DENSITY CARDINALS

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Abstract. How many permutations are needed so that every infinite-coinfinite set of natural numbers with asymptotic density can be rearranged to no longer have the same density? We prove that the density number $\mathfrak{dd}$, which answers this question, is equal to the least size of a non-meager set of reals, $\mathsf{non}(\mathcal{M})$. The same argument shows that a slight modification of the rearrangement number $\mathfrak{rr}$ of [7] is equal to $\mathsf{non}(\mathcal{M})$, and similarly for a cardinal invariant related to large-scale topology introduced by Banakh [3], thus answering a question of the latter. We then consider variants of $\mathfrak{dd}$ given by restricting the possible densities of the original set and / or of the permuted set, providing lower and upper bounds for these cardinals and proving consistency of strict inequalities. We finally look at cardinals defined in terms of relative density and of asymptotic mean, and relate them to the rearrangement numbers of [7].

1. Introduction

Asymptotic density of sets of natural numbers plays an important role in many areas of mathematics; it is also closely connected to Lebesgue measure on the real numbers. For quite some time, density has been investigated from the point of view of set theory: the density ideal $\mathcal{Z}$ is one of the classical analytic P-ideals, which have been characterized by Solecki [18] using lower semicontinuous submeasures, and its cardinal invariants as well as those of the quotient algebra $\mathcal{P}(\omega)/\mathcal{Z}$ have been extensively studied (see e.g. [16] or [17]). More recently, variants of the splitting number and the reaping number defined in terms of density have been introduced and compared with other cardinal invariants of the continuum ([11], see also [13] and [19]). Here we look at density from another point of view: our starting point is the simple observation, inspired by Riemann's Rearrangement Theorem, that any infinite-coinfinite set of natural numbers can be rearranged into a set without density or with arbitrary density in the closed interval $[0, 1]$. The question of how many permutations are necessary to be able to rearrange every infinite-coinfinite set naturally leads to cardinal invariants (which we will call *density numbers*) analogous to the rearrangement numbers studied in [7] (see also [9] for closely related work). We start with basic definitions.

For $A \subseteq \omega$ let

$$d_n(A) = \frac{|A \cap n|}{n}$$

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and define the *lower density*

$$\underline{d}(A) = \liminf_{n\to\infty} d_n(A)$$

and the *upper density*

$$\overline{d}(A) = \limsup_{n\to\infty} d_n(A)$$

of A. If $\underline{d}(A) = \overline{d}(A)$ we call the common value $d(A)$ the *(asymptotic) density* of A. Otherwise A does not have (asymptotic) density. Notice that finite sets have asymptotic density equal to 0 and cofinite sets have asymptotic density equal to 1. On the other hand, the density of infinite-coinfinite sets can be changed by permutations of ω as shown by the following simple result:

Proposition 1. *Given an infinite and coinfinite set A, we have:*

(1) For every $r \in [0,1]$, there is $\pi \in \mathrm{Sym}(\omega)$ such that $d(\pi[A]) = r$;
(2) There is $\pi \in \mathrm{Sym}(\omega)$ such that $\pi[A]$ doesn't admit asymptotic density.

This is reminiscent of Riemann's Rearrangement Theorem:

Theorem 2 (Riemann)**.** *Given a conditionally convergent (c.c. for short) series $\sum_n a_n$, we have:*

(1) For every $r \in \mathbb{R}\cup\{-\infty,+\infty\}$, there is $\pi \in \mathrm{Sym}(\omega)$ such that $\sum_n a_{\pi(n)} = r$;
(2) There is $\pi \in \mathrm{Sym}(\omega)$ such that $\sum_n a_{\pi(n)}$ diverges by oscillation.

The latter means that $-\infty \le \liminf_k \sum_{n=0}^k a_n < \limsup_k \sum_{n=0}^k a_n \le +\infty$. Riemann's Theorem motivates the following definition from [7].

Definition 3. The *rearrangement number* $\mathfrak{rr}$ is the smallest cardinality of a family $\Pi \subseteq \mathrm{Sym}(\omega)$ such that for every c.c. series $\sum_n a_n$ there is $\pi \in \Pi$ such that $\sum_n a_{\pi(n)} \neq \sum_n a_n$ (where we allow the possibility that $\sum_n a_{\pi(n)}$ converges to $\pm\infty$ or diverges by oscillation).

By Proposition 1 we may define analogously:

Definition 4. The *(asymptotic) density number* $\mathfrak{dd}$ is the smallest cardinality of a family $\Pi \subseteq \mathrm{Sym}(\omega)$ such that for every infinite-coinfinite set $A \subseteq \omega$ with asymptotic density there is $\pi \in \Pi$ such that $d(\pi[A]) \neq d(A)$, i.e., either $\pi[A]$ doesn't admit asymptotic density or $\pi[A]$ has asymptotic density distinct from $d(A)$.

