The Amoroso Distribution

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The Amoroso distribution[1,2] is a continuous, univariate, unimodal probability distribution with a semi-infinite range. A surprisingly large menagerie of interesting, univariate probability distributions are special cases or limiting forms of the Amoroso distribution.

$$\begin{aligned}&\text{Amoroso}(x|\nu,\theta,\alpha,\beta) \qquad (1a)\\ &= \frac{1}{\Gamma(\alpha)}\left|\frac{\beta}{\theta}\right|\left(\frac{x-\nu}{\theta}\right)^{\alpha\beta-1}\exp\left\{-\left(\frac{x-\nu}{\theta}\right)^{\beta}\right\}\\ &-\infty \leq \ \nu,\theta,\beta \leq +\infty \qquad \alpha > 0\\ &\quad x \geq \nu \ (\theta > 0) \quad\ x \leq \nu \ (\theta < 0)\end{aligned}$$

This distribution has four real parameters; a location parameter ν, a scale parameter θ, and two shape parameters α and β.

Another useful parameterization is

$$\begin{aligned}&\text{Amoroso}'(x|\mu,\sigma,\alpha,\lambda) \qquad (1b)\\ &= \frac{\alpha^{\alpha}}{\Gamma(\alpha)|\sigma|}\left(1+\lambda\frac{x-\mu}{\sigma}\right)^{\frac{\alpha}{\lambda}-1}\exp\left\{-\alpha\left(1+\lambda\frac{x-\mu}{\sigma}\right)^{\frac{1}{\lambda}}\right\}\\ &= \text{Amoroso}(\mu-\frac{\sigma}{\lambda},\frac{\sigma}{\lambda\alpha^{\lambda}},\alpha,1/\lambda)\end{aligned}$$

In the limit that $\lambda \to 0$, the range becomes $x \in [-\infty,+\infty]$ and

$$\begin{aligned}&\text{Amoroso}'(x|\mu,\sigma,\alpha,0) \qquad (1c)\\ &= \frac{\alpha^{\alpha}}{\Gamma(\alpha)|\sigma|}\exp\left\{\alpha\left(\frac{x-\mu}{\sigma}\right)-\alpha\exp\left(\frac{x-\mu}{\sigma}\right)\right\}\end{aligned}$$

(Recall that $\lim_{a\to 0}(1+ax)^{1/a}=e^x$)

We will define the standard Amoroso distribution as

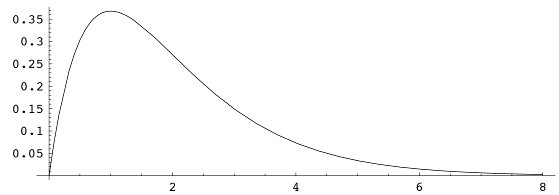
$$\begin{aligned}&\text{StdAmoroso}(x) = xe^{-x} \qquad (1d)\\ &= \text{Amoroso}(x|0,1,2,1)\\ &= \text{Amoroso}'(x|2,2,1,1)\end{aligned}$$

Setting β to -1 yields Pearson's type V (March) distribution[3,4]

$$\begin{aligned}\text{PearsonV}(x|\mu,\theta,\alpha) =& \frac{1}{\Gamma(\alpha)\,|\theta|}\left(\frac{\theta}{x-\nu}\right)^{\alpha+1}e^{-\frac{\theta}{x-\nu}}\\ =&\text{Amoroso}(x|\nu,\theta,\alpha,-1) \qquad (2)\end{aligned}$$

If we set the β shape parameter to unity we obtain Pearson's type III (Vinci) distribution[5–7].

$$\begin{aligned}\text{PearsonIII}(x|\nu,\theta,\alpha) =& \frac{1}{\Gamma(\alpha)\theta}\left(\frac{x-\nu}{\theta}\right)^{\alpha-1}e^{-\left(\frac{x-\nu}{\theta}\right)} \qquad (3)\\ =&\text{Amoroso}(x|\nu,\theta,\alpha,1)\end{aligned}$$



