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For any continuous baseline G distribution [G.M. Cordeiro and M. de Castro, *A new family of generalized distributions*, J. Statist. Comput. Simul. 81 (2011), pp. 883–898], proposed a new generalized distribution (denoted here with the prefix 'Kw-G' (Kumaraswamy-G)) with two extra positive parameters. They studied some of its mathematical properties and presented special sub-models. We derive a simple representation for the Kw-G density function as a linear combination of exponentiated-G distributions. Some new distributions are proposed as sub-models of this family, for example, the Kw-Chen [Z.A. Chen, *A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function*, Statist. Probab. Lett. 49 (2000), pp. 155–161], Kw-XTG [M. Xie, Y. Tang, and T.N. Goh, *A modified Weibull extension with bathtub failure rate function*, Reliab. Eng. System Safety 76 (2002), pp. 279–285] and Kw-Flexible Weibull [M. Bebbington, C.D. Lai, and R. Zitikis, *A flexible Weibull extension*, Reliab. Eng. System Safety 92 (2007), pp. 719–726]. New properties of the Kw-G distribution are derived which include asymptotes, shapes, moments, moment generating function, mean deviations, Bonferroni and Lorenz curves, reliability, Rényi entropy and Shannon entropy. New properties of the order statistics are investigated. We discuss the estimation of the parameters by maximum likelihood. We provide two applications to real data sets and discuss a bivariate extension of the Kw-G distribution.

Keywords: estimation; exponential distribution; extreme values; Kw-G distribution; mean deviation; moment generating function

1. Introduction

Kumaraswamy [1] introduced a two-parameter distribution on (0, 1), the so-called Kumaraswamy distribution, with cumulative distribution function (cdf) given by

$$G(x;\alpha,\beta) = 1-(1-x^{\alpha})^{\beta}, \quad x \in (0,1), \tag{1}$$

where $\alpha > 0$ and $\beta > 0$ are shape parameters. Equation (1) compares extremely favourably in terms of simplicity with the beta cumulative function. The probability density function (pdf)

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corresponding to Equation (1) is

$$g(x;\alpha,\beta)=\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1},\quad x\in(0,1).$$

For a detailed survey of the Kumaraswamy distribution the reader is referred to [2].

For any baseline cumulative function $G(x)$, Cordeiro and de Castro [3] proposed the Kumaraswamy-G ('Kw-G' for short) distribution with pdf $f(x)$ and cdf $F(x)$ given by

$$f(x)=abg(x)G^{a-1}(x)\{1-G^{a}(x)\}^{b-1} \tag{2}$$

and

$$F(x)=1-\{1-G^{a}(x)\}^{b}, \tag{3}$$

respectively, where $g(x)=\mathrm{d}G(x)/\mathrm{d}x$. The Kw-G distribution has the same parameters of the G distribution plus two additional shape parameters $a>0$ and $b>0$. The associated hazard rate function (hrf) is

$$\tau(x)=\frac{abg(x)G^{a-1}(x)}{1-G^{a}(x)}. \tag{4}$$

If X is a random variable with density function (2), then we write $X\sim$ Kw-G(a,b). Each new Kw-G distribution can be obtained from a specified G distribution. For $a=b=1$, the G distribution is a special sub-model of the Kw-G distribution with a continuous crossover towards cases with different shapes (e.g., a particular combination of skewness and kurtosis). One major benefit of the Kw-G family of densities (2) is its ability of fitting skewed data that cannot be properly fitted by existing distributions. Further, this family allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology.

First, we present two examples of the Kw-G distributions. The Kw-Weibull density function (for $x>0$), defined from the Weibull cdf $G(x)=1-\exp\{-(\beta x)^{c}\}$ with parameters $\beta>0$ and $c>0$, reduces to

$$f(x)=abc\beta^{c}x^{c-1}\exp\{-(\beta x)^{c}\}[1-\exp\{-(\beta x)^{c}\}]^{a-1}\{1-[1-\exp\{-(\beta x)^{c}\}]^{a}\}^{b-1}.$$

For $c=1$, we obtain as a special sub-model the Kw-exponential distribution. The Kw-Gumbel density function, defined from the Gumbel cdf $G(x)=1-\exp\{-\exp(-(x-\mu)/\sigma)\}$ (for $x\in R$) with location parameter $\mu>0$ and scale parameter $\sigma>0$, is given by

$$f(x)=\frac{ab}{\sigma}\exp\left\{\frac{x-\mu}{\sigma}-\exp\left(\frac{x-\mu}{\sigma}\right)\right\}\left[1-\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^{a-1}$$
$$\times\left\{1-\left[1-\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^{a}\right\}^{b-1}.$$

Figures 1 and 2 provide some plots of the Kw-Weibull and Kw-Gumbel distributions, respectively.

Secondly, we propose three new distributions in the Kw-G family.

(1) The Kw-Chen density function is given by

$$f(x)=ab\lambda_1\beta_1x^{\beta_1-1}\exp(x^{\beta_1})\exp\{\lambda_1[1-\exp(x^{\beta_1})]\}[1-\exp\{\lambda_1[1-\exp(x^{\beta_1})]\}]^{a-1}$$
$$\times\{1-(1-\exp\{\lambda_1[1-\exp(x^{\beta_1})]\})^{a}\}^{b-1}, \tag{5}$$

where $\lambda_1>0$ and $\beta_1>0$. If X is a random variable with density function (5), then we write $X\sim$ Kw-Chen(a,b,λ_1,β_1). For $a=b=1$, it becomes the Chen distribution [4]. The Kw-Chen survival function is

$$S(x)=[1-(1-\exp\{\lambda_1[1-\exp(x^{\beta_1})]\})^{a}]^{b}.$$

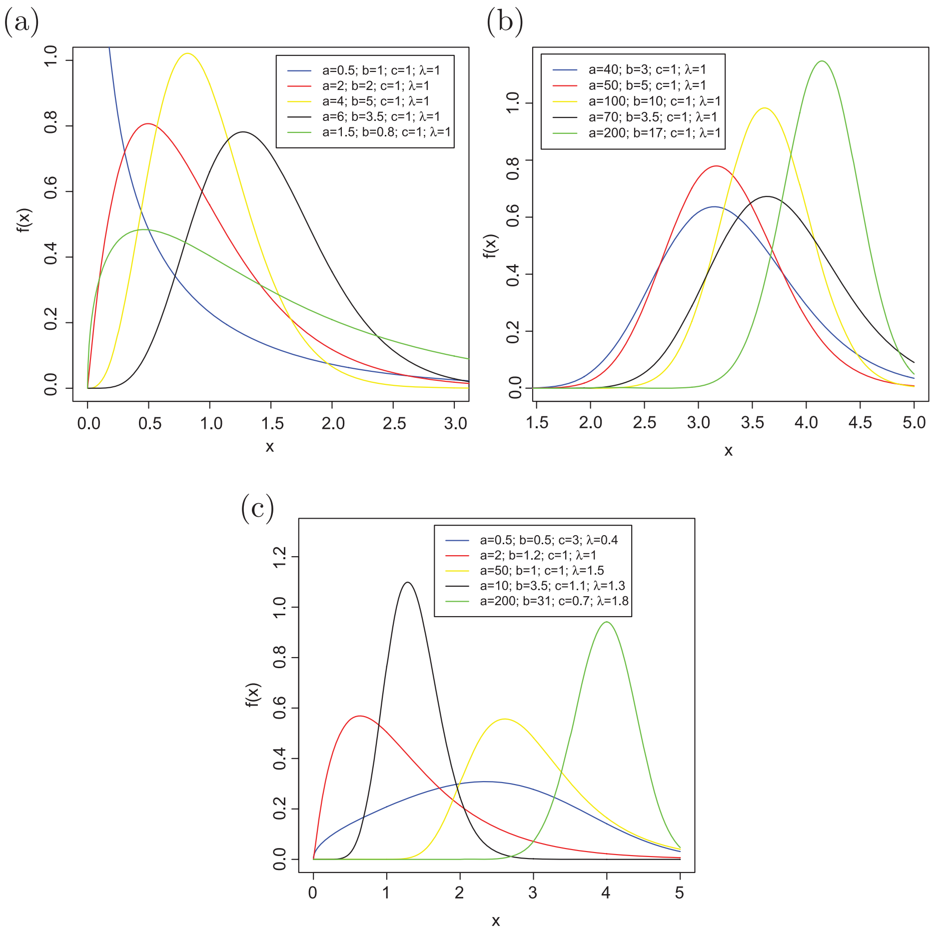

Figure 1. Plots of the Kw-Weibull density function for selected parameter values.

(2) The new Kw-XTG density function has the form

$$f(x) = ab\lambda_2\beta_2\left(\frac{x}{\alpha_2}\right)^{\beta_2-1}\exp\left\{\left(\frac{x}{\alpha_2}\right)^{\beta_2}+\lambda_2\alpha_2\left[1-\exp\left(\left(\frac{x}{\alpha_2}\right)^{\beta_2}\right)\right]\right\}$$
$$\times\left[1-\exp\left\{\lambda_2\alpha_2\left[1-\exp\left(\left(\frac{x}{\alpha_2}\right)^{\beta_2}\right)\right]\right\}\right]^{a-1}$$
$$\times\left\{1-\left(1-\exp\left\{\lambda_2\alpha_2\left[1-\exp\left(\left(\frac{x}{\alpha_2}\right)^{\beta_1}\right)\right]\right\}\right)^{a}\right\}^{b-1}, \qquad (6)$$

where $\lambda_2 > 0$, $\alpha_2 > 0$ and $\beta_2 > 0$. If X is a random variable with density (6), then we write $X \sim \text{Kw-XTG}(a, b, \lambda_2, \alpha_2, \beta_2)$. For $a = b = 1$, it becomes the XTG distribution proposed by Xie *et al.* [5]. The Kw-XTG survival function is

$$S(x) = \left[1-\left(1-\exp\left\{\lambda_2\alpha_2\left[1-\exp\left(\left(\frac{x}{\alpha_2}\right)^{\beta_2}\right)\right]\right\}\right)^{a}\right]^{b}.$$

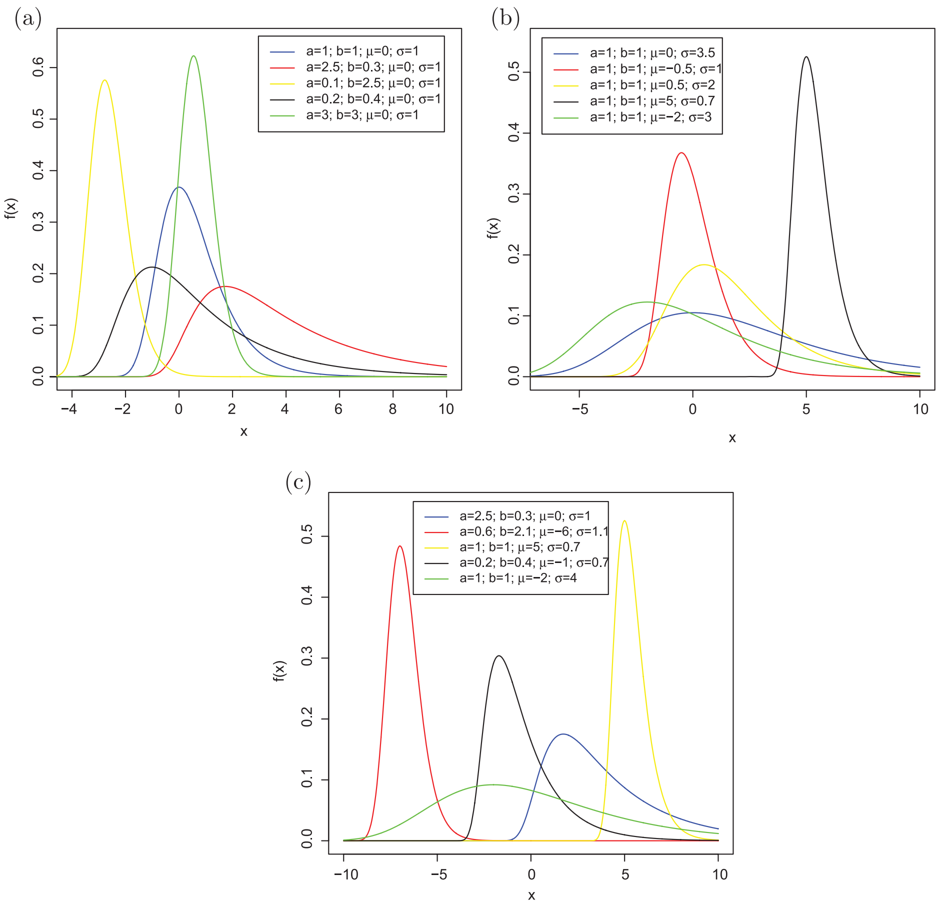

Figure 2. Plots of the Kw-Gumbel density function for selected parameter values.