One may wonder why we require A to have asymptotic density though this is irrelevant in the Proposition. This is because we want to make the definition analogous to $\mathfrak{rr}$. Alternatively, Riemann's Theorem can be reformulated as follows. Call a series $\sum_n a_n$ *potentially conditionally convergent* (*p.c.c.* for short) [7, Definition 30] if there is a permutation $\pi \in \mathrm{Sym}(\omega)$ such that $\sum_n a_{\pi(n)}$ is c.c. Notice that $\sum_n a_n$ is p.c.c. iff $a_n \to 0$, $\sum\{a_n : a_n \ge 0\} = +\infty$, and $\sum\{a_n : a_n \le 0\} = -\infty$. The rephrased version of Riemann's Theorem then says that given any p.c.c. $\sum_n a_n$ and any $r \in \mathbb{R}\cup\{-\infty,+\infty\}$ there is $\pi \in \mathrm{Sym}(\omega)$ such that $\sum_n a_{\pi(n)} = r$ and there is also $\pi \in \mathrm{Sym}(\omega)$ such that $\sum_n a_{\pi(n)}$ diverges by oscillation.

Let us first see that $\mathfrak{dd}$ is indeed an uncountable cardinal.

Lemma 5. *Let $(a_n : n \in \omega)$ be a sequence in ω with the property that for some $n_0 \in \omega$ and every $n \ge n_0$, $a_{n+1} - a_n > 2^n$. Then $A = \{a_n : n \in \omega\}$ is an infinite and coinfinite subset of ω such that $d(A) = 0$.*

Proof. The property of the sequence immediately guarantees that A is infinite and coinfinite. As for the asymptotic density, first notice that a simple inductive argument shows that for $n \geq n_0$, $a_{n+1} > a_{n_0} + 2^n$. Hence, for $a_{n_0} + 2^{n-1} \leq k \leq a_{n_0} + 2^n$, we have that $A \cap k \subseteq \{a_i : i \leq n\}$, so

$$\frac{|A \cap k|}{k} \leq \frac{n+1}{k} \leq \frac{n+1}{a_{n_0} + 2^{n-1}},$$

which converges to 0, when k goes to infinity. □

Proposition 6. $\aleph_1 \leq \mathfrak{dd}$.

Proof. Let $\{\pi_i : i \in \omega\}$ be a countable subset of $\mathrm{Sym}(\omega)$ and, without loss of generality, assume that π_0 is the identity map. We construct, inductively, a sequence of integers $(a_n : n \in \omega)$ such that for every $i \leq n$, $\pi_i(a_{n+1}) - \pi_i(a_n) \geq 2^n$.

Let $a_0 = 1$ and given a_n, let $a_{n+1} > a_n$ be such that $\pi_i(a_{n+1}) \geq \pi_i(a_n) + 2^n$ for all $i \leq n$. This can be done since $\lim_{k\to\infty} \pi_i(k) = \infty$ for every $i \in \omega$.

Let $A = \{a_n : n \in \omega\}$; then it follows from the previous lemma that A is an infinite and coinfinite subset of ω such that $d(\pi_i[A]) = d(A) = 0$ for every $i \in \omega$. □

In the next section, we shall obtain a much better result (Theorem 8): Tukey connections are used to establish that $\mathfrak{dd}$ equals $\mathsf{non}(\mathcal{M})$, the smallest cardinality of a nonmeager set of reals. We use similar methods to prove that a variant of $\mathfrak{rr}$ and a cardinal invariant related to large-scale topology are also equal to $\mathsf{non}(\mathcal{M})$. In Section 3 we introduce variants of $\mathfrak{dd}$ by imposing restrictions on the original density of a set and/or the density of its image after a permutation. Interesting cases occur when these densities are imposed to be in one of the following sets: $\{0, 1\}$, $(0, 1)$, $[0, 1]$ or $\{\mathrm{osc}\}$, where the last case means that the corresponding density does not exist. We then give ZFC lower and upper bounds for these cardinals in terms of well-known cardinals: the already mentioned $\mathrm{non}(\mathcal{M})$; the bounding number $\mathfrak{b}$; the reaping number $\mathfrak{r}$; and the covering numbers of the meager and the null ideals, $\mathrm{cov}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{N})$. This allows us to position our cardinals in Cichoń's diagram. Next, Section 4 provides some consistency results regarding strict inequalities. Finally, in Section 5, we consider variations of the rearrangement and the density cardinals defined within the framework of Section 3 and try to compare cardinals with analogous definitions. The similarity to rearrangement naturally leads us to consider relative density and the asymptotic mean. We also discuss the limitations of these analogies and pose several open questions.

Our notation is standard. For cardinal invariants not defined here we refer to [6]. Only Section 4 requires knowledge of the forcing method. For forcing, and its connection to cardinal invariants, see [5] and [15].

