{c x^c \lambda^{c-1}\exp[-(\lambda x)^c]}{1 - \exp[-(\lambda x)^c]}$$
$$- (b-1)\frac{\alpha\{1 - \exp[-(\lambda x)^c]\}^{\alpha-1} x^c c \lambda^{c-1}\exp[-(\lambda x)^c]}{1 - \{1 - \exp[-(\lambda x)^c]\}^{\alpha}},$$

$$\frac{\partial\ell}{\partial a} = \psi(a+b) - \psi(a) + \alpha\log\{1 - \exp[-(\lambda x)^c]\},$$

$$\frac{\partial\ell}{\partial b} = \psi(a+b) - \psi(b) + \log(1 - \{1 - \exp[-(\lambda x)^c]\}^{\alpha}),$$

$$\frac{\partial\ell}{\partial c} = \frac{1}{c} + \log(\lambda) + \log(x) - (\lambda x)^c\log(\lambda x) + (\alpha a - 1)\frac{(\lambda x)^c\log(\lambda x)\exp\{-(\lambda x)^c\}}{1 - \exp(-(\lambda x)^c)}$$
$$- (b-1)\alpha\frac{\{1 - \exp[-(\lambda x)^c]\}^{\alpha-1}(\lambda x)^c\,\log(\lambda x)\exp(-(\lambda x)^c)}{1 - [1 - \exp\{-(\lambda x)^c\}]^{\alpha}}.$$

The expected value of the score vanishes and then

$$E\{\log[1-\mathrm{e}^{-(\lambda X)^c}]\} = \frac{\psi(a)-\psi(a+b)}{\alpha}, \tag{24}$$

$$E\{\log[1-(1-\mathrm{e}^{-(\lambda X)^c})^{\alpha}]\} = \psi(b)-\psi(a+b),$$

$$E\left\{\frac{[1-\mathrm{e}^{-(\lambda X)^c}]^{\alpha}\log[1-\mathrm{e}^{-(\lambda X)^c}]}{1-[1-\mathrm{e}^{-(\lambda X)^c}]^{\alpha}}\right\} = \frac{1+a\,[\psi(a)-\psi(a+b)]}{\alpha\,(b-1)}. \tag{25}$$

For a random sample $(x_1,\ldots,x_n)$ of size n from X, the total log-likelihood is $\ell_n = \ell_n(\alpha,\lambda,a,b,c) = \sum_{i=1}^{n}\ell^{(i)}$, where $\ell^{(i)}$ is the log-likelihood for the ith observation ($i=1,\ldots,n$). The total score function is $U_n = \sum_{i=1}^{n} U^{(i)}$, where $U^{(i)}$ has the form given before for $i=1,\ldots,n$. The maximum likelihood estimate (MLE) $\hat{\theta}$ of θ is obtained numerically from the nonlinear equations $U_n = 0$.

We maximize the log-likelihood using the nlm function in R [25]. The nlm function was executed for a wide range of initial values. This procedure usually leads to more than one maximum. In these cases, we take the MLEs corresponding to the largest value of the maxima. For a few initial values, no maximum was identified. In those cases, a new initial value was tried to obtain a maximum.

The existence and uniqueness of the MLEs is also of theoretical interest. This issue has been studied by many authors for different distributions, see, for example, Xia *et al.* [26], Zhou [27], Santos Silva and Tenreyro [28] and Seregin [29]. We hope to study this problem for the BEW distribution in a future research.

For interval estimation and tests of hypotheses on the parameters in θ, we require the expected unit information matrix

$$K = K(\theta) = \begin{pmatrix} \kappa_{\alpha,\alpha} & \kappa_{\alpha,\lambda} & \kappa_{\alpha,a} & \kappa_{\alpha,b} & \kappa_{\alpha,c} \\ \kappa_{\lambda,\alpha} & \kappa_{\lambda,\lambda} & \kappa_{\lambda,a} & \kappa_{\lambda,b} & \kappa_{\lambda,c} \\ \kappa_{a,\alpha} & \kappa_{a,\lambda} & \kappa_{a,a} & \kappa_{a,b} & \kappa_{a,c} \\ \kappa_{b,\alpha} & \kappa_{b,\lambda} & \kappa_{b,a} & \kappa_{b,b} & \kappa_{b,c} \\ \kappa_{c,\alpha} & \kappa_{c,\lambda} & \kappa_{c,a} & \kappa_{c,b} & \kappa_{c,c} \end{pmatrix},$$

the elements of which are given in Appendix 2.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})$ is $N_5(0, K(\boldsymbol{\theta})^{-1})$. The asymptotic multivariate normal $N_5(0, K(\boldsymbol{\theta})^{-1})$ distribution of $\hat{\boldsymbol{\theta}}$ can be used to construct approximate confidence intervals and confidence regions for the parameters and for the hazard and survival functions. An asymptotic confidence interval with significance level γ for each parameter θ_r is given by

$$\mathrm{ACI}(\theta_r, 100(1-\gamma)\%) = (\hat{\theta}_r - z_{\gamma/2}\sqrt{\kappa^{\theta_r,\theta_r}}, \hat{\theta}_r + z_{\gamma/2}\sqrt{\kappa^{\theta_r,\theta_r}}),$$

where $\kappa^{\theta_r,\theta_r}$ is the rth diagonal element of $K_n(\theta)^{-1} = [nK(\boldsymbol{\theta})]^{-1}$ for $r=1,\ldots,5$ and $z_{\gamma/2}$ is the quantile $1-\gamma/2$ of the standard normal distribution.