(3) The new Kw-flexible Weibull (Kw-FW) density function is given by

$$f(x) = ab\left(\alpha_3 + \frac{\beta_3}{x^2}\right)\exp\left(\alpha_3 x - \frac{\beta_3}{x}\right)\exp\left\{-\exp\left(\alpha_3 x - \frac{\beta_3}{x}\right)\right\}$$
$$\times\left[1 - \exp\left\{-\exp\left(\alpha_3 x - \frac{\beta_3}{x}\right)\right\}\right]^{a-1}$$
$$\times\left\{1 - \left(1 - \exp\left\{-\exp\left(\alpha_3 x - \frac{\beta}{x}\right)\right\}\right)^{a}\right\}^{b-1}, \qquad (7)$$

where $\alpha_3 > 0$ and $\beta_3 > 0$. If X is a random variable with density (7), then we write $X \sim$ Kw-FW$(a, b, \alpha_3, \beta_3)$. If $a = b = 1$, then it becomes the flexible Weibull (FW) distribution proposed by Bebbington *et al.* [6]. The Kw-FW survival function is

$$S(x) = \left[1 - \left(1 - \exp\left\{-\exp\left(\alpha_3 x - \frac{\beta_3}{x}\right)\right\}\right)^{a}\right]^{b}.$$

Figures 3, 4 and 5 provide some plots of the Kw-Chen, Kw-XTG and Kw-Fw distributions, respectively.

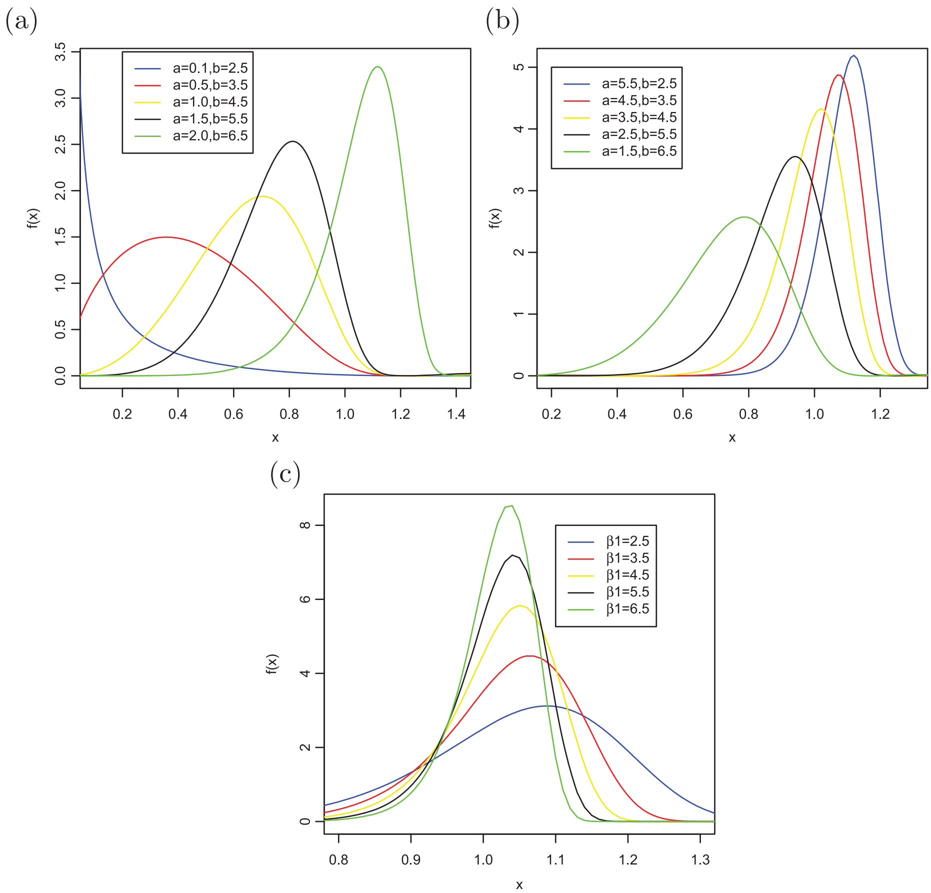

Figure 3. Plots of the Kw-Chen density function for some parameters. (a) For $\lambda_1 = 0.5$ and $\beta_1 = 3.2$. (b) For $\lambda = 0.5$ and $\beta_1 = 3.2$. (c) For $a = 3.5$, $b = 2.5$ and $\lambda_1 = 0.5$.

A physical interpretation of the Kw-G distribution given by Equations (2) and (3) (for a and b positive integers) is as follows. Consider that a system is formed by b independent components and that each component is made up of a independent subcomponents. Suppose the system fails if any of the b components fails and that each component fails if all of the a subcomponents fail. Let $X_{j1}, \ldots, X_{ja}$ denote the lifetimes of the subcomponents within the jth component, $j = 1, \ldots, b$, having a common cdf $G(x)$. Let X_j denote the lifetime of the jth component, for $j = 1, \ldots, b$, and let X denote the lifetime of the entire system. Then, the cdf of X is

$$\begin{aligned}\Pr(X \leq x) &= 1 - \Pr(X_1 > x, \ldots, X_b > x) = 1 - \Pr^b(X_1 > x)\\ &= 1 - \{1 - \Pr(X_1 \leq x)\}^b = 1 - \{1 - \Pr(X_{11} \leq x, \ldots, X_{1a} \leq x)\}^b\\ &= 1 - \{1 - \Pr^a(X_{11} \leq x)\}^b = 1 - \{1 - G^a(x)\}^b.\end{aligned}$$

So, it follows that the Kw-G distribution given by Equations (2) and (3) is precisely the time to failure distribution of the entire system.

The rest of the article is organized as follows. A range of mathematical properties of Equation (2) is considered in Sections 2–5. These include asymptotes and shapes, a simple representation for the Kw-G density function, two methods for simulation, moments, moment generating function (mgf), characteristic function, mean deviations about the mean and the median, Bonferroni and Lorenz

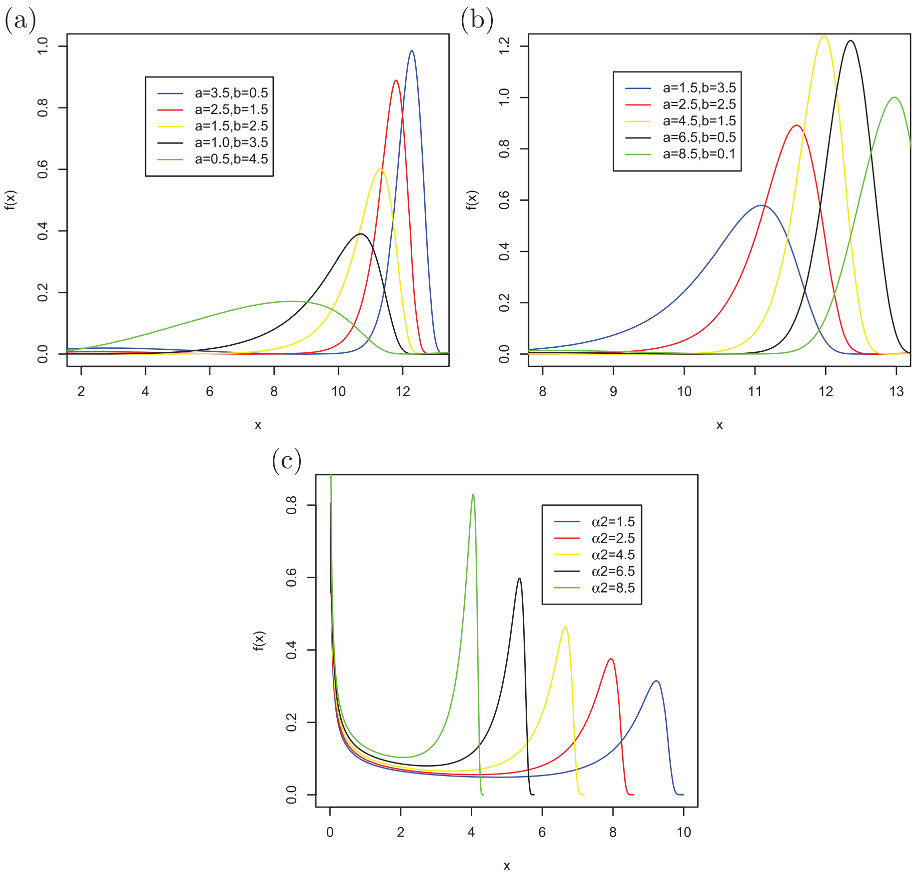

Figure 4. Plots of the Kw-XTG density function for some parameters. (a) For $\lambda_2 = 0.01$, $\alpha_2 = 10$ and $\beta_2 = 6$. (b) For $\lambda_2 = 0.01$, $\alpha_2 = 10$ and $\beta_2 = 6$. (c) For $a = 0.07$, $b = 0.6$, $\lambda_2 = 0.01$ and $\beta_2 = 6$.

curves, asymptotic distributions of the extreme values, Rénvy entropy, Shannon entropy, reliability and some properties of the order statistics. A relation with the so-called beta-G distribution is explored in Section 6. Estimation by the method of maximum likelihood—including the case of censoring and the Fisher information matrix—is presented in Section 7. Applications to two real data sets are illustrated in Section 8. A multivariate generalization of (2) is discussed in Section 9. Finally, some conclusions are noted in Section 10.