		ν	θ	α	β	μ	σ	α	λ
(1)	Amoroso	ν	θ	α	β	μ	σ	α	λ
(2)	Pearson type V	.	.	.	-1	.	.	.	-1
(3)	Pearson type III	.	.	.	1	.	.	.	1
(4)	Nakagami	.	.	.	2	.	.	.	$\frac{1}{2}$
(26)	generalized Fréchet	.	.	n	<0	.	.	n	<0
(28)	generalized Gumbel					.	.	n	0
(25)	generalized Weibull	.	.	n	>0	.	.	n	>0
(27)	generalized extreme value	.	.	1	.	.	.	1	.
(24)	Fréchet	.	.	1	<0	.	.	1	<0
(33)	generalized log gamma					.	.	.	0
(29)	Gumbel					.	.	1	0
(31)	BHP					.	.	$\frac{\pi}{2}$	0
(25)	Weibull	.	.	1	>0	.	.	1	>0
(9)	shifted exponential	.	.	1	1	.	.	1	1
(32)	log gamma					x	.	.	0
(5)	generalized gamma	0	.	.	.				
(19)	scaled inverse-chi	0	.	.	-2				
(16)	inverse gamma	0	.	.	-1				
(34)	Jeffreys	0	.	.	0				
(6)	gamma	0	.	.	1				
(12)	scaled chi	0	.	.	2				
(10)	stretched exponential	0	.	1	.				
(22)	Lévy	0	.	$\frac{1}{2}$	-1				
(13)	half normal	0	.	$\frac{1}{2}$	2				
(21)	inverse Rayleigh	0	.	1	-2				
(20)	inverse exponential	0	.	1	-1				
(8)	exponential	0	.	1	1				
(14)	Rayleigh	0	.	1	2				
(15)	Maxwell	0	.	$\frac{3}{2}$	2				
(6)	Wein	0	.	4	1				
(17)	inverse chi-square	0	$\frac{1}{2}$	.	-1				
(18)	inverse chi	0	$\frac{1}{\sqrt{2}}$	.	-2				
(11)	chi	0	$\sqrt{2}$	.	2				
(7)	chi-square	0	2	.	1				
(8)	standard exponential	0	1	1	1	1	1	1	1
(30)	standard Gumbel					0	1	1	0
(1d)	standard Amoroso	0	1	2	1	2	2	2	1

With $\beta = 2$ we obtain the Nakagami (generalized normal) distribution.

$$\text{Nakagami}(x|\nu,\theta,k/2,2) = \frac{2}{\Gamma(k/2)\theta}\left(\frac{x-\nu}{\theta}\right)^{k-1}\exp\left\{-\left(\frac{x-\nu}{\theta}\right)^2\right\} \tag{4}$$

If we drop the location parameter from Amoroso, then we obtain the generalized gamma (hyper gamma, generalized Weibull) distribution, the parent of the gamma family of distributions[8,9].

$$\begin{aligned}\text{GenGamma}(x|\theta,\alpha,\beta) =& \frac{\beta}{\Gamma(\alpha)\theta}\left(\frac{x}{\theta}\right)^{\alpha\beta-1} e^{-(\frac{x}{\theta})^\beta} \\ & x > 0, \theta > 0 \\ =& \text{Amoroso}(x|0,\theta,\alpha,\beta)\end{aligned} \tag{5}$$

If the β is negative then the distribution is generalized inverse gamma.

Not surprisingly the gamma (scaled-chi-square) distribution[5,7] is a special case of the generalized gamma, where the second shape parameter is set to unity.

$$\begin{aligned}\text{Gamma}(x|\theta,\alpha) &= \frac{1}{\Gamma(\alpha)\theta}\left(\frac{x}{\theta}\right)^{\alpha-1} e^{-x/\theta} \\ &= \text{PearsonIII}(x|0,\theta,\alpha) \\ &= \text{GenGamma}(x|\theta,\alpha,1) \\ &= \text{Amoroso}(x|0,\theta,\alpha,1)\end{aligned} \tag{6}$$

Instances of the gamma distribution often appear in statistical physics. For example the Wein (Vienna) distribution $\text{Wein}(x|T) = \text{Gamma}(x|T,4)$ (An approximation to the relative intensity of black body radiations as a function of the frequency). The Erlang distribution is a gamma distribution with integer α. Note that we obtain Amoroso by adding to the gamma distribution both a location (as in Pearson type III) and an additional shape parameter (as in the generalized gamma).