The likelihood ratio (LR) statistic can be used for comparing the BEW distribution with some of its special sub-models. Considering the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^{\mathrm{T}}, \boldsymbol{\theta}_2^{\mathrm{T}})^{\mathrm{T}}$, tests of hypotheses of the type $H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ versus $H_0 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$ can be performed using LR statistics given by $w = 2\{\ell(\hat{\boldsymbol{\theta}}) - \ell(\tilde{\boldsymbol{\theta}})\}$, where $\hat{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}$ are the MLEs of $\boldsymbol{\theta}$ under H_A and H_0, respectively. Under the null hypothesis, $w \xrightarrow{d} \chi^2_q$, where q is the dimension of the vector $\boldsymbol{\theta}_1$ of interest. The LR test rejects H_0 if $w > \xi_\gamma$, where ξ_γ denotes the upper $100\gamma\%$ point of the χ^2_q distribution. For example, we can verify whether the fit using the BEW distribution is statistically 'superior' to a fit using the BW distribution (for a given data) by testing $H_0 : \alpha = 1$ versus $H_0 : \alpha \neq 1$.

12. The log-beta exponentiated Weibull regression model

If X is a random variable having the BEW density function (5), then $Y = \log(X)$ has a LBEW distribution. The density function of Y, parameterized in terms of $\sigma = c^{-1}$ and $\mu = -\log(\lambda)$, can be expressed as

$$
\begin{aligned}
&f(y; \alpha, \mu, \sigma, a, b) \\
&\quad = \frac{\alpha}{\sigma B(a,b)} \exp\left\{\left(\frac{y-\mu}{\sigma}\right) - \exp\left(\frac{y-\mu}{\sigma}\right)\right\}\left\{1 - \exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{\alpha a-1} \\
&\qquad \times \left\{1 - \left(1 - \exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right)^{\alpha}\right\}^{b-1},
\end{aligned} \tag{26}
$$

where $-\infty < y < \infty$, $\sigma > 0$ and $-\infty < \mu < \infty$. Plots of the density function (26) for selected parameter values are given in Figures 7 and 8. These plots show great flexibility for different values of the shape parameters α, a and b. If Y is a random variable having density function (26), then we write $Y \sim \text{LBEW}(\alpha, \mu, \sigma, a, b)$.

Thus,

$$\text{if } X \sim \text{BEW}(\alpha, \lambda, a, b, c) \quad \text{then } Y = \log(X) \sim \text{LBEW}(\alpha, \mu, \sigma, a, b).$$

The survival function corresponding to Equation (26) is

$$S(y) = 1 - \frac{1}{B(a,b)} \int_0^{\{1-\exp[-\exp(y-\mu/\sigma)]\}^{\alpha}} w^{a-1}(1-w)^{b-1} = 1 - I_{\{1-\exp[-\exp(y-\mu/\sigma)]\}^{\alpha}}(a,b).$$

We define the standardized random variable $Z = (Y - \mu)/\sigma$ with density function

$$
\begin{aligned}
\pi(z; a, b) = {}& \frac{\alpha}{B(a,b)} \exp[z - \exp(z)]\{1 - \exp[-\exp(z)]\}^{\alpha a-1} \\
& \times \{1 - (1 - \exp[-\exp(z)])^{\alpha}\}^{b-1}, \quad -\infty < z < \infty.
\end{aligned} \tag{27}
$$

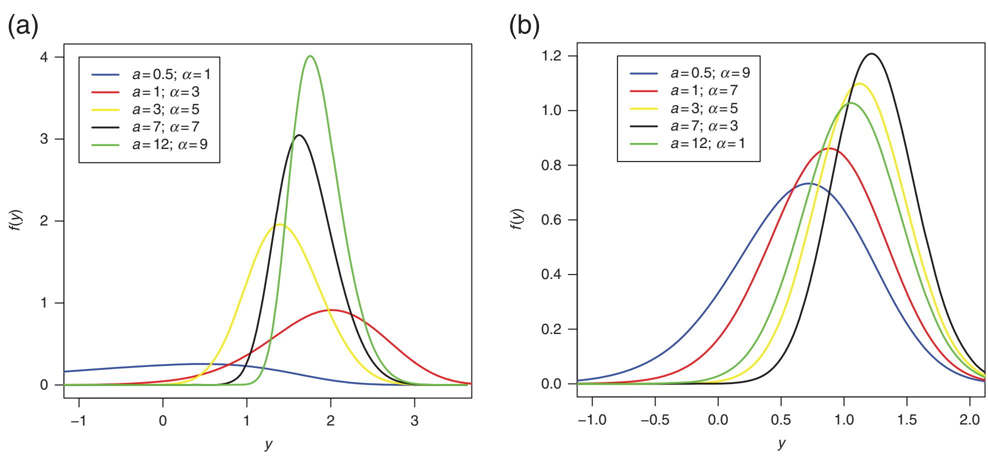

Figure 7. Plots of the LBEW densities: (a) a and α increasing, $b = 0.5$, $\mu = 0$ and $\sigma = 1$ and (b) a increasing and α decreasing, $b = 0.5$, $\mu = 0$ and $\sigma = 1$.