2. The density number and the uniformity of the meager ideal

Since the definitions of $\mathfrak{rr}$ and $\mathfrak{dd}$ are quite similar we expect that there is a close relationship between the cardinals. We shall see in this section (Corollary 9) that $\mathfrak{rr} \leq \mathfrak{dd}$ while it remains open whether equality holds.

It is convenient to formulate order relations between cardinals in terms of *Tukey connections* because this will streamline the arguments and will automatically give results about dual cardinals. Let us quickly review this language ([21], [6, Section 4]). Consider *relational systems* $\mathcal{A} = (A_-, A_+, A)$ such that $A \subseteq A_- \times A_+$ is a relation, for every $a_- \in A_-$ there is $a_+ \in A_+$ with $a_- A a_+$, and for every $a_+ \in A_+$ there is $a_- \in A_-$ such that $a_- A a_+$ fails. For such $\mathcal{A}$ there is the *dual* relational system $\mathcal{A}^\perp = (A_+, A_-, \neg \breve{A})$ where $\breve{A}$ is the converse of A, that is, $a_+ \neg \breve{A} a_-$ if $\neg(a_- A a_+)$. With each relational system $\mathcal{A} = (A_-, A_+, A)$ we associate two cardinals: the *dominating number* $\mathfrak{d}(A_-, A_+, A)$ is the smallest size of a family $C_+ \subseteq A_+$ such that for all $a_- \in A_-$ there is $a_+ \in C_+$ with $a_- A a_+$; the *unbounding number* $\mathfrak{b}(A_-, A_+, A)$ is the least cardinality of a family $C_- \subseteq A_-$ such that for all $a_+ \in A_+$ there is $a_- \in C_-$ such that $a_- A a_+$ fails. Notice that $\mathfrak{d}(\mathcal{A}^\perp) = \mathfrak{b}(\mathcal{A})$ and $\mathfrak{b}(\mathcal{A}^\perp) = \mathfrak{d}(\mathcal{A})$.

Given two relational systems $\mathcal{A} = (A_-, A_+, A)$ and $\mathcal{B} = (B_-, B_+, B)$, $\mathcal{A}$ is *Tukey reducible* to $\mathcal{B}$ ($\mathcal{A} \leq_T B$ in symbols) if there exist functions $\varphi_- : A_- \to B_-$ and $\varphi_+ : B_+ \to A_+$ such that $\varphi_-(a_-) B b_+$ implies $a_- A \varphi_+(b_+)$ for all $a_- \in A_-$ and $b_+ \in B_+$. $\mathcal{A}$ and $\mathcal{B}$ are *Tukey equivalent* ($\mathcal{A} \equiv_T B$ in symbols) if $\mathcal{A} \leq_T \mathcal{B}$ and $\mathcal{B} \leq_T \mathcal{A}$ both hold. $\mathcal{A} \leq_T \mathcal{B}$ is equivalent to $\mathcal{B}^\perp \leq_T \mathcal{A}^\perp$. Either implies that $\mathfrak{d}(\mathcal{A}) \leq \mathfrak{d}(\mathcal{B})$ and $\mathfrak{b}(\mathcal{A}) \geq \mathfrak{b}(\mathcal{B})$. Conversely, if $\mathfrak{d}(\mathcal{A}) > \mathfrak{d}(\mathcal{B})$ is consistent then $\mathcal{A} \leq_T \mathcal{B}$ consistently fails. Note, however, that all Tukey connections we shall exhibit are Borel and thus absolute. Hence, for Borel Tukey reducibilities, the consistency of $\mathfrak{d}(\mathcal{A}) > \mathfrak{d}(\mathcal{B})$ means that $\mathcal{A} \leq_T \mathcal{B}$ fails in ZFC.

The relational system relevant for the density number is $(D, \mathrm{Sym}(\omega), R)$ where D is the collection of infinite-coinfinite sets with density and R is given by $x R \pi$ if $d(\pi[x]) \neq d(x)$ for $x \in D$ and $\pi \in \mathrm{Sym}(\omega)$ (where we allow the possibility that $d(\pi[x])$ is undefined). Clearly $\mathfrak{dd} = \mathfrak{d}(D, \mathrm{Sym}(\omega), R)$. Denote by $\mathfrak{dd}^\perp = \mathfrak{b}(D, \mathrm{Sym}(\omega), R)$ the dual cardinal, that is, the least size of a family D_0 of infinite-coinfinite sets with density such that for all $\pi \in \mathrm{Sym}(\omega)$ there is $x \in D_0$ such that $d(\pi[x]) = d(x)$.