Important special cases of the gamma distribution include the chi-square (χ^2, chi squared) distribution

$$\begin{aligned}\text{ChiSqr}(x|k) &= \frac{1}{2\Gamma(k/2)}\left(\frac{x}{2}\right)^{k/2-1} e^{-x/2} \\ &= \text{Gamma}(x|2,k/2) \\ &= \text{GenGamma}(x|2,k/2,1) \\ &= \text{Amoroso}(x|0,2,k/2,1)\end{aligned} \tag{7}$$

and the exponential (Pearson type X) distribution

$$\begin{aligned}\text{Exp}(x|\theta) &= \theta e^{-\frac{x}{\theta}} \\ &= \text{Gamma}(x|\theta,1) \\ &= \text{Amoroso}(x|0,\theta,1,1)\end{aligned} \tag{8}$$

We can also obtain a shifted exponential distribution as a special case of the Pearson type III distribution

$$\begin{aligned}\text{ShiftExp}(x|\nu,\theta) &= \theta e^{-\frac{x-\nu}{\theta}} \\ &= \text{PearsonIII}(x|\nu,\theta,1) \\ &= \text{Amoroso}(x|\nu,\theta,1,1)\end{aligned} \tag{9}$$

Stretched exponential[10]

$$\begin{aligned}\text{StretchedExp} &= \left|\frac{\beta}{\theta}\right|\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(\frac{x}{\theta})^\beta} \\ &= \text{Amoroso}(x|0,\theta,1,\beta)\end{aligned} \tag{10}$$

Additional special cases of the generalized gamma distribution include the chi (χ) distribution

$$\begin{aligned}\text{Chi}(x|k) &= \frac{1}{\Gamma(k/2)\sqrt{2}}\left(\frac{x}{\sqrt{2}}\right)^{k-1} e^{-x^2/2} \\ &= \text{GenGamma}(x|\sqrt{2},k/2,2) \\ &= \text{Amoroso}(x|0,\sqrt{2},k/2,2)\end{aligned} \tag{11}$$

and scaled-chi (generalized Rayleigh) distribution.

$$\begin{aligned}\text{ScaledChi}(x|s,k) &= \frac{1}{\Gamma(k/2)\sqrt{2s^2}}\left(\frac{x}{\sqrt{2s^2}}\right)^{k-1} e^{-\frac{x^2}{2s^2}} \\ &= \text{GenGamma}(x|\sqrt{2}s,k/2,2) \\ &= \text{Amoroso}(x|0,\sqrt{2}s,k/2,2)\end{aligned} \tag{12}$$

Special cases of the scaled-chi distribution include the half-normal (semi-normal, positive definite normal) distribution,

$$\begin{aligned}\text{HalfNormal}(x|s) &= \frac{2}{\sqrt{2\pi s^2}} e^{-\frac{x^2}{2s^2}} \\ &= \text{ScaledChi}(x|s,1) \\ &= \text{GenGamma}(x|\sqrt{2}s,1/2,2) \\ &= \text{Amoroso}(x|0,\sqrt{2}s,1/2,2)\end{aligned} \tag{13}$$

the Rayleigh distribution

$$\begin{aligned}\text{Rayleigh}(x|s) &= \frac{1}{s} x e^{-\frac{x^2}{2s^2}} \\ &= \text{ScaledChi}(x|s,2) \\ &= \text{GenGamma}(x|\sqrt{2}s,1,2) \\ &= \text{Amoroso}(x|0,\sqrt{2}s,1,2)\end{aligned} \tag{14}$$

and the Maxwell (Maxwell-Boltzmann) distribution

$$\begin{aligned}\text{Maxwell}(x|s) &= \frac{\sqrt{2}}{\sqrt{\pi} s^3} x^2 e^{-\frac{x^2}{2s^2}} \\ &= \text{ScaledChi}(x|s,3) \\ &= \text{GenGamma}(x|\sqrt{2}s,3/2,2) \\ &= \text{Amoroso}(x|0,\sqrt{2}s,3/2,2)\end{aligned} \tag{15}$$