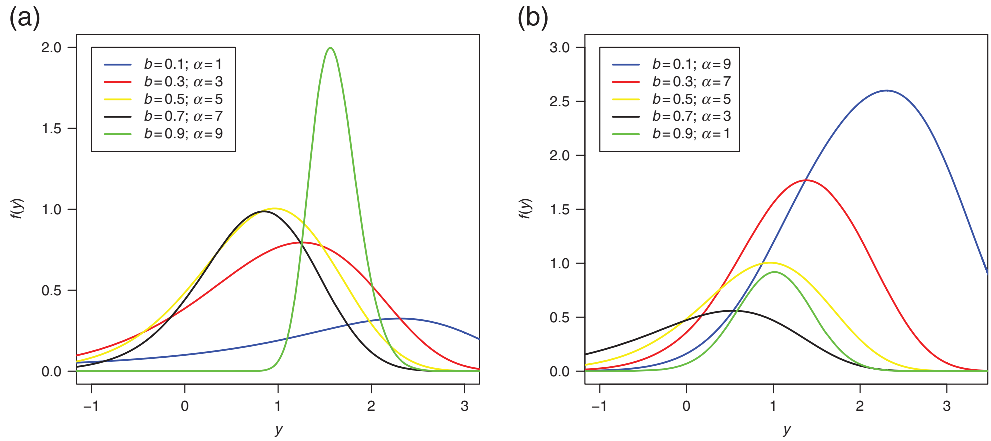

Figure 8. Plots of the LBEW densities: (a) b and α increasing, $a = 0.5$, $\mu = 0$ and $\sigma = 1$ and (b) b increasing and α decreasing, $a = 0.5$, $\mu = 0$ and $\sigma = 1$.

The special case $a = b = 1$ gives the log-exponentiated Weibull (LEW) distribution. If $\alpha = 1$, then we have the log-beta Weibull (LBW) distribution. The special case $\alpha = a = b = 1$ reduces to the log-Weibull (LW) or extreme-value distribution.

The kth ordinary moment of the standardized distribution (27) is

$$\mu'_k = E(Z^k) = \frac{\alpha}{B(a,b)} \times \int_{-\infty}^{\infty} z^k \exp[z - \exp(z)] \\ \times \{1 - \exp[-\exp(z)]\}^{\alpha a-1}\{1 - (1 - \exp[-\exp(z)])^{\alpha}\}^{b-1}\, dz.$$

By expanding

$$G(x)^{a-1}[1 - G(x)^a]^{b-1} = \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} G(x)^{a(i+1)-1}$$

and setting $w = e^z$, we obtain

$$\mu'_k = \frac{\alpha}{B(a,b)} \sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} \int_0^{\infty} \log^k(w) e^{-w}[1 - e^{-w}]^{a(\alpha+i)-1}\, dw. \tag{28}$$

By expanding the binomial in power series, we obtain

$$\mu'_k = \frac{\alpha}{B(a,b)} \sum_{i,j=0}^{\infty}(-1)^{i+j} \binom{b-1}{i} \binom{a(\alpha+j)-1}{j} \int_0^{\infty} \log^k(w) \exp[-(j+1)w]\, dw. \tag{29}$$

By Equation (2.6.21.1) in Prudnikov *et al.* [22, volume 1], this integral can be calculated as

$$I(k,j) = \left(\frac{\partial}{\partial p}\right)^k [(j+1)^{-p}\Gamma(p)]\Big|_{p=1}$$

and then

$$\mu'_k = \frac{1}{B(a,b)} \sum_{i,j=0}^{\infty}(-1)^{i+j} \binom{b-1}{i} \binom{a(\alpha+j)-1}{j} I(k,j). \tag{30}$$

Equation (30) gives the moments of the LBEW distribution.

In many practical applications, the lifetimes x_i are affected by explanatory variables such as the cholesterol level, blood pressure and many others. Let $\mathbf{v}_i = (v_{i1}, \ldots, v_{ip})^{\mathrm{T}}$ be the explanatory variable vector associated with the ith response variable y_i for $i = 1, \ldots, n$. Consider a sample $(y_1, \mathbf{v}_1), \ldots, (y_n, \mathbf{v}_n)$ of n independent observations, where each random response is defined by $y_i = \min\{\log(x_i), \log(c_i)\}$, and $\log(x_i)$ and $\log(c_i)$ are the log-lifetime and log-censoring, respectively. We consider non-informative censoring such that the observed lifetimes and censoring times are independent.