Let $\mathcal{M}$ and $\mathcal{N}$ be the meager and null ideals on the Cantor space 2^ω, respectively. Consider the relational system $(\mathcal{M}, 2^\omega, \not\ni)$. Then $\mathfrak{d}(\mathcal{M}, 2^\omega, \not\ni) = \mathsf{non}(\mathcal{M})$ is the uniformity of the meager ideal and $\mathfrak{b}(\mathcal{M}, 2^\omega, \not\ni) = \mathsf{cov}(\mathcal{M})$ is the covering number of the meager ideal.

We will also need the following relational system. Let $h \in \omega^\omega$ be a function with $|h(n)| \geq 1$ for all n. A function $\phi : \omega \to [\omega]^{<\omega}$ is called an *h-slalom* if $|\phi(n)| \leq h(n)$ for all n. Let Φ_h be the collection of all h-slaloms. Consider the triple $(\omega^\omega, \Phi_h, \in^\infty)$ where $f \in^\infty \phi$ if $f(n) \in \phi(n)$ holds for infinitely many n, for $f \in \omega^\omega$ and $\phi \in \Phi_h$. The connection between the invariants of $(\omega^\omega, \Phi_h, \in^\infty)$ and the invariants of $\mathcal{M}$ is established by the following classical result of Bartoszyński:

Theorem 7 (Bartoszyński [4], [5, Lemmas 2.4.2 and 2.4.8])**.** *For any $h \in \omega^\omega$ satisfying $|h(n)| \geq 1$ for all $n \in \omega$ we have that $\mathsf{non}(\mathcal{M}) = \mathfrak{d}(\omega^\omega, \Phi_h, \in^\infty)$ and $\mathsf{cov}(\mathcal{M}) = \mathfrak{b}(\omega^\omega, \Phi_h, \in^\infty)$.*

However, and this is the tricky part, $(\omega^\omega, \Phi_h, \in^\infty)$ and $(\mathcal{M}, 2^\omega, \not\ni)$ are not Tukey equivalent. While $(\omega^\omega, \Phi_h, \in^\infty) \leq_T (\mathcal{M}, 2^\omega, \not\ni)$ is easy to see (see, again, [4] or [5, Lemmas 2.4.2 and 2.4.8]), the converse fails. (This follows from Zapletal's result [22] saying that there is a forcing adding a half-Cohen real (i.e., an infinitely often equal real) without adding a Cohen real.) But $(\mathcal{M}, 2^\omega, \not\ni)$ is Tukey reducible to a sequential composition of $(\omega^\omega, \Phi_h, \in^\infty)$ ([4] or [5, Lemmas 2.4.2 and 2.4.8], see also [6, Theorem 5.9] for the special case h is the constant function with value 1).

In [7, Theorems 7, 8, and 11], $\max\{\mathsf{cov}(\mathcal{N}), \mathfrak{b}\} \leq \mathfrak{rr} \leq \mathsf{non}(\mathcal{M})$ was proved, and the same inequalities hold for $\mathfrak{dd}$ instead of $\mathfrak{rr}$. There is, however, a better result:

Theorem 8. *$(\omega^\omega, \Phi_h, \in^\infty) \leq_T (D, \mathrm{Sym}(\omega), R) \leq_T (\mathcal{M}, 2^\omega, \not\ni)$ for any $h \in \omega^\omega$ growing fast enough.*

Corollary 9. $\mathfrak{dd} = \mathsf{non}(\mathcal{M})$ *and* $\mathfrak{dd}^\perp = \mathsf{cov}(\mathcal{M})$. *A fortiori,* $\mathfrak{rr} \leq \mathfrak{dd}$ *holds.*

Proof. This is immediate by Theorems 8 and 7. $\square$

Proof of Theorem 8. We first show the easier $(D, \mathrm{Sym}(\omega), R) \leq_T (\mathcal{M}, 2^\omega, \not\ni)$. Notice that for the meager ideal it does not matter whether we consider 2^ω or the Baire space ω^ω; furthermore $\mathrm{Sym}(\omega)$ is homeomorphic to ω^ω; so we may as well work with the meager ideal on $\mathrm{Sym}(\omega)$ and with the triple $(\mathcal{M}, \mathrm{Sym}(\omega), \not\ni)$ instead of $(\mathcal{M}, 2^\omega, \not\ni)$. We need to define $\varphi_- : D \to \mathcal{M}$ and $\varphi_+ : \mathrm{Sym}(\omega) \to \mathrm{Sym}(\omega)$ such that for all $x \in D$ and $\pi \in \mathrm{Sym}(\omega)$, if $\pi \notin \varphi_-(x)$ then $xR\varphi_+(\pi)$.