2. Asymptotes, shapes and simulation

2.1. *Asymptotes and shapes*

The asymptotes of Equations (2)–(4) as $x \to 0, \infty$ are given by

$$f(x) \sim abg(x)G^{a-1}(x) \quad \text{as } x \to 0, \quad f(x) \sim abg(x)\{1 - G^a(x)\}^{b-1} \quad \text{as } x \to \infty,$$

$$F(x) \sim aG^a(x) \quad \text{as } x \to 0, \quad 1 - F(x) \sim \{1 - G^a(x)\}^b \quad \text{as } x \to \infty,$$

$$\tau(x) \sim abg(x)G^{a-1}(x) \quad \text{as } x \to 0, \quad \tau(x) \sim \frac{abg(x)}{1 - G^a(x)} \quad \text{as } x \to \infty.$$

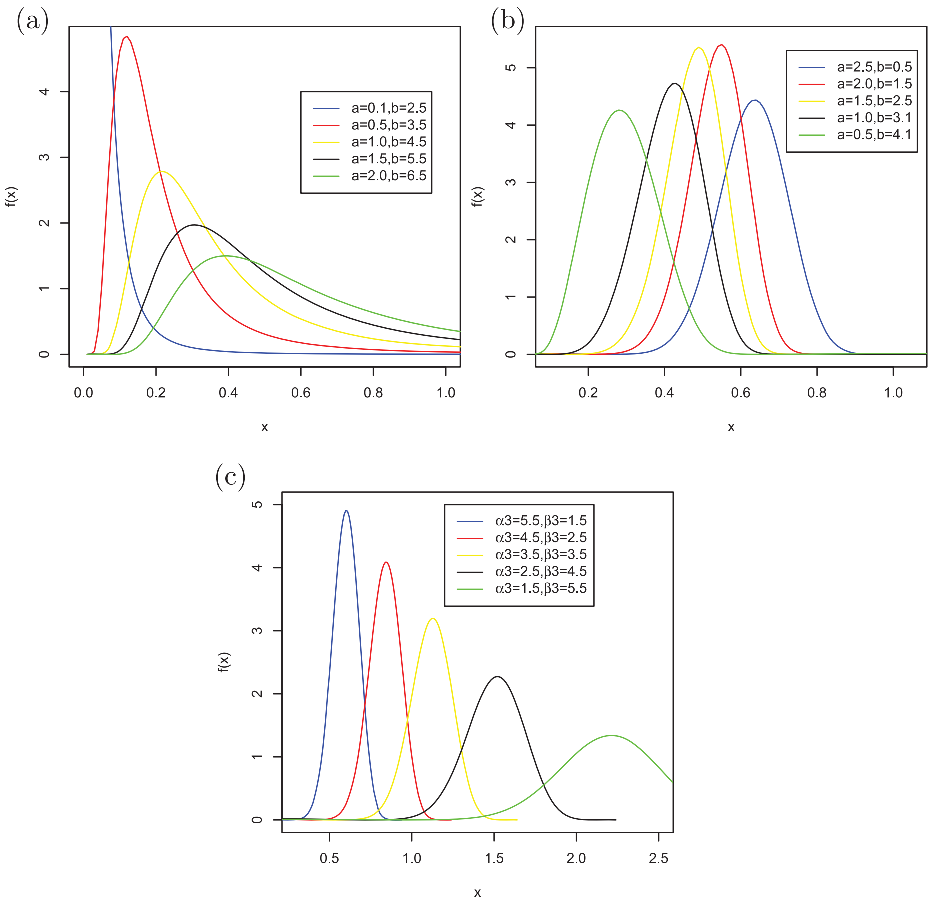

Figure 5. Plots of the Kw-FW density function for the indicated parameter. (a) For $\alpha_3 = 0.5$ and $\beta_3 = 3.2$. (b) For $\alpha_3 = 0.5$ and $\beta_3 = 3.2$. (c) For $a = 3.5$, $b = 2.5$ and $\lambda_1 = 0.5$.

The shapes of the pdf (2) and the hrf (4) can be described analytically. The critical points of the pdf are the roots of the equation:

$$\frac{g'(x)}{g(x)} + (a-1)\frac{g(x)}{G(x)} = a(b-1)\frac{g(x)G^{a-1}(x)}{1-G^a(x)}. \tag{8}$$

There may be more than one root to Equation (8). If $x = x_0$ is a root of Equation (8), then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\lambda(x_0) < 0$, $\lambda(x_0) > 0$ or $\lambda(x_0) = 0$, where

$$\lambda(x) = \frac{g(x)g''(x) - (g'(x))^2}{g^2(x)} + (a-1)\frac{G(x)g'(x) - g^2(x)}{G^2(x)}$$
$$- a(b-1)\frac{G^{a-2}(x)\{(a-1)g^2(x) + G(x)g'(x)\}}{1-G^a(x)} - a^2(b-1)\frac{G^{2a-2}(x)g^2(x)}{\{1-G^a(x)\}^2}.$$

The critical points of the hrf are the roots of the equation:

$$\frac{g'(x)}{g(x)} + (a-1)\frac{g(x)}{G(x)} = -a\frac{g(x)G^{a-1}(x)}{1-G^a(x)}. \tag{9}$$

There may be more than one root to Equation (9). If $x = x_0$ is a root of Equation (9), then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\lambda(x_0) < 0$, $\lambda(x_0) > 0$ or $\lambda(x_0) = 0$, where

$$\lambda(x) = \frac{g(x)g''(x) - (g'(x))^2}{g^2(x)} + (a-1)\frac{G(x)g'(x) - g^2(x)}{G^2(x)} - a\frac{G^{a-2}(x)\{(a-1)g^2(x) + G(x)g'(x)\}}{1 - G^a(x)} - a^2\frac{G^{2a-2}(x)g^2(x)}{\{1 - G^a(x)\}^2}.$$

2.2. *Simple representation*

Some series expansions for Equations (2) and (3) can be derived using the concept of exponentiated distributions. For an arbitrary baseline cdf $G(x)$, a random variable is said to have the exponentiated-G distribution with parameter $a > 0$, say $X \sim \text{Exp-G}(a)$, if its pdf and cdf are

$$h_a(x) = ag(x)G^{a-1}(x) \quad \text{and} \quad H_a(x) = G^a(x),$$

respectively. The properties of exponentiated distributions have been studied by many authors in recent years, see Mudholkar *et al.* [7], for exponentiated Weibull, Gupta *et al.* [8] for exponentiated Pareto, Gupta and Kundu [9] for exponentiated exponential and Nadarajah and Gupta [10] for exponentiated gamma distribution.

Expanding the binomial terms in Equations (2) and (3), the pdf and the cdf of the Kw-G distribution can be expressed as

$$f(x) = a^{-1}\sum_{k=0}^{\infty}\frac{w_k}{(k+1)}h_{a(k+1)}(x) \tag{10}$$

and

$$F(x) = 1 - \sum_{k=0}^{\infty}\binom{b}{k}(-1)^k H_{ka}(x), \tag{11}$$

where $w_k = (-1)^k ab\binom{b-1}{k}$, $h_{a(k+1)}(x)$ and $H_{k\,a}(x)$ are the pdf and cdf of the Exp-G$(a(k+1))$ and Exp-G(ka) distributions, respectively. So, the properties of the Kw-G distribution can be obtained by knowing those of the exponentiated-G distribution, see, for example, Mudholkar *et al.* [7], Gupta and Kundu [9], Nadarajah and Kotz [10], among others.

2.3. *Simulation*

We present two methods for simulation from the Kw-G distribution. The first uses the inversion method. The quantile function corresponding to Equation (3) is directly obtained from the quantile function associated with $G(x)$ by

$$F^{-1}(x) = G^{-1}\{[1 - (1-x)^{1/b}]^{1/a}\}. \tag{12}$$

So, one can generate Kw-G variates by

$$X = G^{-1}\{[1 - (1-U)^{1/b}]^{1/a}\},$$

where U is a uniform variate on the unit interval $[0, 1]$.

Our second method for simulation from the Kw-G distribution is based on the rejection method. It holds if $a \geq 1$ and $b \geq 1$. Define a constant M by

$$M = \frac{a^b b(a-1)^{1-1/a}(b-1)^{b-1}}{(ab-1)^{b-1/a}}.$$

Then the following scheme holds for simulating Kw-G variates:

(1) simulate $X = x$ from the pdf g;
(2) simulate $Y = UMg(x)$, where U is a uniform variate on the unit interval [0, 1];
(3) accept $X = x$ as a Kw-G variate if $Y < f(x)$. If $Y \geq f(x)$ return to step 2.

3. Moments and moment generating function

3.1. *Moments*

From now on, let $X \sim$ Kw-G(a, b). Cordeiro and de Castro [3] derived explicit expressions for the moments of X as linear functions of probability weighted moments of X. A first representation for the nth moment of X can be obtained from Equation (10) as

$$E(X^n) = a^{-1} \sum_{i=0}^{\infty} \frac{w_i}{(i+1)} E(Y_i^n),$$

where $Y_i \sim$ Exp-G$(a(i+1))$. Expressions for moments of several exponentiated distributions are given by Nadarajah and Kotz [11] which can be used to produce $E(X^n)$.

For $b > 0$ real non-integer, we can rewrite Equation (2) as

$$f(x) = \sum_{i=0}^{\infty} w_i G(x)^{a(i+1)-1} g(x). \tag{13}$$

If b is an integer, the index i in the previous sum stops at $b-1$. A second representation for $E(X^n)$ follows from Equation (13) as

$$E(X^n) = \sum_{i=0}^{\infty} w_i \tau(n, a(i+1)-1), \tag{14}$$

where the integral $\tau(n, a) = \int_{-\infty}^{\infty} x^n G(x)^a g(x)\,\mathrm{d}x$ can be expressed in terms of the baseline quantile function $Q(x) = G^{-1}(x)$ as

$$\tau(n, a) = \int_0^1 Q(u)^n u^a \,\mathrm{d}u. \tag{15}$$

The ordinary moments of several Kw-G distributions can be calculated directly from Equations (14) and (15). For example, the moments of the Kw-exponential (with parameter $\lambda > 0$) and Kw-Pareto, where $G(x) = 1 - (1+x)^{-\nu}$ and $\nu > 0$, are

$$E(X^n) = n!\lambda^n \sum_{i,j=0}^{\infty} \frac{(-1)^{n+j}\binom{a(i+1)-1}{j} w_i}{(j+1)^{n+1}},$$

$$E(X^n) = \sum_{i,j=0}^{\infty} (-1)^{n+j} w_i \binom{n}{j} B(a(i+1), 1 - j\nu^{-1}),$$

respectively, where $B(\cdot,\cdot)$ is the beta function. For the Kw-standard logistic, where $G(x) = \{1 + \exp(-x)\}^{-1}$, we obtain from Prudnikov *et al.* [12, Section 2.6.13, Equation 4]

$$E(X^n) = \sum_{i=0}^{\infty} w_i \left(\frac{\partial}{\partial t}\right)^n B(t + a(i+1), 1 - t)|_{t=0}.$$

The skewness and kurtosis measures can be computed from the ordinary moments using well-known relationships. Plots of the skewness and kurtosis for some choices of the parameter b as function of a, and for some choices of a as function of b are given below.

- *The Kw-Chen distribution*. For $\lambda_1 = 0.2$ and $\beta_1 = 0.3$, Figures 6 and 7 show that the skewness curves decrease and increase with b (a fixed) and a (b fixed), respectively. The kurtosis curves decrease with b (a fixed) and both decrease and increase with a (b fixed), respectively.
- *The Kw-XTG distribution*. For $\lambda_2 = 0.01$, $\alpha_2 = 1.5$ and $\beta_2 = 0.05$, Figures 8 and 9 show that the skewness curves decrease and increase with b (a fixed) and decrease with a (b fixed), respectively. The kurtosis curves decrease and increase with b (a fixed) and decrease with a (b fixed), respectively.
- *The Kw-FW distribution*. For $\alpha_3 = 0.01$ and $\beta_3 = 0.0001$, Figures 10 and 11 reveal that the skewness curves decrease and increase with b (a fixed) and a (b fixed), respectively. The kurtosis curves decrease and increase with b (a fixed) and a (b fixed), respectively.

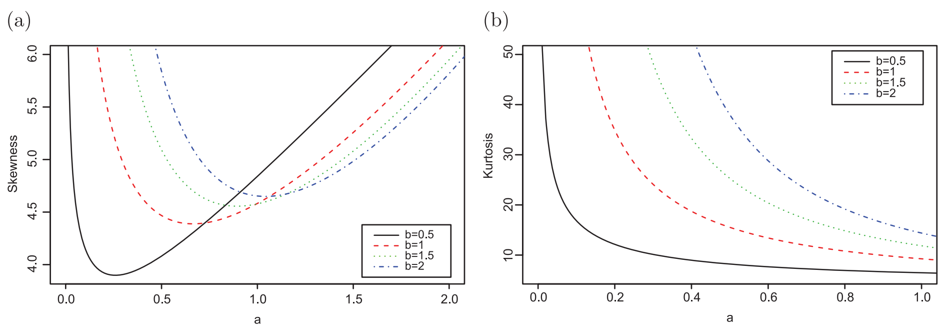

Figure 6. Skewness and kurtosis of the Kw-Chen distribution as a function of b, for some values of a with $\lambda_1 = 0.2$ and $\beta_1 = 0.3$.