For the first time, we construct a linear regression model for the response variable y_i based on the LBEW distribution given by

$$y_i = \mathbf{v}_i^{\mathrm{T}}\boldsymbol{\beta} + \sigma z_i, \quad i = 1, \ldots, n, \tag{31}$$

where the random error z_i has the density function (27), $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^{\mathrm{T}}$, $\sigma > 0$, $a > 0$ and $b > 0$ are unknown scalar parameters and $\mathbf{v}_i$ is the vector of explanatory variables modelling the location parameter $\mu_i = \mathbf{v}_i^{\mathrm{T}}\boldsymbol{\beta}$. Hence, the location parameter vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^{\mathrm{T}}$ of the LBEW model has a linear structure $\boldsymbol{\mu} = \mathbf{v}\boldsymbol{\beta}$, where $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)^{\mathrm{T}}$ is a known model matrix. The LW (or the extreme value) regression model is defined by Equation (31) with $a = b = 1$.

Let F and C be the sets of individuals for which y_i is the log-lifetime or log-censoring, respectively. The total log-likelihood function for the model parameters $\boldsymbol{\theta} = (\alpha, a, b, \sigma, \boldsymbol{\beta}^{\mathrm{T}})^{\mathrm{T}}$ can be obtained from Equations (27) and (31) as

$$\begin{aligned} l(\boldsymbol{\theta}) &= q\{\log(\alpha) - \log[\sigma B(a,b)]\} + \sum_{i \in F}\left\{\left(\frac{y_i - \mathbf{v}_i^{\mathrm{T}}\boldsymbol{\beta}}{\sigma}\right) - \exp\left(\frac{y_i - \mathbf{v}_i^{\mathrm{T}}\boldsymbol{\beta}}{\sigma}\right)\right\} \\ &\quad + (a\alpha - 1)\sum_{i \in F}\log\left\{1 - \exp\left[-\exp\left(\frac{y_i - \mathbf{v}_i^{\mathrm{T}}\boldsymbol{\beta}}{\sigma}\right)\right]\right\} \\ &\quad + (b - 1)\sum_{i \in F}\log\left\{1 - \left(1 - \exp\left[-\exp\left(\frac{y_i - \mathbf{v}_i^{\mathrm{T}}\boldsymbol{\beta}}{\sigma}\right)\right]\right)^{\alpha}\right\} \\ &\quad + \sum_{i \in C}\log\left\{1 - I_{\{1-\exp[-\exp(y_i - \mathbf{v}_i^{\mathrm{T}}\boldsymbol{\beta}/\sigma)]\}^{\alpha}}(a,b)\right\}, \end{aligned} \tag{32}$$

where q is the observed number of failures. The MLE $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ can be obtained by maximizing the log-likelihood function (32). From the fitted model (31), the survival function for y_i can be estimated by

$$\hat{S}(y_i; \hat{\alpha}, \hat{a}, \hat{b}, \hat{\sigma}, \hat{\boldsymbol{\beta}}^{\mathrm{T}}) = 1 - I_{\{1-\exp[-\exp(y_i - \mathbf{v}_i^{\mathrm{T}}\hat{\boldsymbol{\beta}}/\hat{\sigma})]\}^{\hat{\alpha}}}(\hat{a}, \hat{b}). \tag{33}$$

Under general regularity conditions, the asymptotic distribution of $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is multivariate normal $N_{p+4}(0, K(\boldsymbol{\theta})^{-1})$, where $K(\boldsymbol{\theta})$ is the expected information matrix. The asymptotic covariance matrix $K(\boldsymbol{\theta})^{-1}$ of $\hat{\boldsymbol{\theta}}$ can be approximated by the inverse of the $(p+4) \times (p+4)$ observed information matrix $J(\boldsymbol{\theta})$ and then the inference on the parameter vector $\boldsymbol{\theta}$ can be based on the normal approximation $N_{p+4}(0, J(\boldsymbol{\theta})^{-1})$ for $\hat{\boldsymbol{\theta}}$. This multivariate normal $N_{p+4}(0, J(\boldsymbol{\theta})^{-1})$ distribution can be used to construct approximate confidence regions for some parameters in $\boldsymbol{\theta}$ and for the hazard and survival functions. In fact, an $100(1-\alpha)\%$ asymptotic confidence interval for each parameter θ_r is given by

$$\mathrm{ACI}_r = \left(\hat{\theta}_r - z_{\alpha/2}\sqrt{-\hat{J}^{r,r}}, \hat{\theta}_r + z_{\alpha/2}\sqrt{-\hat{J}^{r,r}}\right),$$