Let φ_+ be the identity function, $\varphi_+(\pi) = \pi$ for $\pi \in \mathrm{Sym}(\omega)$, and, for $x \in D$, let $\varphi_-(x) = \bigcup_{k,\ell\in\omega} F_{k,\ell}$, where

$$F_{k,\ell} = \left\{ \pi \in \mathrm{Sym}(\omega) : \left(\forall n \geq \ell \ \frac{|\pi(x) \cap n|}{n} \geq \frac{1}{k} \text{ or } \forall n \geq \ell \ \frac{|\pi(x) \cap n|}{n} \leq 1 - \frac{1}{k} \right) \right\}.$$

To see that $\varphi_-(x)$ is meager in $\mathrm{Sym}(\omega)$, it suffices to show that each $A_{k,\ell} = \mathrm{Sym}(\omega) \setminus F_{k,\ell}$ is open and dense. But, as

$$A_{k,\ell} = \left\{ \pi \in \mathrm{Sym}(\omega) : \left(\exists n \geq \ell \ \frac{|\pi(x) \cap n|}{n} \leq \frac{1}{k} \text{ and } \exists n \geq \ell \ \frac{|\pi(x) \cap n|}{n} \geq 1 - \frac{1}{k} \right) \right\},$$

if $\pi \in A_{k,\ell}$, it is clear that any $\pi' \in \mathrm{Sym}(\omega)$ that agrees with π in a sufficiently large initial segment of ω also belongs to $A_{k,\ell}$. Bearing in mind the topology of $\mathrm{Sym}(\omega)$, one sees that $A_{k,\ell}$ is open. The fact that $A_{k,\ell}$ is dense amounts to showing that, for every $\pi \in \mathrm{Sym}(\omega)$ and $m \in \omega$, there is $\pi' \in A_{k,\ell}$ such that $\pi \restriction_m = \pi' \restriction_m$, but that is also clear.

Finally, if $\pi \notin \varphi_-(x)$, then $\underline{d}(\pi[x]) = 0$ and $\overline{d}(\pi[x]) = 1$ so that the density of $\pi[x]$ is not defined. A fortiori, $xR\varphi_+(\pi)$ holds.

Next let $h \in \omega^\omega$ be such that $h(n) \geq 2^n + n + 1$ for all n. To prove that $(\omega^\omega, \Phi_h, \in^\infty) \leq_T (D, \mathrm{Sym}(\omega), R)$ we need to define $\varphi_- : \omega^\omega \to D$ and $\varphi_+ : \mathrm{Sym}(\omega) \to \Phi_h$ such that for all $g \in \omega^\omega$ and all $\pi \in \mathrm{Sym}(\omega)$, if $\varphi_-(g)R\pi$ then $g \in^\infty \varphi_+(\pi)$.

Let $\varphi_+(\pi)(n) = 2^n \cup \{\pi^{-1}(k) : k \leq n\}$. Clearly $|\varphi_+(\pi)(n)| \leq 2^n + n + 1 \leq h(n)$. If $g \in \omega^\omega$ satisfies $g(n) \geq 2^n$ for almost all n, recursively define $\varphi_-(g) = \{a_n^g : n \in \omega\}$ as follows: let n_0 be minimal such that $g(n) \geq 2^n$ for all $n \geq n_0$; then let $a_0^g = 0$ and

$$a_{n+1}^g = \begin{cases} 2^{a_n^g} & \text{if } n < n_0 \\ g(a_n^g) & \text{if } n \geq n_0 \end{cases}$$

Note that $a_{n+1}^g \geq 2^{a_n^g} \geq 2^n$ holds for all n, that the a_n^g therefore form a strictly increasing sequence, and that $d(\varphi_-(g)) = 0$ holds. (If g is not of this form, the definition of $\varphi_-(g)$ is irrelevant.)

Assume that $g \in \omega^\omega$ and $\pi \in \mathrm{Sym}(\omega)$ are such that $g \in^\infty \varphi_+(\pi)$ fails. Then $g(n) \notin \varphi_+(\pi)(n)$ holds for almost all n. In particular $g(n) \geq 2^n$ for almost all n so that the definition of $\varphi_-(g)$ in the preceding paragraph applies. Let n_0 be minimal such that $g(n) \notin \varphi_+(\pi)(n)$ for all $n \geq n_0$. Then $a_{n+1}^g = g(a_n^g) \notin \varphi_+(\pi)(a_n^g)$ holds for all $n \geq n_0$. Hence $a_{n+1}^g \neq \pi^{-1}(k)$ for $k \leq a_n^g$, and $\pi(a_{n+1}^g) > a_n^g$ follows. In particular $\pi(a_{n+2}^g) > a_{n+1}^g \geq 2^n$ for $n \geq n_0$, and $d(\pi[\varphi_-(g)]) = 0$ is immediate. Thus $\varphi_-(g)$ and $\pi[\varphi_-(g)]$ have the same density, and $\varphi_-(g)R\pi$ fails, as required. This completes the proof of the theorem. $\square$

Combinatorial characterizations of $\mathsf{non}(\mathcal{M})$ like $\mathfrak{d}(\omega^\omega, \Phi_h, \in^\infty)$ have been used before to show that cardinals defined in terms of permutations of ω are larger than or equal to $\mathsf{non}(\mathcal{M})$, see [12, Theorems 2.2 and 2.4] of which the present proof is reminiscent.