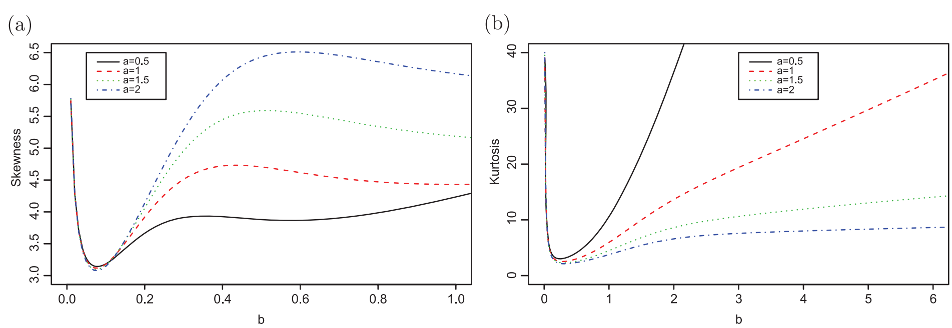

Figure 7. Skewness and kurtosis of the Kw-Chen distribution as a function of a, for some values of b with $\lambda_1 = 0.2$ and $\beta_1 = 0.3$.

3.2. *Moment generating function*

Let $X \sim$ Kw-G(a, b). We provide four representations for the mgf $M(t) = E[\exp(tX)]$ of X. Clearly, the first one is

$$M(t) = \sum_{s=0}^{\infty} \frac{\mu'_s}{s!} t^s, \tag{16}$$

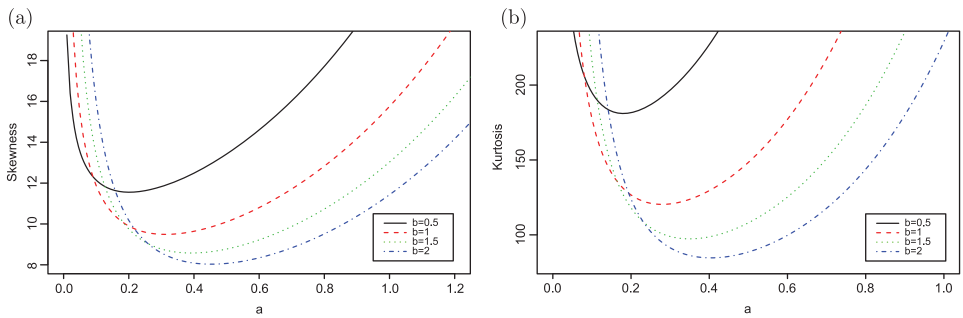

Figure 8. Skewness and kurtosis of the Kw-XTG distribution as a function of b, for some values of a with $\lambda_2 = 0.01$, $\alpha_2 = 1.5$ and $\beta_2 = 0.05$.

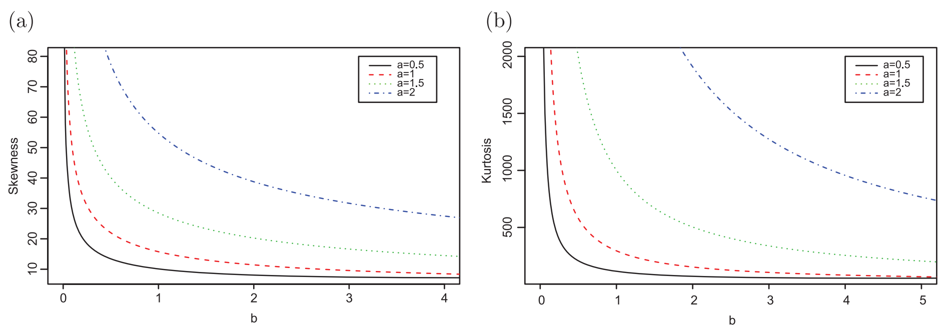

Figure 9. Skewness and kurtosis of the Kw-XTG distribution as a function of a, for some values of b with $\lambda_2 = 0.01$, $\alpha_2 = 1.5$ and $\beta_2 = 0.05$.

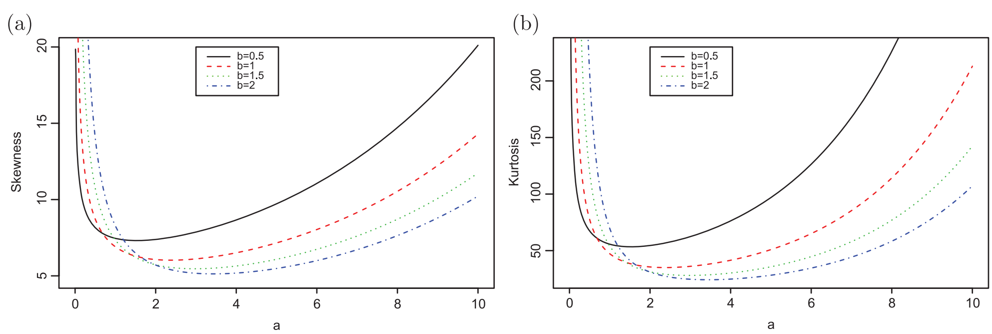

Figure 10. Skewness and kurtosis of the Kw-FW distribution as a function of b, for some values of a with $\alpha_3 = 0.01$ and $\beta_3 = 0.0001$.

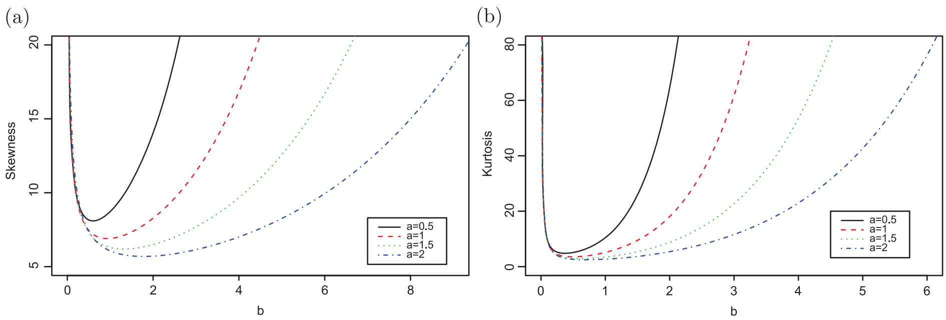

Figure 11. Skewness and kurtosis of the Kw-FW distribution as a function of a, for some values of b with $\alpha_3 = 0.01$ and $\beta_3 = 0.0001$.

where $\mu'_s = E(X^s)$. Further, the second one comes from

$$M(t) = abE[\exp(tX)G^{a-1}(X)\{1 - G^a(X)\}^{b-1}] = ab\sum_{k=0}^{\infty}\binom{b-1}{k}(-1)^k E\left[\frac{\exp(tX)}{U^{-[a(k+1)-1]}}\right],$$

where U is a uniform random variable on the unit interval. Note that X and U are not independent.

A third representation for $M(t)$ is obtained from (10)

$$M(t) = a^{-1}\sum_{i=0}^{\infty}\frac{w_i}{(i+1)}M_i(t),$$

where $M_i(t)$ is the mgf of $Y_i \sim$ Exp-G$(a(i+1))$. Hence, for several Kw-G distributions, $M(t)$ can be immediately determined from the mgf of the G distribution.

A fourth representation for $M(t)$ can be derived from Equation (13) as

$$M(t) = \sum_{i=0}^{\infty} w_i\rho(t, a(i+1) - 1), \tag{17}$$

where the function $\rho(t, a) = \int_{-\infty}^{\infty}\exp(tx)G(x)^a g(x)\,dx$ can be calculated from the baseline quantile function $Q(x) = G^{-1}(x)$ by

$$\rho(t, a) = \int_0^1 \exp\{tQ(u)\}u^a\,du. \tag{18}$$

We can obtain the mgf of several Kw-G distributions from Equation (17) and (18). For example, the mgf's of the Kw-exponencial (with parameter λ), Kw-standard logistic and Kw-Pareto (with parameter $\nu > 0$) are determined from Equation (17) as

$$M(t) = \sum_{i=0}^{\infty} w_i B(a(i+1), 1 - \lambda t), \quad M(t) = \sum_{i=0}^{\infty} w_i B(t + a(i+1), 1 - t)$$

and

$$M(t) = \exp(-t)\sum_{i,r=0}^{\infty}\frac{w_i B(a(i+1), 1 - r\nu^{-1})}{r!}t^r,$$

respectively.

Clearly, four representations for the characteristic function (chf) $\phi(t) = E[\exp(\mathrm{i}tX)]$ of the Kw-G distribution are immediately obtained from the above representations for the mgf by $\phi(t) = M(\mathrm{i}t)$, where $\mathrm{i} = \sqrt{-1}$.

4. Other measures

4.1. *Mean deviations*

Let $X \sim \text{KwG}(a, b)$. The mean deviations about the mean ($\delta_1(X)$) and about the median ($\delta_2(X)$) can be expressed as

$$\delta_1(X) = E(|X - \mu_1'|) = 2\mu_1' F(\mu_1') - 2T(\mu_1') \quad \text{and} \quad \delta_2(X) = E(|X - M|) = \mu_1' - 2T(M), \tag{19}$$

respectively, where $\mu_1' = E(X)$, $M = \text{Median}(X)$ denotes the median, $F(\mu_1')$ comes from Equation (3) and $T(z) = \int_{-\infty}^{z} x f(x)\,\mathrm{d}x$. The median M follows from Equation (12) as

$$M = G^{-1}\{[1 - 2^{-1/b}]^{1/a}\}.$$

Setting $u = G(x)$ in Equation (13) yields

$$T(z) = \sum_{k=0}^{\infty} w_k T_k(z), \tag{20}$$

where the integral $T_k(z)$ can be expressed in terms of $Q(u) = G^{-1}(u)$ by

$$T_k(z) = \int_0^{G(z)} Q(u) u^{a(k+1)-1}\,\mathrm{d}u. \tag{21}$$

The mean deviations of any Kw-G distribution can be computed from Equations (19)–(21). For example, the mean deviations of the Kw-exponencial (with parameter λ), Kw-standard logistic and Kw-Pareto (with parameter $\nu > 0$) are immediately calculated (after using the generalized binomial expansion) from the functions

$$T_k(z) = \lambda^{-1}\Gamma(a(k+1)+1) \sum_{j=0}^{\infty} \frac{(-1)^j \{1 - \exp(-j\lambda z)\}}{\Gamma(a(k+1)+1-j)(j+1)!},$$

$$T_k(z) = \frac{1}{\Gamma(k)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(a(k+1)+j)\{1 - \exp(-jz)\}}{(j+1)!}$$

and

$$T_k(z) = \sum_{j=0}^{\infty} \sum_{r=0}^{j} \frac{(-1)^j \binom{a(k+1)}{j}\binom{j}{r}}{(1 - r\nu)} z^{1-r\nu},$$

respectively.