where $-\hat{J}^{r,r}$ represents the rth diagonal element of the inverse of the estimated observed information matrix $J(\hat{\boldsymbol{\theta}})^{-1}$ and $z_{\alpha/2}$ is the quantile $1-\alpha/2$ of the standard normal distribution. The LR statistic can be used to discriminate between the LEW and LBEW regression models since they are nested models. In this case, the hypotheses to be tested are $H_0 : (a, b)^{\mathrm{T}} = (1, 1)^{\mathrm{T}}$ versus $H_1 : H_0$ *is not true*, and the LR statistic reduces to $w = 2\{l(\hat{\boldsymbol{\theta}}) - l(\tilde{\boldsymbol{\theta}})\}$, where $\tilde{\boldsymbol{\theta}}$ is the MLE of $\boldsymbol{\theta}$ under H_0. The null hypothesis is rejected if $w > \chi^2_{1-\alpha}(2)$, where $\chi^2_{1-\alpha}(2)$ is the quantile of the chi-square distribution with two degrees of freedom.

13. Applications

In this section, we illustrate the usefulness of the BEW distribution applied to two real data sets.

13.1. *Cigarettes data*

First, we work with carbon monoxide (CO) measurements made in several brands of cigarettes in 1998. The data have been collected by the Federal Trade Commission (FTC), an independent agency of the United States government, whose main mission is the promotion of consumer protection.

For three decades, the FTC has regularly released reports on the nicotine and tar content of cigarettes. The reports show that nicotine levels, on average, had remained stable since 1980, after falling in the preceding decade. The report entitled 'Tar, Nicotine, and Carbon Monoxide of the Smoke of 1206 Varieties of Domestic Cigarettes for the year of 1998' at http://www.ftc.gov/reports/tobacco includes the data sets and some information about the source of the data, smoker's behaviour and beliefs about nicotine, tar and CO contents in cigarettes.

The CO data set can be found at http://home.att.net/ rdavis2/cigra.html. The data include $n = 384$ records of measurements of CO content, in milligrams, in cigarettes of several brands. Some summary statistics for the CO data are: mean $= 11.34$, median $= 12.00$, minimum $= 0.05$ and maximum $= 22.00$.

We fitted the BEW, BEE and BE distributions to these data by the method of maximum likelihood. The MLEs of the parameters (with their standard errors) and the Akaike information criterion (AIC) for the fitted models are displayed in Table 2.

The LR statistics for testing the hypotheses H_0 : BEE $\times$ H_a : BEW and H_0 : BE $\times$ H_a : BEW are 31.75273 (p-value $= 1.7510339\text{e} - 08$) and 166.1401 ($p = 0$), respectively. So, we reject the null hypotheses in both cases in favour of the BEW distribution. The plots of the fitted BEW, BEE and BE densities are shown in Figure 9. They indicate that the proposed distribution provides a better fit than the other two sub-models. The required numerical evaluations were implemented by using an R program (sub-routine `nlminb` can be found at http://cran.r-project.org).

Table 2. MLEs of the model parameters for the cigarretes data, the corresponding SEs (given in parentheses) and the AIC measures.

Model	a	b	λ	α	c	AIC
BEW	0.2086	54.0078	0.0414	3.3583	2.9977	1950.53
	(0.0042)	(2.4282)	(0.0005)	(0.1398)	(0.0731)	
BEE	0.2032	75.2096	0.0772	14.4885	1	1980.28
	(0.0025)	(1.4950)	(0.0007)	(0.1997)	(–)	
BE	4.3680	3.1425	0.0847	1	1	2112.67
	(0.0182)	(0.0365)	(0.0008)	(–)	(–)	

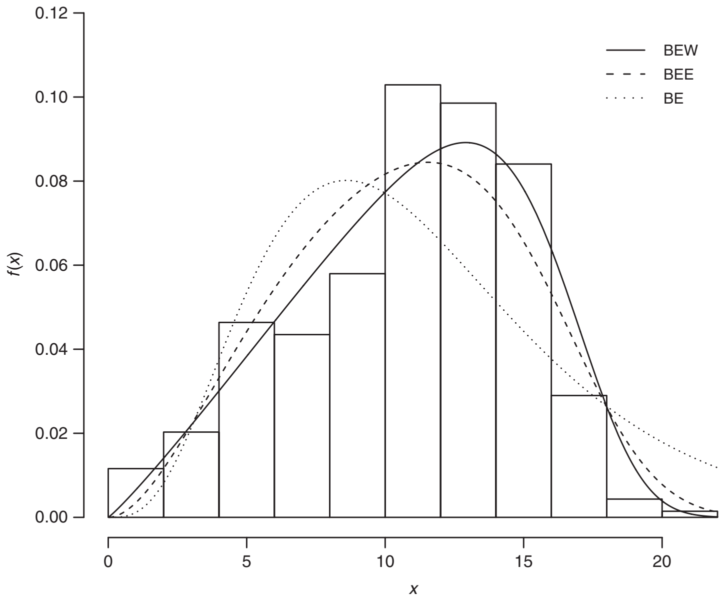

Figure 9. Fitted BEW, BEE and BE densities for CO contents in cigarettes of different brands. Source: FTC (2000).

