Discussions on Professor Fraser's article on "Is Bayes posterior just quick and dirty confidence?"

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We congratulate Professor Fraser for this very engaging article. It gives us an opportunity to gaze at the past and future of Bayes and confidence. It is well known that a Bayes posterior can only provide credible intervals and has no assurance of frequentist coverage (known as confidence). Professor Fraser's article provides a detailed and insightful exploration into the root of this issue. It turns out that the Bayes posterior is exactly a confidence in the linear case (a mathematics coincidence), and Professor Fraser's insightful and far-reaching examples demonstrate how the departure from linearity induces the departure of a posterior, in a proportionate way, from being a confidence. Of course, Bayesian inference is not bounded by frequestist criteria or geared to provide confidence statements, even though in some applications researchers have treated the Bayes credible intervals as confidence intervals on asymptotic grounds. It is debatable whether this departure of Bayesian inference from confidence should be a concern or not. But, nevertheless, the article provides us a powerful exploration and demonstration which can help us better comprehend the two statistical philosophies and the 250-year debate between Bayesians and frequentists.

In the midst of the 250-year debate, Fisher's "fiducial distribution" played a prominent role, which however is now referred to as the "biggest blunder" of the father of modern statistical inference [1]. Both developments of the confidence distribution and Fisher's fiducial distribution share the common goal of providing distribution estimation for parameters without using priors, and their performances are often judged by the (asymptotic or exact) probability coverage of their corresponding intervals. Maybe partly due to this reason and partly due to Fisher's "feud" and "longtime dispute" with Neyman (cf., [13]), the confidence distribution has historically often been misconstrued as a fiducial concept. On page 10, Professor Fraser states that "In the frequentist framework, the function $p(\theta)$ can be viewed as a distribution of confidence, as introduced by Fisher [3] but originally called fiducial." It seems to suggest that the concept of confidence distribution is exchangeable with Fisher's fiducial distribution. But recent resurging interest and research on confidence distributions calls for a disagreement with this more classical assertion. We would like to take the opportunity to raise this point for discussion. Professor Fraser has more in-depth understanding of the issue and he may well wish to correct us, if we are mistaken or have missed something.

First of all, in the recent developments on confidence distributions, the concept is developed strictly within the frequentist domain and resides entirely within the frequentist logic, and there is no involvement of any fiducial reasoning; see, e.g. [6], [8], [9]. This can in fact also be seen in all of Professor Fraser's illustrative examples in the article, in which no fiducial argument is adopted. To us, a confidence distribution, which uses a sample dependent distribution on the parameter space to estimate the parameter of interest, is no different from a point estimator, which uses a (sample dependent) point in the parameter space to estimate the parameter of interest. Neither is it different from a confidence interval, which uses two sample dependent points in the parameter space to estimate the parameter of interest. In this interpretation, a confidence distribution is no longer viewed as an inherent distribution of the fixed (nonrandom) parameter θ and, unlike the fiducial distribution, it is a probability distribution in the frequentist sense. The nice thing about treating confidence distribution as a purely frequentist concept is that the confidence distribution is now a clean and coherent frequestist concept (similar to a point estimator) and it frees itself from those restrictive, if not controversial, constraints set forth by Fisher on fiducial distributions. Table 1 uses an analogy to describe the relation between the new concept of confidence distribution and the fiducial distribution. A similar analogy was also described in [12] and [10].

The concept of confidence distribution has attracted a surge of renewed attention in recent years. The renewed interest in confidence distributions starts with Efron [2], who asserted that bootstrap distributions are "distribution estimators" and "confidence distributions". He predicted that "something like fiducial inference" may "become a big hit in the 21st century". The goal of these new developments is not to derive any new fiducial inference that is paradox free. Rather, it is to provide useful statistical inference tools for problems where frequentist methods with good properties were previously unavailable or hard to obtain. It seems relevant, without going into details of specific examples, to indicate a variety of recent studies involving confidence distributions, ranging from confidence distribution and its inference, approximate likelihood inference, incorporation of expert opinions in a frequesntist setting, combination of information from independent studies, confidence curves, to applications in survival analysis and others. We refer interested readers to [2], [6], [8], [11] and also a review article [10].

As pointed out by Professor Fraser, confidence distribution is a very old concept first suggested by Neyman in 1937 [5] and some similar ideas can be traced back even earlier to Bayes [1] and Fisher [3]. A nagging question that comes to our mind is why is this concept largely unknown in the statistical community and why have statisticians never regarded it as a valuable tool? We believe that inference based on confidence distributions deserve a place in the statistician's toolbox and that distributional inference by confidence distributions should be more widely known and used. Many recent research activities on the topic are aimed at achieving just that. As for Bayesian analysis, confidence or not, the impact on sciences of this seemingly modest discovery of Thomas Bayes, in terms of updating (revising) information in light of new data evidence, is nothing short of miraculous. With the advent of Bayesian learning, its future couldn't be brighter. Let us be grateful to Professor Fraser for giving us this opportunity to revisit and examine the past and future of Bayes and confidence.

References:

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Table 1: An analogy between "confidence distribution versus fiducial distribution" and "consistent and asymptotically efficient point estimator versus MLE"

	CD ("Distribution Estimation")	Analogy in Point Estimation
CD definition & Definition of point estimator	Any sample dependent distribution function on the parameter space can in principle be used to estimate the parameter, but we impose a certain requirement (i.e., as a function of the sample, the CD is Uniform[0,1]-distributed at the true parameter value; see, e.g., [6] & [8]) to ensure that the statistical inferences (e.g., point estimates, confidence intervals, p-values, etc) derived from the CD have desired frequentist property.	Any single point (a real value or a statistic) on the parameter space can in principle be used to estimate a parameter, but we impose restrictions so that the point estimator can have certain desired properties, such as unbiasedness, consistency, efficiency etc.
CD *versus* Fiducial distribution & Consistent and asymptotically efficient estimator *versus* MLE	Under some suitable conditions, fiducial distributions satisfy the frequentist coverage property (see, e.g., [3]), which typically make them CD functions. Thus, the fiducial approach can provide a standard procedure to obtain a CD function.	Under some regularity conditions, the MLEs typically have certain desired frequentist properties (e.g., consistency, asymptotic efficiency, etc.) Thus, the MLE approach provides a standard procedure to obtain desirable point estimators.
	A CD does not have to be a fiducial distribution or involve any fiducial reasoning.	A point estimator with desirable properties does not have to be an MLE.