An alternative representation for $T(z)$ can be derived from Equation (10) as

$$T(z) = \int_{-\infty}^{z} x f(x)\,\mathrm{d}x = a^{-1} \sum_{k=0}^{\infty} \frac{w_k}{k+1} J_k(z), \tag{22}$$

where

$$J_k(z) = \int_{-\infty}^{z} x h_{a(k+1)}(x)\,\mathrm{d}x. \tag{23}$$

Equation (23) is the basic quantity to compute the mean deviations of the exponentiated-G distributions. Hence, the Kw-G mean deviations depend only on the mean deviations of the exponentiated-G distribution. So, alternative representations for $\delta_1(X)$ and $\delta_2(X)$ are

$$\delta_1(X) = 2\mu_1' F(\mu_1') - 2a^{-1}\sum_{k=0}^{\infty}\frac{w_k}{k+1}J_k(\mu_1')\delta_2(X) = \mu_1' - 2a^{-1}\sum_{k=0}^{\infty}\frac{w_k}{k+1}J_k(M).$$

A simple application is provided for the Kw-Weibull distribution. The exponentiated Weibull with parameter $a(k+1)$ has density function (for $x > 0$) given by

$$h_{a(k+1)}(x) = a(k+1)c\beta^c x^{c-1}\exp\{-(\beta x)^c\}[1-\exp\{-(\beta x)^c\}]^{a(k+1)-1}$$

and then

$$\begin{aligned} J_k(z) &= a(k+1)c\beta^c\int_0^z x^c \exp\{-(\beta x)^c\}[1-\exp\{-(\beta x)^c\}]^{a(k+1)-1}\,\mathrm{d}x \\ &= a(k+1)c\beta^c\sum_{r=0}^{\infty}(-1)^r\binom{a(k+1)-1}{r}\int_0^z x^c\exp\{-(r+1)(\beta x)^c\}\,\mathrm{d}x. \end{aligned}$$

We can calculate the last integral using the incomplete gamma function $\gamma(\alpha, x) = \int_0^x w^{\alpha-1}\exp(-w)\,\mathrm{d}w$ (for $\alpha > 0$) and then

$$J_k(z) = a(k+1)\beta^{-1}\sum_{r=0}^{\infty}\frac{(-1)^r\binom{a(k+1)-1}{r}}{(r+1)^{1+c^{-1}}}\gamma(1+c^{-1}, (r+1)(\beta z)^c).$$

Equations (20) and (22) are the main results of this section. Applications of these equations can be given to obtain Bonferroni and Lorenz curves defined for a given probability p by

$$B(p) = \frac{T(q)}{p\mu_1'} \quad \text{and} \quad L(p) = \frac{T(q)}{\mu_1'},$$

respectively, where $\mu_1' = E(X)$ and $q = F^{-1}(p) = G^{-1}\{[1-(1-p)^{1/b}]^{1/a}\}$.

4.2. *Extreme values*

If $\bar{X} = (X_1 + \cdots + X_n)/n$ denotes the sample mean from i.i.d. random variables following (2), then by the usual central limit theorem $\sqrt{n}(\bar{X} - E(X))/\sqrt{\mathrm{Var}(X)}$ approaches the standard normal distribution as $n \to \infty$ under suitable conditions. Sometimes one would be interested in the asymptotics of the extreme values $M_n = \max(X_1, \ldots, X_n)$ and $m_n = \min(X_1, \ldots, X_n)$.

First, suppose that G belongs to the max domain of attraction of the Gumbel extreme value distribution. Then by Leadbetter *et al.* [13, Chapter 1], there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t\to\infty}\frac{1-G(t+xh(t))}{1-G(t)} = \exp(-x)$$

for every $x \in (-\infty, \infty)$. But

$$\begin{aligned} \lim_{t\to\infty}\frac{1-F(t+xh(t))}{1-F(t)} &= \lim_{t\to\infty}\left\{\frac{1-G^a(t+xh(t))}{1-G^a(t)}\right\}^b \\ &= \lim_{t\to\infty}\left\{\frac{1-G(t+xh(t))}{1-G(t)}\right\}^b = \exp(-bx) \end{aligned}$$

for every $x \in (-\infty, \infty)$. So, it follows by Leadbetter *et al.* [13, Chapter 1] that F also belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n\to\infty} \Pr\{a_n(M_n - b_n) \le x\} = \exp\{-\exp(-bx)\}$$

for some suitable norming constants $a_n > 0$ and b_n.

Second, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then by Leadbetter *et al.* [13, Chapter 1], there must exist a $\beta > 0$, such that

$$\lim_{t\to\infty} \frac{1 - G(tx)}{1 - G(t)} = x^{\beta}$$

for every $x > 0$. But

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t\to\infty} \left\{ \frac{1 - G^a(tx)}{1 - G^a(t)} \right\}^b = \lim_{t\to\infty} \left\{ \frac{1 - G(tx)}{1 - G(t)} \right\}^b = x^{\beta b}$$

for every $x > 0$. So, it follows by Leadbetter *et al.* [13, Chapter 1] that F also belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty} \Pr\{a_n(M_n - b_n) \le x\} = \exp(-x^{\beta b})$$

for some suitable norming constants $a_n > 0$ and b_n.

Third, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then by Leadbetter *et al.* [13, Chapter 1], there must exist a $\alpha > 0$, such that

$$\lim_{t\to 0} \frac{G(tx)}{G(t)} = x^{\alpha}$$

for every $x < 0$. But

$$\lim_{t\to 0} \frac{F(tx)}{F(t)} = \lim_{t\to 0} \left\{ \frac{G(tx)}{G(t)} \right\}^a = x^{\alpha a}$$

for every $x < 0$. So, it follows by Leadbetter *et al.* [13, Chapter 1] that F also belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n\to\infty} \Pr\{a_n(M_n - b_n) \le x\} = \exp\{-(-x)^{\alpha a}\}$$

for some suitable norming constants $a_n > 0$ and b_n.

The same argument applies to min domains of attraction. That is, F belongs to the same min domain of attraction as that of G.

4.3. *Shannon entropy*

The entropy of a random variable X with density function $f(x)$ is a measure of variation of the uncertainty. Shannon entropy is defined by $E[-\log f(X)]$. Suppose $X \sim$ Kw-G(a, b). Then,

$$\begin{aligned} E[-\log f(X)] = &-\log(ab) - E[\log g(X)] - (a-1)E[\log G(X)] \\ &- (b-1)E[\log\{1 - G^a(X)\}]. \end{aligned} \tag{24}$$

The three expectations in Equation (24) can be expressed as

$$E[\log g(X)] = ab\int_{-\infty}^{\infty}\log g(x)g(x)G^{a-1}(x)\{1-G^a(x)\}^{b-1}\,\mathrm{d}x = \sum_{k=0}^{\infty} w_k I_k,$$

$$E[\log G(X)] = \frac{b}{a}\int_0^1 \log u(1-u)^{b-1}\,\mathrm{d}u = \frac{b}{a}\left.\frac{\partial B(\alpha+1,b)}{\partial\alpha}\right|_{\alpha=0} = -\frac{C+\psi(b+1)}{a}$$

and

$$E[\log\{1-G^a(X)\}] = b\int_0^1 \log(u)u^{b-1}\,\mathrm{d}u = -\frac{1}{b}.$$

Here, C is Euler's constant and I_k can be rewritten as

$$I_k = \int_{-\infty}^{\infty}\log\{g(x)\}g(x)G^{a(k+1)-1}(x)\,\mathrm{d}x = \int_0^1 \log\{g(Q(u))\}u^{a(k+1)-1}\,\mathrm{d}u,$$

where $g(Q(u))$ is the quantile density function. So, it follows from Equation (24) that

$$E[-\log f(X)] = -\log(ab) - \sum_{k=0}^{\infty} w_k I_k + \frac{(a-1)\{C+\psi(b+1)\}}{a} + \frac{b-1}{b}.$$

We can determine I_k for some Kw-G models. For the Kw-exponencial distribution with parameter λ, the quantile density function is $g(Q(u)) = \lambda(1-u)$ and then

$$I_k = \frac{\lambda}{a(k+1)} + \int_0^1 \log(1-u)u^{a(k+1)-1}\,\mathrm{d}u.$$

The last integral can be calculated for a real non-integer by changing variable $v = 1-u$, using the generalized binomial expansion and noting that $\int_0^1 \log(v)\,v^a dv = -(a+1)^{-2}$. We have

$$I_k = \frac{\lambda}{a(k+1)} + \sum_{j=0}^{\infty}\frac{(-1)^{j+1}\Gamma(a(k+1))}{\Gamma(a(k+1)-j)j!(j+1)^2}.$$

If a is an integer, then the index j stops at $a(k+1)-1$.

For the Kw-standard logistic distribution, $g(Q(u)) = u(1-u)$ and, in the same way, we obtain

$$I_k = \frac{-1}{a^2(k+1)^2} + \sum_{j=0}^{\infty}\frac{(-1)^{j+1}\Gamma(a(k+1))}{\Gamma(a(k+1)-j)j!(j+1)^2}.$$

For the Kw-Pareto distribution with parameter ν, we obtain $g(Q(u)) = \nu(1-u)^{1+\nu^{-1}}$ and then

$$I_k = \frac{\log(\nu)}{a(k+1)} + (1+\nu^{-1})\sum_{j=0}^{\infty}\frac{(-1)^{j+1}\Gamma(a(k+1))}{\Gamma(a(k+1)-j)j!(j+1)^2}.$$

An alternative expression for the first expectation in Equation (24) follows from Equation (22) as

$$E[\log g(X)] = a^{-1}\sum_{k=0}^{\infty}\frac{w_k}{k+1}E[\log\{g(Y_k)\}],$$

where $Y_k \sim$ Exp-G$(a(k+1))$. So, one can also express Equation (24) as

$$E[-\log\{f(X)\}] = -\log(ab) - a^{-1}\sum_{k=0}^{\infty}\frac{w_k}{k+1}E[\log g(Y_k)] + \frac{(a-1)\{C+\psi(b+1)\}}{a} + \frac{b-1}{b}.$$

The above expectation can be computed for several Exp-G distributions.

4.4. *Rényi entropy*

The entropy of a random variable X with density function $f(x)$ is a measure of variation of the uncertainty. One of the popular entropy measure is the Rényi entropy given by

$$\mathcal{J}_R(c) = \frac{1}{1-c}\log\left[\int_{-\infty}^{\infty} f^c(x)\,\mathrm{d}x\right], \quad c > 0, c \neq 1. \tag{25}$$

The integral can be written as

$$\int_{-\infty}^{\infty} f^c(x)\,\mathrm{d}x = (ab)^c\int_{-\infty}^{\infty} g^c(x)G^{(a-1)c}(x)\{1-G^a(x)\}^{(b-1)c}\,\mathrm{d}x$$

and then expanding the binomial and changing the variable

$$\int_{-\infty}^{\infty} f^c(x)\,\mathrm{d}x = (ab)^c\sum_{j=0}^{\infty}\binom{(b-1)c}{j}(-1)^j K(c,j). \tag{26}$$

Here, $K(c, j)$ denotes the integral

$$K(c,j) = \int_0^1 g^{c-1}(Q(u))u^{a(c+j)-c}\,\mathrm{d}u,$$

to be calculated for each Kw-model. For the Kw-exponential (with parameter λ), Kw-standard logistic and Kw-Pareto (with parameter ν) distributions, we obtain

$$K(c,j) = \lambda^{c-1}B(a(c+j)-c, c-1), \quad K(c,j) = B(a(c+j)-1, c-1)$$

and

$$K(c,j) = \nu^{c-1}B(a(c+j)-c, (1+\nu^{-1})(c-1)),$$

respectively. Equation (26) is the main result of this section.

4.5. *Reliability*

Here, we derive the reliability $R = \Pr(X_2 < X_1)$ when $X_1 \sim$ Kw-G(a_1, b_1) and $X_2 \sim$ Kw-G(a_2, b_2) are independent random variables with a positive support. It has many applications especially in engineering concepts. Let f_i denote the pdf of X_i and F_i denote the cdf of X_i.