13.2. *Class-H insulation data*

As an application of the LBEW regression model, we consider the data set given in Nelson [30, p. 115], concerning 'hours to failure of motorettes with a new Class-H insulation'. An experiment has been designed in order to evaluate the effect of temperature on the failure time. Four test temperatures were considered: 190, 220, 240 and 260°C, and 10 motorettes were randomly assigned to each test temperature. The motorettes were periodically examined for insulation failure. The failure time (in hours) of observation i, t_i, was defined as the midway between the inspection time when the failure was found and the time of the previous inspection, and x_{i1} is the temperature (for other details, see [30]).

We adopt the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \sigma z_i,$$

where the random variable $y_i = \log(t_i)$ follows the LBEW distribution (26) for $i = 1, \ldots, 40$.

The MLEs of the model parameters are calculated using the procedure NLMixed in SAS. The initial values for the parameters $\boldsymbol{\beta}$ and σ in the iterative algorithm were taken as the estimated values obtained from the fitted LW regression model ($\alpha = a = b = 1$). The MLEs of the parameters and the maximized log-likelihood for the fitted models are listed in Table 3.

These results indicate that the LBEW model has the lowest AIC value among those of the fitted models. The values of these statistics indicate that LBEW model provides the best fit for the data. Further, we note from the fitted LBEW regression model that x_1 is significant at 1% and that there is a significant difference between the temperatures levels 190, 220, 240 and 260 for the failure times. A graphical comparison of the fitted LBEW, LBW and LW models (see Figure 10) indicates that the model LBEW provides a superior fit. The curves displayed in Figure 10 represent the empirical survival function and the estimated survival functions obtained from Equation (33).

Table 3. MLEs of the parameters from some fitted regression models to the class-H insulation life data set, the corresponding SEs (given in parentheses), p-value in [.] and the AIC measure.

Model	α	σ	a	b	β_0	β_1	AIC
LBEW	70.6776 (0.0431)	2.2205 (0.1902)	70.7983 (0.0429)	1.2809 (0.5179)	10.2313 (2.3136) [<0.0001]	−0.0317 (0.0018) [<0.0001]	16.9
LBW	1 (−)	0.7738 (0.3033)	13.0544 (4.3069)	1.2110 (0.4013)	13.8269 (0.5974) [<0.0001]	−0.0298 (0.0021) [<0.0001]	18.7
LW	1 (−)	0.2720 (0.0315)	1 (−)	1 (−)	14.3024 (0.3176) [0]	−0.0279 (0.0014) [0]	22.4

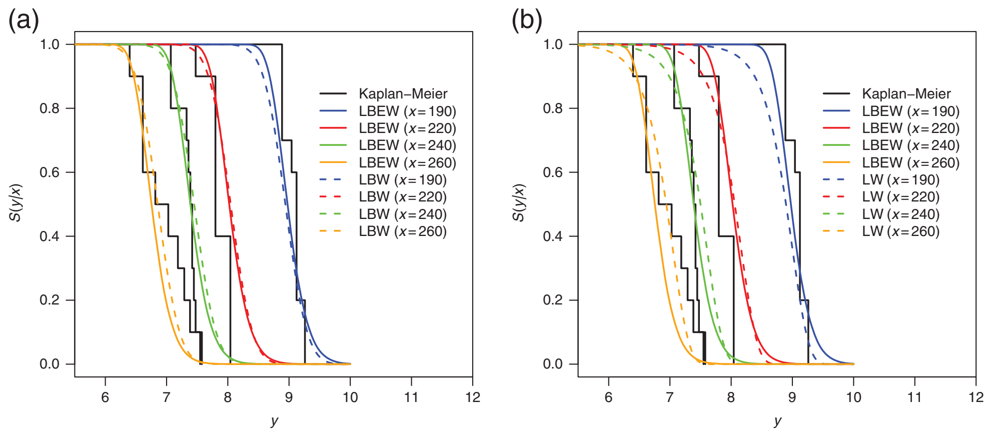

Figure 10. Estimated survival functions and the empirical survival for class-H insulation life data. (a) LBEW versus LBW regression model. (b) LBEW versus LW regression model.

14. Conclusion

We study general mathematical properties of a new model called the BEW distribution because of the wide usage of the Weibull distribution and the fact that the current generalization provides means of its continuous extension to still more complex situations. The new distribution represents a generalization of several models previously considered in the literature such as the Weibull, EW [4–6,15], EE [16], BE [2], BW [7] and beta-generalized exponential [13] distributions. This generalization provides a continuous crossover towards distributions with different shapes (e.g. skewness and kurtosis). The new distribution is quite flexible to analyse positive data and is an important alternative model to the sub-models mentioned before.