With negative shape parameters, the generalized gamma generates various inverse distributions, including the inverse gamma (scaled inverse chi-square[11]) distribution,

$$\mathrm{InvGamma}(x|\theta,\alpha) = \frac{1}{\Gamma(\alpha)\theta}\left(\frac{\theta}{x}\right)^{\alpha+1} e^{-\theta/x} \tag{16}$$
$$= \mathrm{GenGamma}(x|\theta,\alpha,-1)$$
$$= \mathrm{PearsonV}(x|0,\theta,\alpha)$$
$$= \mathrm{Amoroso}(x|0,\theta,\alpha,-1)$$

the inverse-chi-square distribution,

$$\mathrm{InvChiSqr}(x|k) = \frac{2}{\Gamma(k/2)}\left(\frac{1}{2x}\right)^{\frac{k}{2}+1} e^{-\frac{1}{2x}} \tag{17}$$
$$= \mathrm{InvGamma}(x|1/2,k/2)$$
$$= \mathrm{GenGamma}(x|1/2,k/2,-1)$$
$$= \mathrm{PearsonV}(x|0,1/2,k/2)$$
$$= \mathrm{Amoroso}(x|0,1/2,k/2,-1)$$

the inverse-chi distribution,

$$\mathrm{InvChi}(x|k) = \frac{2\sqrt{2}}{\Gamma(k/2)}\left(\frac{1}{\sqrt{2}x}\right)^{k+1} e^{-\frac{1}{2x^2}} \tag{18}$$
$$= \mathrm{GenGamma}(x|1/\sqrt{2},k/2,-2)$$
$$= \mathrm{Amoroso}(x|0,1/\sqrt{2},k/2,-2)$$

scaled inverse-chi distribution,

$$\mathrm{ScaledInvChi}(x|s,k) = \frac{2\sqrt{2s^2}}{\Gamma(k/2)}\left(\frac{1}{\sqrt{2s^2}x}\right)^{k+1} e^{-\frac{1}{2s^2x^2}} \tag{19}$$
$$= \mathrm{GenGamma}(x|1/\sqrt{2s^2},k/2,-2)$$
$$= \mathrm{Amoroso}(x|0,1/\sqrt{2s^2},k/2,-2)$$

inverse exponential,

$$\mathrm{InvExp}(x|\theta) = \frac{\theta}{x^2}e^{-\theta/x} \tag{20}$$
$$= \mathrm{InvGamma}(x|\theta,1)$$
$$= \mathrm{GenGamma}(x|\theta,1,-1)$$
$$= \mathrm{Amoroso}(x|0,\theta,1,-1)$$

and inverse Rayleigh.

$$\mathrm{InvRayleigh}(x|\theta) = \frac{1}{8s^2}\left(\frac{1}{x}\right)^{3} e^{-\frac{1}{2s^2x^2}} \tag{21}$$
$$= \mathrm{GenGamma}(x|1/\sqrt{2s^2},1,-2)$$
$$= \mathrm{Amoroso}(x|0,1/\sqrt{2s^2},1,-2)$$

The Lévy distribution (Van der Waals profile) is a special case of the inverse gamma distribution. The Lévy distribution is notable for being stable; a linear combination of identically distributed Lévy distributions is again a Lévy distribution. The other stable distributions with analytic forms are the normal (which we encounter below) and the Cauchy distribution, which is not a member of the Amoroso family.

$$\mathrm{Lévy}(x|c) = \sqrt{\frac{c}{2\pi}}\frac{e^{-c/2x}}{x^{3/2}} \tag{22}$$
$$= \mathrm{InvGamma}(x|c/2,1/2)$$
$$= \mathrm{GenGamma}(x|c/2,1/2,-1)$$
$$= \mathrm{PearsonV}(x|0,c/2,1/2)$$
$$= \mathrm{Amoroso}(x|0,c/2,1/2,-1)$$

The Weibull (extreme value type III , Fisher-Tippett type III, Gumbel type III) distribution[12,13] occurs with the shape parameter $\alpha = 1$. This is the limiting distribution of the minimum of a large number identically distributed random variables that are at least ν. (Maximum if θ is negative.)

$$\mathrm{Weibull}(x|\nu,\theta,\beta) = \frac{\beta}{\theta}\left(\frac{x-\nu}{\theta}\right)^{\beta-1} e^{-(\frac{x-\nu}{\theta})^{\beta}} \tag{23}$$
$$= \mathrm{Amoroso}(x|\nu,\theta,1,\beta)$$

Special cases of the Weibull distribution include the exponential ($\beta = 1$) and Rayleigh ($\beta = 2$) distributions, and the standard Weibull ($\nu = 0$).