By expanding the binomials in f_1 and F_2, we obtain

$$R = 1 - a_1 b_1 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{b_1 - 1}{j} \binom{b_2}{k} \frac{(-1)^{j+k}}{[a_1(j+1) + a_2 k]}. \quad (27)$$

In the particular case $a_1 = a_2 = a$, one can reduce Equation (27) to

$$R = 1 - b_1 \sum_{j=0}^{\infty} \binom{b_1 + b_2 - 1}{j} \frac{(-1)^j}{(j+1)}.$$

Further, if $a_1 = a_2$ and $b_1 = b_2$, then $R = \frac{1}{2}$. An alternative expression for R, obtained using Equations (10) and (11), is

$$R = 1 - a \sum_{k,m=0}^{\infty} \binom{b-1}{k} \binom{b-1}{m} \frac{(-1)^{k+m} R_{k,m}}{(k+1)},$$

where $R_{k,m}$ becomes

$$R_{k,m} = \int_0^{\infty} h_{a(k+1)}(x) H_{a\,m}(x)\, \mathrm{d}x.$$

Clearly, $R_{k,m}$ denotes the reliability function of independent random variables (with a positive support) following exponentiated-G distributions with parameters $a(k+1)$ and $a\,m$. Hence, the reliability for the Kw-G(a_1, b_1) and Kw-G(a_2, b_2) independent random variables reduces to a linear combination of the reliability functions $R_{k,m}$'s.

5. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. The density function $f_{i:n}(x)$ of the ith order statistic $X_{i:n}$, for $i = 1, \ldots, n$, from i.i.d. random variables $X_1, \ldots, X_n$ following any Kw-G distribution, is simply given by

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{i+j-1}. \quad (28)$$

Cordeiro and de Castro [3] showed that

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} q_{r,k}\, G(x)^{a(k+1)+r-1} g(x), \quad (29)$$

where

$$q_{r,k} = \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j} w_k\, p_{r,i+j-1}}{B(i, n-i+1)}$$

and the quantities $p_{r,u}(a, b)$ (for $r, u = 0, 1, \ldots$) are given by

$$p_{r,u}(a, b) = \sum_{k=0}^{u} (-1)^k \binom{u}{k} \sum_{m=0}^{\infty} \sum_{l=r}^{\infty} (-1)^{mr+l} \binom{kb}{m} \binom{ma}{l} \binom{l}{r}.$$

From the definition of $\tau(n, a)$ given in Equation (15), the sth moment of the order statistic $X_{i:n}$ can be written as

$$E(X_{i:n}^s) = \sum_{r,k=0}^{\infty} q_{r,k}\tau(s, a(k+1) + r - 1). \tag{30}$$

The mgf of $X_{i:n}$ can be obtained from Equations (18) and (29) as

$$M_{i:n}(t) = \sum_{r,k=0}^{\infty} q_{r,k}\rho(t, a(k+1) + r - 1). \tag{31}$$

Now, we provide a simple application. The quantities $E(X_{i:n}^s)$ and $M_{i:n}(t)$ for the Kw-exponencial distribution with parameter $\lambda > 0$ follow from Equations (30) and (31), respectively, as

$$E(X_{i:n}^s) = s!\lambda^s \sum_{r,k,j=0}^{\infty} q_{r,k} \frac{(-1)^{s+j}\binom{a(k+1)+r-1}{j}}{(j+1)^{s+1}}$$

and

$$M_{i:n}(t) = \sum_{r,k=0}^{\infty} q_{r,k} B(a(k+1) + r, 1 - \lambda t).$$

These quantities are easily derived for the Kw-G distributions cited before.

Alternatively, we can express $f_{i:n}(x)$ as a linear combination of exponentiated-G density functions. Equation (29) can be rewritten as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} q_{r,k}^{\star} h_{a(k+1)+r}, \tag{32}$$

where

$$q_{r,k}^{\star} = \frac{q_{r,k}}{a(k+1) + r}.$$

Clearly, $\sum_{r,k=0}^{\infty} q_{r,k}^{\star} = 1$. Some mathematical properties of the Kw-G order statistics can be immediately derived from Equation (32) by knowing those of the exponentiated-G distribution including moments, inverse and factorial moments, mgf, mean deviations and Bonferroni and Lorenz curves. Equations (29)–(32) are the main results of this section.

6. Relation with the beta G distribution

Consider starting from the baseline cdf $G(x)$ and pdf $g(x)$, Eugene *et al.* [14] defined the beta-G(a, b) density function by

$$f(x) = \frac{1}{B(a, b)} G(x)^{a-1}\{1 - G(x)\}^{b-1} g(x), \tag{33}$$

where $a > 0$ and $b > 0$ are two additional parameters to those parameters of G and $B(a, b) = \Gamma(a + b)/[\Gamma(a)\Gamma(b)]$ is the beta function. Eugene *et al.* [14] and Nadarajah and Kotz [11,15] defined the beta normal, beta Gumbel and beta exponencial distributions by taking $G(x)$ as the cdf of the normal, Gumbel and exponential distributions. We can easily see that if $Z \sim$ Exp-G(a) with cdf $H_a(x) = G(x)^a$, then the beta-H$_a(1, b)$ distribution is identical to the Kw-G(a, b) distribution.

So, the beta-G$(1, b)$ distribution applied to the Exp-G(a) yields the Kw-G(a, b) distribution. Some properties of special Kw-G sub-models can be derived in this way. Clearly, if Z has the beta-G$(1, b)$ distribution, then $X = G^{-1}(Z^{1/a})$ will have the Kw-G(a, b) distribution.

Now, we obtain some properties of the Kw-exponential(a, b) using the results of Barreto-Souza *et al.* [16] who investigated the beta exponentiated exponential distribution. Consider the exponential distribution with parameter λ. The properties of the Kw-exponential(a, b, λ) can be immediately derived from those of the four parameter beta exponentiated exponential distribution, defined here as a three parameter BEE$(1, b, \lambda, a)$ model [16]. The density function of the Kw-exponential random variable X (with parameters a, b and λ) is

$$f(x) = ab\lambda\, \mathrm{e}^{-\lambda x}(1 - \mathrm{e}^{-\lambda x})^{a-1}\{1 - (1 - \mathrm{e}^{-\lambda x})^{a}\}^{b-1}, \quad x > 0. \tag{34}$$

For a $b > 0$ a real non-integer, the moments of X are

$$\mu_r' = E(X^r) = \frac{a\Gamma(b+1)}{\lambda^r} \sum_{j=0}^{\infty} (-1)^{j+r} \binom{b-1}{j} \left.\frac{\mathrm{d}^r B(p, a(j+1))}{\mathrm{d}p^r}\right|_{p=1}.$$

If $b > 0$ is an integer, then the above sum stops at $b - 1$.

When $t < \lambda$, the mgf follows immediately from Barreto-Souza *et al.* [16] for a $b > 0$ real non-integer as

$$M(t) = ab \sum_{j=0}^{\infty} \binom{b-1}{j} (-1)^j B\left(\frac{1-t}{\lambda, a(j+1)}\right).$$

If $b > 0$ is an integer, then assuming $t < \lambda$, the above sum stops at $b - 1$.

The rth moment of $X_{i:n}$, for a $b > 0$ real non-integer becomes

$$E(X_{i:n}^r) = \sum_{k=0}^{n-i} \sum_{m_1=0}^{\infty} \ldots \sum_{m_{k+i-1}=0}^{\infty} \delta_k E(X_k^r)$$

where

$$\delta_k = \frac{(-1)^{k+\sum_{j=1}^{k+i-1} m_j} \binom{n-i}{k} B(\alpha\{a(i+1) + \sum_{j=1}^{k+i-1} m_j\}, b)}{B(a, b)^{k+i} B(i, n-i+1)} \prod_{j=1}^{k+j-1} \frac{\binom{b-1}{m_j}}{(a + m_j)}$$

and $X_k \sim$ BGE$(a\{(i+1) + \sum_{j=1}^{k+i-1} m_j\}, b, \lambda, a)$. For a $b > 0$ integer, the indices in the above sums stop at $b - 1$. Clearly, the properties for any Kw-G distribution are required from our previous results if there is no beta-H construction for H defined as an exponentiated-G distribution.

7. Estimation

Here, we consider estimation by the method of maximum likelihood and provide expressions for the associated Fisher information matrix. We also consider estimation issues for censored data.

Suppose $x_1, \ldots, x_n$ is a random sample from the Kw-G distribution (2). Suppose too that g is parameterized by a vector $\boldsymbol{\theta}$ of length p. The log-likelihood (LL) function of the parameters,

$(a, b, \boldsymbol{\theta})$, is

$$\log L(a, b, \boldsymbol{\theta}) = n \log a + n \log b + \sum_{j=1}^{n} \log g(x_j; \boldsymbol{\theta}) + (a-1) \sum_{j=1}^{n} \log G(x_j; \boldsymbol{\theta}) + (b-1) \sum_{j=1}^{n} \log\{1 - G^a(x_j; \boldsymbol{\theta})\}. \quad (35)$$

The maximum likelihood estimates (MLEs) are the simultaneous solutions of the equations:

$$\frac{n}{a} + \sum_{j=1}^{n} \log G(x_j; \boldsymbol{\theta}) - (b-1) \sum_{j=1}^{n} \frac{G^a(x_j; \boldsymbol{\theta}) \log G(x_j; \boldsymbol{\theta})}{1 - G^a(x_j; \boldsymbol{\theta})} = 0,$$

$$\frac{n}{b} + \sum_{j=1}^{n} \log\{1 - G^a(x_j; \boldsymbol{\theta})\} = 0$$

and

$$\sum_{j=1}^{n} \frac{1}{g(x_j; \boldsymbol{\theta})} \frac{\partial g(x_j; \boldsymbol{\theta})}{\partial \theta_k} + (a-1) \sum_{j=1}^{n} \frac{1}{G(x_j; \boldsymbol{\theta})} \frac{\partial G(x_j; \boldsymbol{\theta})}{\partial \theta_k} - a(b-1) \sum_{j=1}^{n} \frac{G^{a-1}(x_j; \boldsymbol{\theta})}{1 - G^a(x_j; \boldsymbol{\theta})} \frac{\partial G(x_j; \boldsymbol{\theta})}{\partial \theta_k} = 0.$$

For an interval estimation of the parameters, $(a, b, \boldsymbol{\theta})$, and tests of hypotheses, one requires the Fisher information matrix. Interval estimation for the model parameters can be obtainable with standard likelihood theory. The elements of this matrix for Equation (35) can be worked out as

$$E\left(-\frac{\partial^2 \log L}{\partial a^2}\right) = \frac{n}{a^2} + nE\left[\frac{(\log G(X; \boldsymbol{\theta}))^2 G^a(X; \boldsymbol{\theta})}{\{1 - G^a(X; \boldsymbol{\theta})\}^2}\right],$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial b}\right) = nE\left[\frac{\log G(X; \boldsymbol{\theta}) G^a(X; \boldsymbol{\theta})}{1 - G^a(X; \boldsymbol{\theta})}\right],$$

$$E\left(-\frac{\partial^2 \log L}{\partial a \partial \theta_j}\right) = -nE\left[\frac{1}{G(X; \boldsymbol{\theta})} \frac{\partial G(X; \boldsymbol{\theta})}{\partial \theta_j}\right] + n(b-1)E\left[\frac{G^{a-1}(X; \boldsymbol{\theta})\{a \log G(X; \boldsymbol{\theta}) + 1\}}{1 - G^a(X; \boldsymbol{\theta})} \frac{\partial G(X; \boldsymbol{\theta})}{\partial \theta_j}\right] - naE\left[\frac{G^{2a-1}(X; \boldsymbol{\theta}) \log G(X; \boldsymbol{\theta})}{\{1 - G^a(X; \boldsymbol{\theta})\}^2} \frac{\partial G(X; \boldsymbol{\theta})}{\partial \theta_j}\right],$$

$$E\left(-\frac{\partial^2 \log L}{\partial b^2}\right) = \frac{n}{b^2},$$

$$E\left(-\frac{\partial^2 \log L}{\partial b \partial \theta_j}\right) = naE\left[\frac{G^{a-1}(X; \boldsymbol{\theta})}{1 - G^a(X; \boldsymbol{\theta})} \frac{\partial G(X; \boldsymbol{\theta})}{\partial \theta_j}\right]$$