In view of the analogy between permutations of density and rearrangement of series, it may now be natural to conjecture:

Problem 10 ([7, Question 46])**.** *Does* $\mathfrak{rr} = \mathsf{non}(\mathcal{M})$ *hold?*

However, it is far from clear that this is true, for while sets of density strictly between 0 and 1 seem to correspond to c.c. series, sets of density 0 or 1 rather correspond to p.c.c. series converging to $\pm\infty$:

p.c.c. series $\sum_n a_n$	infinite-coinfinite set A
c.c. series	$d(A) \in (0,1)$
$\sum_n a_n = \pm\infty$	$d(A) \in \{0,1\}$
$\sum_n a_n$ diverges by oscillation	$d(A)$ is undefined

Table 1. Analogy between rearrangement and density

Thus the following rearrangement analogue of the density number $\mathfrak{dd}$ is natural:

Definition 11. $\mathfrak{rr}'$ is the smallest cardinality of a family $\Pi \subseteq \mathrm{Sym}(\omega)$ such that for every p.c.c. series $\sum_n a_n \in \mathbb{R} \cup \{\pm\infty\}$ there is $\pi \in \Pi$ such that $\sum_n a_{\pi(n)} \neq \sum_n a_n$, where, as usual, we allow the possibility that $\sum_n a_{\pi(n)}$ diverges by oscillation.

Clearly $\mathfrak{rr} \leq \mathfrak{rr}' \leq \mathsf{non}(\mathcal{M})$ (the proof of the second inequality is exactly like in [7, Theorem 8]; this was originally proved by Agnew [1]). The point is that:

Theorem 12. $\mathfrak{rr}' = \mathsf{non}(\mathcal{M})$.

Proof Sketch. The proof of $\mathsf{non}(\mathcal{M}) \leq \mathfrak{rr}'$ is analogous to the second part of the proof of Theorem 8, and we therefore confine ourselves to pointing out the necessary changes. We show $\mathfrak{d}(\omega^\omega, \Phi_h, \in^\infty) \leq \mathfrak{rr}'$ where h is as in the latter proof. Given $\Pi \subseteq \mathrm{Sym}(\omega)$ with $|\Pi| < \mathfrak{d}(\omega^\omega, \Phi_h, \in^\infty)$, find $g \in \omega^\omega$ such that $g(n) \notin \varphi_+(\pi)(n)$ for almost all n, for all $\pi \in \Pi$. Let $A := \varphi_-(g) = \{a^g_n : n \in \omega\}$. Let $\{b^g_n : n \in \omega\}$ be the increasing enumeration of the complement $B := \omega \setminus A$. Let

$$c_k = \begin{cases} \frac{1}{n} & \text{if } k = b^g_n \\ -\frac{1}{n} & \text{if } k = a^g_n \end{cases}$$

Clearly $\sum_k c_k$ is p.c.c. with $\sum_k c_k = \infty$. The argument of the proof of Theorem 8 now shows that $\sum_k c_{\pi(k)} = \infty$ still holds for all $\pi \in \Pi$. Thus Π is not a witness for $\mathfrak{rr}'$, and $\mathsf{non}(\mathcal{M}) \leq \mathfrak{rr}'$ follows. $\square$

For simplicity, we gave a direct, Tukey-free, proof of this result. For a discussion of the rearrangement number and its relatives in the framework of Tukey connections, see van der Vlugt's master thesis [20].

We finally show that a similar method can be used to prove that a cardinal invariant related to large-scale topology is equal to $\mathsf{non}(\mathcal{M})$. Consider the triple $([\omega]^\omega, \mathrm{Sym}(\omega), S)$ where $xS\pi$ if the set $\{n \in x : n \neq \pi(n) \in x\}$ is infinite. Following Banakh [3] let $\Delta = \mathfrak{d}([\omega]^\omega, \mathrm{Sym}(\omega), S)$ and $\hat{\Delta} = \mathfrak{b}([\omega]^\omega, \mathrm{Sym}(\omega), S)$. That is, Δ is the least cardinality of a $\Pi \subseteq \mathrm{Sym}(\omega)$ such that for all $x \in [\omega]^\omega$ there is $\pi \in \Pi$ such that the set $\{n \in x : n \neq \pi(n) \in x\}$ is infinite while $\hat{\Delta}$ is the least cardinality of an $X \subseteq [\omega]^\omega$ such that for all $\pi \in \mathrm{Sym}(\omega)$ there is $x \in X$ such that the set $\{n \in x : n \neq \pi(n) \in x\}$ is finite. Banakh [3, see Theorems 3.2 and 7.1] proved $([\omega]^\omega, \mathrm{Sym}(\omega), S) \leq_T (\mathcal{M}, 2^\omega, \not\ni)$ and established a connection of Δ to large-scale topology. We prove:

Theorem 13. $(\omega^\omega, \Phi_h, \in^\infty) \leq_T ([\omega]^\omega, \mathrm{Sym}(\omega), S)$ *for any* $h \in \omega^\omega$ *growing fast enough.*

Corollary 14. $\Delta = \mathsf{non}(\mathcal{M})$ *and* $\hat{\Delta} = \mathsf{cov}(\mathcal{M})$.