The Fréchet (extreme value type II , Fisher-Tippett type II, Gumbel type II, inverse Weibull) distribution is the limiting distribution of the largest of a large number identically distributed random variables whose moments are not all finite and are bounded from below by ν. (If the shape parameter θ is negative then minimum rather than maxima.)

$$\mathrm{Fréchet}(x|\nu,\theta,\beta) = \left|\frac{\beta}{\theta}\right|\left(\frac{\theta}{x-\nu}\right)^{\beta'+1} e^{-(\frac{\theta}{x-\nu})^{\beta'}}$$
$$= \mathrm{Amoroso}(x|\nu,\theta,1,-\beta') \tag{24}$$

Special cases of the Fréchet distribution include the inverse exponential ($\beta' = 1$) and inverse Rayleigh ($\beta' = 2$) and the standard Fréchet ($\nu = 0$) distribution.

Instead of asking for the minimum or maximum of a large number of random variables, we instead ask for the nth largest we obtain the generalized Weibull distribution

$$\mathrm{GenWeibull}(x|\nu,\theta,n,\beta) = \frac{\beta}{\theta}\left(\frac{x-\nu}{\theta}\right)^{n\beta-1} e^{-(\frac{x-\nu}{\theta})^{n\beta}}$$
$$= \mathrm{Amoroso}(x|\nu,\theta,n,\beta) \tag{25}$$

and the generalized Fréchet distribution.

$$\mathrm{GenFréchet}(x|\nu,\theta,n,\beta) = \frac{\beta}{\theta}\left(\frac{\theta}{x-\nu}\right)^{n\beta'+1} e^{-(\frac{\theta}{x-\nu})^{n\beta'}}$$
$$= \mathrm{Amoroso}(x|\nu,\theta,n,-\beta') \tag{26}$$

The generalized extreme value (GEV, von Mises-Jenkinson) distribution is the superclass of type I, II and

III extreme value distributions.

$$\begin{aligned}&\text{GenExtremeValue}(x|\mu,\sigma,\lambda) \quad (27)\\&= \frac{1}{|\sigma|}\left(1+\lambda\frac{x-\mu}{\sigma}\right)^{\frac{1}{\lambda}-1}\exp\left\{-\left(1+\lambda\frac{x-\mu}{\sigma}\right)^{\frac{1}{\lambda}}\right\}\\&= \text{Amoroso}'(x|\mu,\sigma,1,\lambda)\\&= \text{Amoroso}(\mu-\frac{\sigma}{\lambda},\frac{\sigma}{\lambda\alpha^{\lambda}},1,1/\lambda)\end{aligned}$$

The generalized Gumbel (generalized log-gamma) distribution is the limiting distribution of the nth largest value of a large number of unbounded identically distributed random variables.

$$\begin{aligned}&\text{GenGumbel}(x|\mu,\sigma,n) \quad (28)\\&= \frac{1}{\sigma}\exp\left\{n\left(\frac{\mu-x}{\sigma}\right)-n\exp\left(\frac{\mu-x}{\sigma}\right)\right\}\\&= \text{Amoroso}'(x|\mu,\sigma,n,0)\end{aligned}$$

If we limit $n=1$ then we obtain the Gumbel (Fisher-Tippett (type I), Fisher-Tippett-Gumbel, FTG, Gumbel-Fisher-Tippett, log-Weibull, extreme value (type I), doubly exponential) distribution

$$\begin{aligned}&\text{Gumbel}(x|\mu,\sigma) \quad (29)\\&= \frac{1}{\sigma}\exp\left\{\left(\frac{x-\mu}{\sigma}\right)-\exp\left(\frac{x-\mu}{\sigma}\right)\right\}\\&= \text{Amoroso}'(x|\mu,\sigma,1,0)\end{aligned}$$

With negative scale $\sigma<0$, this is an extreme value distribution of maximum, with $\sigma>0$ an extreme value distribution of minima. (Note that often the Gumbel is defined with the negative of the scale used here.) A Gompertz distribution is a truncated Gumbel.