The BEW density can be represented as a linear combination of Weibull densities which allow us to derive several of its structural properties. The properties studied include ordinary moments, generating and quantile functions, mean deviations about the mean and about the median, Bonferroni and Lorenz curves, Rényi entropy, Shannon entropy, moments of order statistics, L moments and reliability. The density of the BEW order statistics can also be expressed as a linear combination of Weibull densities. The estimation of parameters is approached by the method of maximum likelihood. The expected information matrix is derived. Further, we define a LBEW distribution and derive an expansion for its moments. Based on this new distribution, we propose a LBEW regression model as an alternative to model lifetime censored data when the hazard rate function presents bathtub, unimodal, increasing and decreasing shapes. The usefulness of the BEW distribution is illustrated in two applications to real data sets. In conclusion, we introduce a rather general and flexible lifetime model, which provides a rather flexible mechanism for fitting a wide spectrum of real world data sets.

Note

1. http://functions.wolfram.com/06.23.06.0004.01.

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Appendix 1

LEMMA *Let X be a random variable with density function* (5). *Then,*

$$
\begin{aligned}
&E[(\lambda X)^j\{\log(\lambda X)\}^k\exp\{-l(\lambda X)^c\}]\\
&=\frac{\alpha}{c^kB(a,b)}\sum_{m,r=0}^{\infty}\sum_{p=0}^{k}(-1)^{m+r+k-p}\binom{b-1}{m}\binom{\alpha(a+m)-1}{r}\\
&\quad\times\binom{k}{p}\frac{\{\log(l+r+1)\}^{k-p}}{(l+r+1)^{j/c+1}}\Gamma^{(p)}\left(\frac{j}{c}+1\right)\\
&=\frac{1}{c^k}\sum_{m,r=0}^{\infty}\frac{w_{m,r}[\alpha(a+m)-r]}{(l+r+1)^{j/c+1}}\sum_{p=0}^{k}(-1)^{k-p+1}\binom{k}{p}\{\log(l+r+1)\}^{k-p}\Gamma^{(p)}\left(\frac{j}{c}+1\right)
\end{aligned}
$$

for positive integers j, k and l, where $\Gamma^{(p)}(\cdot)$ *denotes the pth derivative of the gamma function. In particular,*

$$
\begin{aligned}
E[(\lambda X)^j]&=\frac{\alpha}{B(a,b)}\sum_{m,r=0}^{\infty}(-1)^{m+r}\binom{b-1}{m}\binom{\alpha(a+m)-1}{r}\frac{1}{(r+1)^{j/c+1}}\Gamma\left(\frac{j}{c}+1\right)\\
&=\sum_{m,r=0}^{\infty}\frac{w_{m,r}\{\alpha(a+m)-r\}}{(r+1)^{j/c+1}},\\
E[\{\log(\lambda X)\}^k]&=\frac{\alpha}{c^kB(a,b)}\sum_{m,r=0}^{\infty}\sum_{p=0}^{k}(-1)^{m+r+k-p}\binom{b-1}{m}\binom{\alpha(a+m)-1}{r}\binom{k}{p}\\
&\quad\times\frac{\{\log(r+1)\}^{k-p}}{r+1}\Gamma^{(p)}(1)\\
&=\frac{1}{c^k}\sum_{m,r=0}^{\infty}\frac{w_{m,r}(\alpha(a+m)-r)}{r+1}\sum_{p=0}^{k}(-1)^{k-p+1}\binom{k}{p}\{\log(r+1)\}^{k-p}\Gamma^{(p)}(1)
\end{aligned}
$$

and

$$
\begin{aligned}
E[\exp\{-l(\lambda X)^c\}]&=\frac{\alpha}{B(a,b)}\sum_{m,r=0}^{\infty}\frac{(-1)^{m+r}}{(l+r+1)}\binom{b-1}{m}\binom{\alpha(a+m)-1}{r}\\
&=\sum_{m,r=0}^{\infty}\frac{w_{m,r}\{\alpha(a+m)-r\}}{(l+r+1)}.
\end{aligned}
$$

Proof First, using the binomial expansion for $\{1-[1-\exp\{-(\lambda x)^c\}]^\alpha\}^{b-1}$, and then for $[1-\exp\{-(\lambda x)^c\}]^{\alpha(a+m)-1}$, and setting $y=(l+r+1)(\lambda x)^c$, we obtain

$$
\begin{aligned}
&E[(\lambda X)^j\{\log(\lambda X)\}^k\exp\{-l(\lambda X)^c\}]\\
&=\frac{c\alpha\lambda^{c+j}}{B(a,b)}\sum_{m=0}^{\infty}(-1)^m\binom{b-1}{m}\int_0^\infty x^{j+c-1}\{\log(\lambda x)\}^k\exp\{-(l+1)(\lambda x)^c\},\\
&\quad\times[1-\exp\{-(\lambda x)^c\}]^{\alpha a-1}[1-\exp\{-(\lambda x)^c\}]^{m\alpha}\,\mathrm{d}x\\
&=\frac{c\alpha\lambda^{c+j}}{B(a,b)}\sum_{m,r=0}^{\infty}(-1)^{m+r}\binom{b-1}{m}\binom{\alpha(a+m)-1}{r}\\
&\quad\times\int_0^\infty x^{j+c-1}\{\log(\lambda x)\}^k\exp\{-(l+r+1)(\lambda x)^c\}\,\mathrm{d}x
\end{aligned}
$$