This answers a question of Banakh [3, Problem 3.8].

Proof of Corollary 14. This is immediate by Theorems 13 and 7, and by Banakh's Theorems 3.2 and 7.1 [3]. □

Proof of Theorem 13. The proof is very similar to the second part of the proof of Theorem 8. Let $h \in \omega^\omega$ be such that $h(n) \geq 3(n+1)$ for all n. We need to define $\varphi_- : \omega^\omega \to [\omega]^\omega$ and $\varphi_+ : \mathrm{Sym}(\omega) \to \Phi_h$ such that for all $g \in \omega^\omega$ and $\pi \in \mathrm{Sym}(\omega)$, if $\varphi_-(g) S \pi$ then $g \in^\infty \varphi_+(\pi)$.

Let $\varphi_+(\pi)(n) = (n+1) \cup \{\pi^{-1}(k), \pi(k) : k \leq n\}$. Clearly $|\varphi_+(\pi)(n)| \leq 3(n+1) \leq h(n)$. If $g \in \omega^\omega$ satisfies $g(n) \geq n+1$ for almost all n, recursively define $\varphi_-(g) = \{a^g_n : n \in \omega\}$ as follows: let n_0 be minimal such that $g(n) \geq n+1$ for all $n \geq n_0$; then let $a^g_0 = 0$ and

$$a^g_{n+1} = \begin{cases} a^g_n + 1 & \text{if } n < n_0 \\ g(a^g_n) & \text{if } n \geq n_0 \end{cases}$$

Note that $a^g_{n+1} \geq a^g_n + 1 \geq n+1$ holds for all n, that the a^g_n thus form a strictly increasing sequence, and that $\varphi_-(g)$ is infinite. (If g is not of this form, the definition of $\varphi_-(g)$ is irrelevant.)

Assume that $g \in \omega^\omega$ and $\pi \in \mathrm{Sym}(\omega)$ are such that $g \in^\infty \varphi_+(\pi)$ fails. Then $g(n) \notin \varphi_+(\pi)(n)$ holds for almost all n. In particular $g(n) \geq n+1$ for almost all n so that the definition of $\varphi_-(g)$ in the preceding paragraph applies. Let n_0 be minimal such that $g(n) \notin \varphi_+(\pi)(n)$ for all $n \geq n_0$. Fix such n with $\pi(a^g_{n+1}) \neq a^g_{n+1}$. Then $a^g_{n+1} = g(a^g_n) \notin \varphi_+(\pi)(a^g_n)$ holds. Hence $a^g_{n+1} \neq \pi^{-1}(k)$ for $k \leq a^g_n$, and $\pi(a^g_{n+1}) > a^g_n$ follows. Now assume $m > n+1$. Then $a^g_m = g(a^g_{m-1}) \notin \varphi_+(\pi)(a^g_{m-1})$ holds. Hence $a^g_m \neq \pi(k)$ for $k \leq a^g_{m-1}$, and $\pi(a^g_{n+1}) \neq a^g_m$ follows. Therefore $\pi(a^g_{n+1}) \notin \varphi_-(g)$. Unfixing n we see that the set $\{n : a^g_n \neq \pi(a^g_n) \in \varphi_-(g)\}$ is finite, and $\varphi_-(g) S \pi$ fails, as required. This completes the proof of the theorem. □

3. Variations on the density number: ZFC results

Inspired by the variants of the rearrangement number considered in [7] – and also by $\mathfrak{rr}'$ of the previous section – we now look at natural variants of the density number. A further motivation for studying these variants comes from an analysis of the proof of Theorem 8; in fact, we shall use the framework we will develop to see what this proof really shows (Theorem 16).

Let osc be a symbol denoting "oscillation". For a (necessarily infinite-coinfinite) set $A \subset \omega$, say that $d(A) = \mathsf{osc}$ if $\underline{d}(A) < \overline{d}(A)$. Let $\mathsf{all} = [0,1] \cup \{\mathsf{osc}\}$. This is the set of all possible densities of (infinite-coinfinite) subsets A of ω, where we include the possibility that the density of A is undefined. We are ready for the main definition of this section.