and

$$E\left(-\frac{\partial^2 \log L}{\partial\theta_j\partial\theta_k}\right) = nE\left[\frac{1}{g^2(X;\boldsymbol{\theta})}\frac{\partial g(X;\boldsymbol{\theta})}{\partial\theta_j}\frac{\partial g(X;\boldsymbol{\theta})}{\partial\theta_k}\right]$$
$$+ n(a-1)E\left[\frac{1}{G^2(X;\boldsymbol{\theta})}\frac{\partial G(X;\boldsymbol{\theta})}{\partial\theta_j}\frac{\partial G(X;\boldsymbol{\theta})}{\partial\theta_k}\right]$$
$$- nE\left[\frac{1}{g(X;\boldsymbol{\theta})}\frac{\partial^2 g(X;\boldsymbol{\theta})}{\partial\theta_j\partial\theta_k}\right] - nE\left[\frac{1}{G(X;\boldsymbol{\theta})}\frac{\partial^2 G(X;\boldsymbol{\theta})}{\partial\theta_j\partial\theta_k}\right]$$
$$+ na(a-1)(b-1)E\left[\frac{G^{a-2}(X;\boldsymbol{\theta})}{1-G^a(X;\boldsymbol{\theta})}\frac{\partial G(X;\boldsymbol{\theta})}{\partial\theta_j}\frac{\partial G(X;\boldsymbol{\theta})}{\partial\theta_k}\right]$$
$$+ na^2(b-1)E\left[\frac{G^{2(a-1)}(X;\boldsymbol{\theta})}{\{1-G^a(X;\boldsymbol{\theta})\}^2}\frac{\partial G(X;\boldsymbol{\theta})}{\partial\theta_j}\frac{\partial G(X;\boldsymbol{\theta})}{\partial\theta_k}\right]$$
$$+ na(b-1)E\left[\frac{G^{a-1}(X;\boldsymbol{\theta})}{1-G^a(X;\boldsymbol{\theta})}\frac{\partial^2 G(X;\boldsymbol{\theta})}{\partial\theta_j\partial\theta_k}\right].$$

The expectations in the first two elements can be calculated as

$$E\left[\frac{(\log G(X;\boldsymbol{\theta}))^2 G^a(X;\boldsymbol{\theta})}{\{1-G^a(X;\boldsymbol{\theta})\}^2}\right] = \frac{bB(2,b-2)}{6a^2} \times N,$$

where $N = \{\pi^2 - 6\psi^{'}(b) - 12C - 12\psi(b) + 6C^2 + 12C\psi(b) + 6\psi^2(b)\}$ and

$$E\left[\frac{\log G(X;\boldsymbol{\theta}) G^a(X;\boldsymbol{\theta})}{1-G^a(X;\boldsymbol{\theta})}\right] = \frac{bB(2,b-1)}{a}\{1 - C - \psi(b+1)\}.$$

The remaining expectations can be computed numerically.

Often with lifetime data, one encounters censored data. There are different forms of censoring: type I censoring, type II censoring, etc. Here, we consider the general case of multicensored data: there are n subjects of which

- n_0 are known to have failed at the times $x_1, \ldots, x_{n_0}$.
- n_1 are known to have failed in the interval $[s_{j-1}, s_j]$, $j = 1, \ldots, n_1$.
- n_2 survived to a time r_j, $j = 1, \ldots, n_2$ but not observed any longer.

Note that $n = n_0 + n_1 + n_2$. Note too that type I censoring and type II censoring are contained as particular cases of multicensoring. The LL function of the parameters, $(a, b, \boldsymbol{\theta})$, for this multicensoring data is

$$\log L(a,b,\boldsymbol{\theta}) = n_0 \log a + n_0 \log b + \sum_{j=1}^{n_0} \log g(x_j;\boldsymbol{\theta}) + (a-1)\sum_{j=1}^{n_0} \log G(x_j;\boldsymbol{\theta})$$
$$+ (b-1)\sum_{j=1}^{n_0} \log\{1 - G^a(x_j;\boldsymbol{\theta})\}$$
$$+ \sum_{j=1}^{n_1} \log\{[1 - G^a(s_{j-1};\boldsymbol{\theta})]^b - [1 - G^a(s_j;\boldsymbol{\theta})]^b\}$$
$$+ b\sum_{j=1}^{n_2} \log\{1 - G^a(r_j;\boldsymbol{\theta})\}. \quad (36)$$

It follows that the MLEs are the simultaneous solutions of the equations:

$$\frac{n_0}{a} + \sum_{j=1}^{n_0} \log G(x_j;\boldsymbol{\theta}) - (b-1)\sum_{j=1}^{n_0} \frac{G^a(x_j;\boldsymbol{\theta})\log G(x_j;\boldsymbol{\theta})}{1-G^a(x_j;\boldsymbol{\theta})}$$
$$+ b\sum_{j=1}^{n_1} \frac{U(s_j;\boldsymbol{\theta}) - U(s_{j-1};\boldsymbol{\theta})}{[1-G^a(s_{j-1};\boldsymbol{\theta})]^b - [1-G^a(s_j;\boldsymbol{\theta})]^b} - b\sum_{j=1}^{n_2} \frac{G^a(r_j;\boldsymbol{\theta})\log G(r_j;\boldsymbol{\theta})}{1-G^a(r_j;\boldsymbol{\theta})} = 0,$$

$$\frac{n_0}{b} + \sum_{j=1}^{n_0} \log\{1-G^a(x_j;\boldsymbol{\theta})\} - \sum_{j=1}^{n_1} \frac{V(s_j;\boldsymbol{\theta}) - V(s_{j-1};\boldsymbol{\theta})}{[1-G^a(s_{j-1};\boldsymbol{\theta})]^b - [1-G^a(s_j;\boldsymbol{\theta})]^b}$$
$$+ \sum_{j=1}^{n_2} \log\{1-G^a(r_j;\boldsymbol{\theta})\} = 0$$

and

$$\sum_{j=1}^{n_0} \frac{1}{g(x_j;\boldsymbol{\theta})}\frac{\partial g(x_j;\boldsymbol{\theta})}{\partial\theta_k} + (a-1)\sum_{j=1}^{n_0} \frac{1}{G(x_j;\boldsymbol{\theta})}\frac{\partial G(x_j;\boldsymbol{\theta})}{\partial\theta_k}$$
$$- a(b-1)\sum_{j=1}^{n_0} \frac{G^{a-1}(x_j;\boldsymbol{\theta})}{1-G^a(x_j;\boldsymbol{\theta})}\frac{\partial G(x_j;\boldsymbol{\theta})}{\partial\theta_k} + ab\sum_{j=1}^{n_1} \frac{W(s_j;\boldsymbol{\theta}) - W(s_{j-1};\boldsymbol{\theta})}{[1-G^a(s_{j-1};\boldsymbol{\theta})]^b - [1-G^a(s_j;\boldsymbol{\theta})]^b}$$
$$- ab\sum_{j=1}^{n_2} \frac{G^{a-1}(r_j;\boldsymbol{\theta})\partial G(r_j;\boldsymbol{\theta})/\partial\theta_k}{1-G^a(r_j;\boldsymbol{\theta})} = 0,$$

where $U(s) = \{1-G^a(s)\}^{b-1}G^a(s)\log G(s)$, $V(s) = \{1-G^a(s)\}^b \log\{1-G^a(s)\}$ and $W(s) = \{1-G^a(s)\}^{b-1}G^{a-1}(s)\partial G(s)/\partial\theta_k$. The Fisher information matrix corresponding to Equation (36) is too complicated to be presented here.

8. Applications

We illustrate the superiority of some new Kw-G distributions proposed here as compared with some of their sub-models. We give two applications (uncensored and censored data) using well-known data sets to demonstrate the applicability of the proposed regression model.

8.1. *Uncensored data: voltage*

Here, we compare the results of the fits of some distributions to a data set [17, p. 383], which gives the times of failure and running times for a sample of devices from a field-tracking study of a larger system. At a certain point in time, 30 units were installed in normal service conditions. Two causes of failure were observed for each unit that failed: the failure caused by an accumulation of randomly occurring damage from power-line voltage spikes during electric storms and failure caused by normal product wear. The required numerical evaluations were implemented using the SAS procedure NLMIXED. Table 1 lists the MLEs (and the corresponding standard errors in parentheses) of the parameters and the values of the following statistics for some fitted models: AIC (Akaike information criterion), BIC (Bayesian information criterion) and CAIC (Consistent Akaike information criterion). These results indicate that the Kw-Weibull model has the lowest

Table 1. MLEs of the parameters for some fitted models to the voltage data (the standard errors are given in parentheses) and the values of the AIC, CAIC and BIC statistics.

Model	a	b	λ_2	α_2	β_2	AIC	CAIC	BIC
Kw-XTG	0.0725 (0.0169)	0.6298 (0.0125)	1.01e−6 (0.0000)	209.43 (0.0027)	6.0854 (0.1707)	337.8	340.3	344.8
XTG	1	1	0.0016 (0.0007)	85.4922 (9.8953)	0.8020 (0.2543)	364.4	365.3	368.6
	a	b	α_3	β_3				
Kw-FW	0.0603 (0.0004)	0.0738 (0.0134)	0.0115 (0.00003)	69.0275 (2.0281)		356.4	358.0	362.0
FW	1	1	0.0033 (0.0005)	15.8889 (5.2693)		387.6	388.1	390.4
	a	b	λ_1	β_1				
Kw-Chen	0.1051 (0.0437)	0.3855 (0.1762)	1e−8 (1e−10)	0.5165 (0.0059)		357.5	359.1	363.1
Chen	1	1	0.0051 (0.0034)	0.3125 (0.0205)		366.0	366.5	368.8
	a	b	c	β				
Kw-Weibull	0.0516 (0.0241)	0.2288 (0.0905)	7.7026 (0.2191)	0.0043 (0.0003)		352.3	353.9	357.9
	a	b	c	λ				
Beta Weibull	0.1467 (0.0280)	30.0404 (2.3411)	6.3920 (0.1765)	0.0017 (0.0002)		362.6	364.2	368.2

AIC, CAIC and BIC values among all fitted models, and so it could be chosen as the best model. The beta Weibull density is given by

$$f(x) = \frac{c\lambda^c}{B(a,b)} x^{c-1} \exp\{-b(\lambda x)^c\}[1 - \exp\{-(\lambda x)^c\}]^{a-1}, \quad x, a, b, c, \lambda > 0.$$

In order to assess whether the model is appropriate, plots of the histogram of the data and the fitted Kw-XGT, XGT, Kw-FW, FW, Kw-Chen, Chen, Kw-Weibull and beta Weibull distributions are given in Figure 12. Figure 13 provides the empirical and estimated survival functions of these distributions. We conclude that the Kw-XGT distribution fits well to these data.

8.2. *Censored data: radiotherapy*

The data set refers to the survival time (days) for cancer patients ($n = 51$) undergoing radiotherapy [18]. The percentage of censored observations is 17.65%. Thus, the Kw-G family seems to be

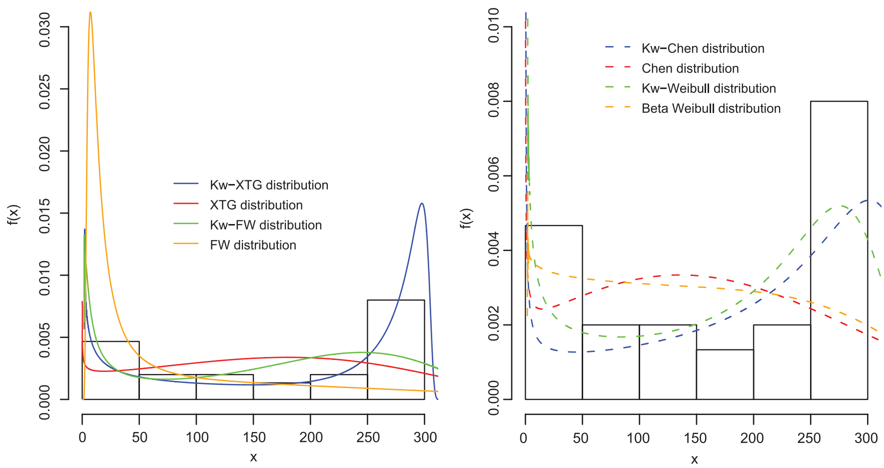

Figure 12. Estimated densities for some models fitted to the voltage data.