The standard Gumbel (Gumbel) distribution is

$$\begin{aligned}\text{StdGumbel}(x) &= \exp\{x-e^{x}\} \quad (30)\\&= \text{Amoroso}'(x|0,1,1,0)\end{aligned}$$

Another special case of the generalized Gumbel is the BHP (Bramwell-Holdsworth-Pinton) distribution[14,15]

$$\begin{aligned}&\text{BHP}(x|\mu,\sigma) \quad (31)\\&= \frac{1}{\sigma}\exp\left\{\frac{\pi}{2}\left(\frac{x-\mu}{\sigma}\right)-\frac{\pi}{2}\exp\left(\frac{x-\mu}{\sigma}\right)\right\}\\&= \text{Amoroso}'(x|\mu,\sigma,\frac{\pi}{2},0)\end{aligned}$$

Log-gamma

$$\begin{aligned}\text{LogGamma}(x|\sigma,\alpha) &= \frac{1}{\Gamma(\alpha)\sigma}\exp\left\{\alpha\left(\frac{x}{\sigma}\right)-\exp\left(\frac{x}{\sigma}\right)\right\}\\&= \text{Amoroso}'(x|\sigma\ln\alpha,\sigma,\alpha,0) \quad (32)\end{aligned}$$

Generalized Log-Gamma(Coale-McNeil[16,17])

$$\begin{aligned}&\text{GenLogGamma}(x|\mu',\sigma,\alpha) \quad (33)\\&= \frac{1}{\Gamma(\alpha)\sigma}\exp\left\{\alpha\left(\frac{x-\mu'}{\sigma}\right)-\exp\left(\frac{x-\mu'}{\sigma}\right)\right\}\\&= \text{Amoroso}'(x|\mu+\sigma\ln\alpha,\sigma,\alpha,0)\end{aligned}$$

If we let $\beta=0$ then we obtain Jeffreys distribution[18], an improper (unnormalizable) distribution widely used as an uninformative prior in Bayesian probability[19].

$$\begin{aligned}\text{Jeffreys}(x) &\propto \frac{1}{x} \quad (34)\\&= \text{Amoroso}(0,\theta,\alpha,0))\end{aligned}$$

If θ and α are finite, their exact values are irrelevant. If we take the limit $\alpha\to\infty$ but keep the product $\alpha\beta=1-p$ constant then we can obtain a variety of improper power-law (Pearson type XI[20], fractal) distributions.

$$\begin{aligned}\text{PowerLaw}(x|p) &\propto \frac{1}{x^{p}} \quad (35)\\&= \lim_{\alpha\to\infty}\text{Amoroso}(0,\theta,\alpha,(1-p)/\alpha)\end{aligned}$$

If $p=0$ we obtain the half-uniform distribution over the positive numbers.

The normal (Gauss, Gaussian, bell curve) distribution can be obtained in several limits. For example,

$$\begin{aligned}&\text{Normal}(x|\mu,\sigma) \quad (36)\\&= \frac{1}{\sqrt{2\pi\sigma^{2}}}\exp\left\{-\frac{(x-\mu)^{2}}{2\sigma'^{2}}\right\}\\&= \lim_{\alpha\to\infty}\text{Amoroso}'(x|\mu,\sigma'/\sqrt{\alpha},\alpha,0)\end{aligned}$$

In the limit that $\sigma'\to\infty$ we obtain an unbounded uniform distribution, and in the limit $\sigma'\to 0$ we obtain a delta function distribution.

Properties

$$E[\left(\frac{x-\nu}{\theta}\right)^{n}] = \frac{\Gamma(\alpha+\frac{n}{\beta})}{\Gamma(\alpha)} \quad (37)$$

$$\text{mean} = \nu+\theta\frac{\Gamma(\alpha+\frac{1}{\beta})}{\Gamma(\alpha)} \quad (38)$$

$$\text{variance} = \theta^{2}\left[\frac{\Gamma(\alpha+\frac{2}{\beta})}{\Gamma(\alpha)}-\frac{\Gamma(\alpha+\frac{1}{\beta})^{2}}{\Gamma(\alpha)^{2}}\right] \quad (39)$$

$$\text{Entropy} = \log\frac{\theta\Gamma(\alpha)}{\beta}+\alpha+\left(\frac{1}{\beta}-\alpha\right)\psi(\alpha) \quad (40)$$

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