$$= \frac{\alpha}{c^k B(a,b)} \sum_{m,r=0}^{\infty} (-1)^{m+r} \binom{b-1}{m} \binom{\alpha(a+m)-1}{r}$$
$$\times \int_0^{\infty} \frac{y^{j/c}}{(l+r+1)^{j/c+1}} \{\log y - \log(l+r+1)\}^k \mathrm{e}^{-y}\, \mathrm{d}y$$
$$= \frac{\alpha}{c^k\, B(a,b)} \sum_{m,r=0}^{\infty} \sum_{p=0}^{k} \frac{(-1)^{m+r+k-p}}{(l+r+1)^{j/c+1}} \binom{b-1}{m} \binom{\alpha(a+m)-1}{r}$$
$$\times \binom{k}{p} \{\log(l+r+1)\}^{k-p} \Gamma^{(p)}\left(\frac{j}{c}+1\right).$$

The result then follows. ∎

Appendix 2

The elements of the 5×5 unit expected information matrix are

$$\kappa_{\alpha,\alpha} = \frac{1}{\alpha^2}\{1+(b-1)T_{2,0,0,0,2,0,1}\}, \quad \kappa_{\alpha,a} = \frac{\psi(a+b)-\psi(a)}{\alpha}, \quad \kappa_{\alpha,b} = \frac{1+a[\psi(a)-\psi(a+b)]}{\alpha(b-1)},$$
$$\kappa_{\alpha,\lambda} = \frac{c}{\lambda}\{a\,T_{0,1,1,1,0,0,0} - (b-1)(T_{1,1,1,1,1,0,1} + T_{1,1,1,1,0,0,1} + T_{2,1,1,1,1,0,2})\},$$
$$\kappa_{\alpha,c} = \frac{1}{c}\{aT_{0,1,1,1,0,1,0} - (b-1)[T_{1,1,1,1,0,1,1} + T_{2,1,1,1,1,1,1}]\},$$
$$\kappa_{\lambda,\lambda} = -\frac{c}{\lambda^2}\{1+(c-1)\,T_{0,0,0,1,0,0,0}\} + \frac{(\alpha a-1)c}{\lambda^2}\{(c-1)T_{0,1,1,1,0,0,0} + cT_{0,1,2,2,0,0,0}\}$$
$$+ \frac{(b-1)\alpha c}{\lambda^2}\{cT_{1,1,2,1,0,0,1} - T_{1,1,1,1,0,0,1} - cT_{1,1,2,2,0,0,1} + \alpha c T_{2,2,2,2,0,0,2}\},$$
$$\kappa_{\lambda,a} = \frac{\alpha c}{\lambda} T_{0,1,1,1,0,0,0}, \quad \kappa_{\lambda,b} = -\frac{\alpha c}{\lambda} T_{1,1,1,1,0,0,1}, \quad \kappa_{b,b} = \psi'(b) - \psi'(a+b),$$
$$\kappa_{\lambda,c} = -\frac{1}{\lambda}\{1 + T_{0,0,0,1,0,1,0} + T_{0,0,0,1,0,0,0} - (\alpha\, a-1)[T_{0,1,1,1,0,0,0} + T_{0,1,1,1,0,1,0} + T_{0,1,1,2,0,1,0} + T_{0,2,2,2,0,1,0}]$$
$$- (b-1)\alpha[(\alpha-1)T_{1,2,2,2,0,1,1} - T_{1,1,1,1,0,1,1} - T_{1,1,1,2,0,1,1} - T_{1,1,1,1,0,0,1} + \alpha T_{2,2,2,2,0,1,2}]\},$$
$$\kappa_{a,a} = \psi'(a) - \psi'(a+b), \quad \kappa_{a,b} = -\psi'(a+b), \quad \kappa_{a,c} = \frac{\alpha}{c}\, T_{0,1,1,1,0,1,0}, \quad \kappa_{b,c} = -\frac{\alpha}{c} T_{1,1,1,1,0,1,1}$$

and

$$\kappa_{c,c} = \frac{1}{c^2}\{1 - T_{0,0,0,1,0,2,0} + (\alpha\, a-1)[T_{0,1,1,1,0,2,0} + T_{0,1,2,2,0,2,0}]$$
$$+ (b-1)\alpha[\alpha T_{1,2,2,2,0,2,1} - T_{1,1,1,1,0,2,1} - T_{1,1,2,2,0,2,1} + \alpha T_{2,2,2,2,0,2,2}]\},$$

where

$$T_{i,j,k,l,m,p,q} = E\{(1-V)^{-i}[1-V^{1/\alpha}]^j V^{q-k/\alpha}[\log(1-V^{1/\alpha})]^l [\log(V)]^m [\log(-(\log(1-V^{1/\alpha}))]^p\},$$

V is a beta random variable with parameters a and b and $i,j,k,l,m,p,q \in \{0,1,2\}$. The total information matrix is then $K_n = K_n(\theta) = nK(\theta)$.