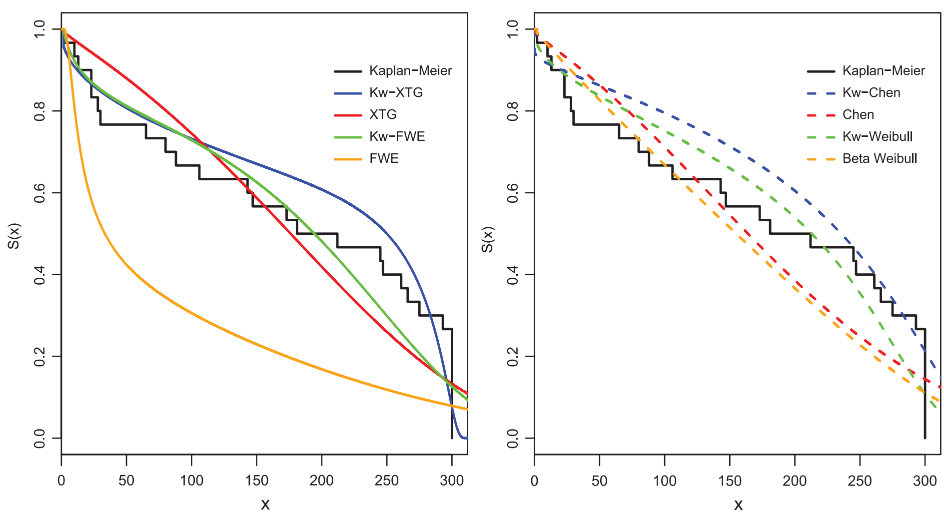

Figure 13. Estimated survival functions for some models and the empirical survival function for voltage data.

an appropriate model for fitting such data. Table 2 lists the MLEs of the model parameters. The values of the three statistics are smaller for the Kw-Weibull, Kw-Chen and Kw-XGT distributions compared with those values of the other distributions. In these terms, the three distributions are very competitive models for lifetime data analysis. In Figure 14, we plot the empirical survival function and the estimated survival function for the Kw-XGT, Kw-Chen and Kw-Weibull distributions which give satisfactory fits.

9. Multivariate generalization

The Kw-G distribution defined by Equation (3) can be generalized to the bivariate and multivariate cases in a natural way. Consider the bivariate case for simplicity. Let G denote a bivariate cdf on $(0, \infty) \times (0, \infty)$ with joint pdf g, marginal pdfs g_i, $i = 1, 2$ and marginal cdfs G_i, $i = 1, 2$. Then a bivariate Kw-G distribution can defined by the cdf

$$F(x_1, x_2) = 1 - \{1 - G^a(x_1, x_2)\}^b \tag{37}$$

for $a > 0$ and $b > 0$. The marginal pdfs f_i, $i = 1, 2$ and marginal cdfs F_i, $i = 1, 2$ of F are

$$f_i(x) = abg_i(x)G_i^{a-1}(x)\{1 - G_i^a(x)\}^{b-1}$$

and

$$F_i(x) = 1 - \{1 - G_i^a(x)\}^b$$

for $i = 1, 2$. The conditional cdfs of F are

$$F(x_2 \mid x_1) = \frac{1 - \{1 - G^a(x_1, x_2)\}^b}{1 - \{1 - G_1^a(x_1)\}^b}$$

and

$$F(x_1 \mid x_2) = \frac{1 - \{1 - G^a(x_1, x_2)\}^b}{1 - \{1 - G_2^a(x_2)\}^b}.$$

The joint pdf of F is

$$f(x_1, x_2) = \frac{abG^{a-2}(x_1, x_2)\{A(x_1, x_2) + B(x_1, x_2) + C(x_1, x_2)\}}{\{1 - G^a(x_1, x_2)\}^{1-b}},$$

where

$$A(x_1, x_2) = -\frac{a(b-1)G^a(x_1, x_2)}{1 - G^a(x_1, x_2)} \frac{\partial G(x_1, x_2)}{\partial x_1} \frac{\partial G(x_1, x_2)}{\partial x_2},$$

$$B(x_1, x_2) = (a-1)\frac{\partial G(x_1, x_2)}{\partial x_1} \frac{\partial G(x_1, x_2)}{\partial x_2}$$

and

$$C(x_1, x_2) = G(x_1, x_2)g(x_1, x_2).$$

The marginal pdfs of F are

$$f(x_1 \mid x_2) = \frac{\{1 - G_2^a(x_2)\}^{1-b}G^{a-2}(x_1, x_2)\{A(x_1, x_2) + B(x_1, x_2) + CA(x_1, x_2)\}}{\{1 - G^a(x_1, x_2)\}^{1-b}g_2(x_2)G_2^{a-1}(x_2)}$$

Table 2. MLEs of the parameters for the Kw-G family fitted to the radiotherapy data (the standard errors are in parentheses) and the value of the AIC, CAIC and BIC statistics.

Model	a	b	λ_2	α_2	β_2	AIC	CAIC	BIC
Kw-XTG	89.0008 (7.3293)	1.3733 (0.1554)	3.1464 (1.4881)	0.3046 (0.1397)	0.0837 (0.0102)	596.2	597.5	605.9
XTG	1	1	20.8961 (2.0322)	0.00002 ($18e-6$)	0.1212 (0.0011)	605.5	606.0	611.3
	a	b	α_3	β_3				
Kw-FW	0 .3069 (0.0841)	0.5716 (0.1646)	0.0009 (0.0003)	254.76 (7.0300)		606.9	607.8	614.6
FW	1	1	0.0007 (0.0002)	109.43 (18.1249)		609.7	610.0	613.6
	a	b	λ_1	β_1				
Kw-Chen	105.43 (47.67)	1.4934 (0.8845)	1.2080 (0.5839)	0.0833 (0.0013)		594.2	595.1	601.9
Chen	1	1	0.0185 (0.0067)	0.2260 (0.0119)		608.0	608.3	611.9
	a	b	c	β				
Kw-Weibull	21.5936 (4.7489)	1.1589 (0.6803)	0.2668 (0.0447)	0.3617 (0.1307)		594.1	595.0	601.8
	a	b	c	λ				
Beta Weibull	0.00004 (0.000012)	0.7021 (0.3196)	1.7670 (0.4176)	34.7002 (4.84070)		599.2	600.0	606.9

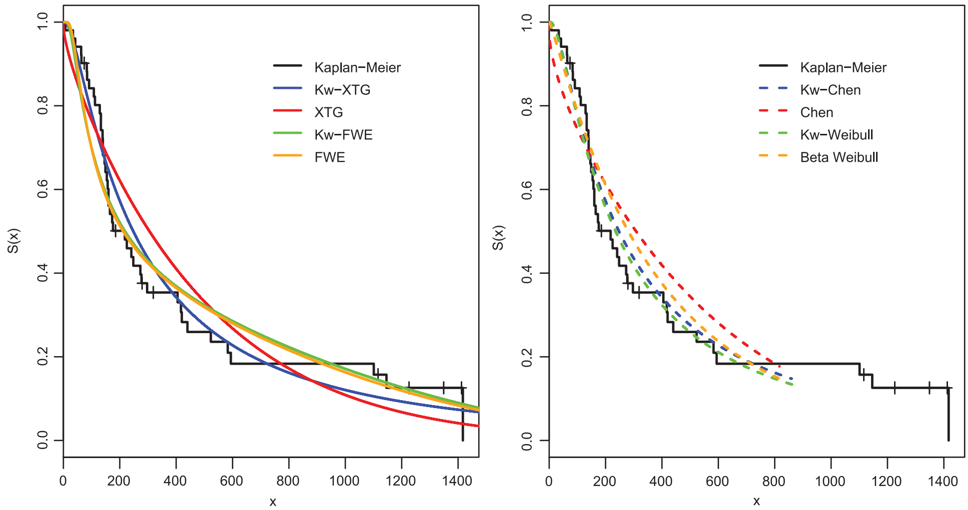

Figure 14. Estimated survival functions for some fitted models and the empirical survival function for radiotherapy data.

and

$$f(x_2 \mid x_1) = \frac{\{1 - G_1^a(x_1)\}^{1-b} G^{a-2}(x_1, x_2)\{A(x_1, x_2) + B(x_1, x_2) + CA(x_1, x_2)\}}{\{1 - G^a(x_1, x_2)\}^{1-b} g_1(x_1) G_1^{a-1}(x_1)}.$$

The properties of Equation (37) can be studied as in Sections 2–15. For instance, if G is parameterized by a vector $\boldsymbol{\theta}$ of length p, then the MLEs of the parameters, $(a, b, \boldsymbol{\theta})$, are the simultaneous solutions of the equations:

$$\frac{n}{a} + \sum_{j=1}^{n} \log G(x_{1j}, x_{2j}; \boldsymbol{\theta}) - (b-1) \sum_{j=1}^{n} \frac{G^a(x_{1j}, x_{2j}; \boldsymbol{\theta}) \log G(x_{1j}, x_{2j}; \boldsymbol{\theta})}{1 - G^a(x_{1j}, x_{2j}; \boldsymbol{\theta})} = 0,$$

$$\frac{n}{b} + \sum_{j=1}^{n} \log\{1 - G^a(x_{1j}, x_{2j}; \boldsymbol{\theta})\} = 0$$

and

$$\sum_{j=1}^{n} \frac{1}{g(x_{1j}, x_{2j}; \boldsymbol{\theta})} \frac{\partial g(x_{1j}, x_{2j}; \boldsymbol{\theta})}{\partial \theta_k} + (a-1) \sum_{j=1}^{n} \frac{1}{G(x_{1j}, x_{2j}; \boldsymbol{\theta})} \frac{\partial G(x_{1j}, x_{2j}; \boldsymbol{\theta})}{\partial \theta_k}$$
$$- a(b-1) \sum_{j=1}^{n} \frac{G^{a-1}(x_{1j}, x_{2j}; \boldsymbol{\theta})}{1 - G^a(x_{1j}, x_{2j}; \boldsymbol{\theta})} \frac{\partial G(x_{1j}, x_{2j}; \boldsymbol{\theta})}{\partial \theta_k} = 0,$$

where $(x_{11}, x_{21}), (x_{12}, x_{22}), \ldots, (x_{1n}, x_{2n})$ is a random sample from Equation (37). We hope to provide a comprehensive treatment of the bivariate and multivariate cases in a future paper.

10. Conclusions

Cordeiro and de Castro [3] proposed the Kw-G family of distributions to extend several widely known distributions such as the normal, Weibull, gamma and Gumbel distributions. We demonstrate that the probability density function (pdf) of any Kw-G distribution can be expressed as a linear combination of exponentiated-G density functions. This result allows us to derive general explicit expressions for several measures of the Kw-G distributions such as moments, generating function, mean deviations, Bonferroni and Lorenz curves, Shannon entropy, Rénvy entropy and reliability. Further, we demonstrate that pdf of the Kw-G order statistics can be expressed as a linear combination of exponentiated-G density functions. Our formulas related with any Kw-G model are manageable, and with the use of modern computer resources with analytic and numerical capabilities, may turn into adequate tools comprising the arsenal of applied statisticians.

The estimation of parameters is approached by the method of maximum likelihood. The usefulness of the Kw-G models is illustrated in two analysis of real data using the AIC, BIC and CAIC. Applications of some Kw-G distributions to two real data sets are given to show their usefulness. In conclusion, the Kw-G family of distributions provides a rather flexible mechanism for fitting a wide spectrum of positive real world data.

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