Definition 15. Assume $X, Y \subseteq \mathsf{all}$ are such that $X \neq \emptyset$ and for all $x \in X$ there is $y \in Y$ with $y \neq x$ (equivalently either $|Y| \geq 2$ or $Y \setminus X \neq \emptyset$). The (X,Y)-*density number* $\mathfrak{dd}_{X,Y}$ is the smallest cardinality of a family $\Pi \subseteq \mathrm{Sym}(\omega)$ such that for every infinite-coinfinite set $A \subseteq \omega$ with $d(A) \in X$ there is $\pi \in \Pi$ such that $d(\pi[A]) \in Y$ and $d(\pi[A]) \neq d(A)$.

Note in particular that if $d(A) = \mathsf{osc}$ we require that $d(\pi[A])$ has density. Clearly, if $X' \subseteq X$ and $Y' \supseteq Y$ then $\mathfrak{dd}_{X',Y'} \leq \mathfrak{dd}_{X,Y}$. Also $\mathfrak{dd} = \mathfrak{dd}_{[0,1],\mathsf{all}}$.

Let us first provide the Tukey framework for $\mathfrak{dd}_{X,Y}$: for X, Y as in Definition 15, consider triples $(D_X, \mathrm{Sym}(\omega), R_Y)$ where D_X is the collection of infinite-coinfinite sets A with $d(A) \in X$, the relation R_Y is given by $A R_Y \pi$ if $d(\pi[A]) \in Y$ and $d(\pi[A]) \neq d(A)$, for $A \in D_X$ and $\pi \in \mathrm{Sym}(\omega)$. Then $\mathfrak{dd}_{X,Y} = \mathfrak{d}(D_X, \mathrm{Sym}(\omega), R_Y)$. Let $\mathfrak{dd}^{\perp}_{X,Y} := \mathfrak{b}(D_X, \mathrm{Sym}(\omega), R_Y)$. We are ready to rephrase the main result of the last section in this framework:

Theorem 16. *(1) If $\mathsf{osc} \notin X$ and $\mathsf{osc} \in Y$, then $(D_X, \mathrm{Sym}(\omega), R_Y) \leq_T (\mathcal{M}, 2^\omega, \not\ni)$.*

(2) If $0 \in X$ or $1 \in X$, then $(\omega^\omega, \Phi_h, \in^\infty) \leq_T (D_X, \mathrm{Sym}(\omega), R_Y)$ for $h \in \omega^\omega$ with $|h(n)| \geq 2^n + n + 1$ for all n.

Proof. This is immediate from the proof of Theorem 8. □

We shall see later that the assumptions in both parts of this theorem are necessary. For (1), this follows from Theorem 22 and the consistency of $\mathsf{non}(\mathcal{M}) < \mathsf{cov}(\mathcal{M})$ (true in the Cohen model), and for (2), from Theorem 26 and the consistency of $\mathfrak{r} < \mathsf{non}(\mathcal{M})$ (which holds e.g. in the Blass-Shelah model [8], [5, pp. 370]).

Corollary 17. $\mathfrak{dd} = \mathfrak{dd}_{\{0,1\},\mathsf{all}} = \mathfrak{dd}_{[0,1],\{\mathsf{osc}\}} = \mathfrak{dd}_{\{0,1\},\{\mathsf{osc}\}} = \mathsf{non}(\mathcal{M})$*, with the dual result holding for the dual cardinals.*

It is natural to look for other lower and upper bounds of $\mathfrak{dd}_{X,Y}$ for various X and Y; in particular we may ask:

- what if $0, 1 \notin X$?
- what if $\mathsf{osc} \in X$ or $\mathsf{osc} \notin Y$?

We will mostly obtain lower bounds, but there is one more upper bound too, Theorem 26. First consider the relational system $(2^\omega, \mathcal{N}, \in)$. Then $\mathfrak{d}(2^\omega, \mathcal{N}, \in) = \mathsf{cov}(\mathcal{N})$ and $\mathfrak{b}(2^\omega, \mathcal{N}, \in) = \mathsf{non}(\mathcal{N})$.

Theorem 18. *If $X \cap [0,1] \neq \emptyset$, then $(2^\omega, \mathcal{N}, \in) \leq_T (D_X, \mathrm{Sym}(\omega), R_Y)$.*

We do not know whether this is also true if $X = \{\mathsf{osc}\}$ (Question 37).

Corollary 19. *If $X \cap [0,1] \neq \emptyset$, then $\mathsf{cov}(\mathcal{N}) \leq \mathfrak{dd}_{X,Y}$ and $\mathfrak{dd}^{\perp}_{X,Y} \leq \mathsf{non}(\mathcal{N})$.*

Proof of Theorem 18. If $0 \in X$ or $1 \in X$ this follows from Theorem 16 because it is well-known and easy to see that $(2^\omega, \mathcal{N}, \in) \leq_T (\omega^\omega, \Phi_h, \in^\infty)$ (where h is as in Theorem 16). Indeed, let us show that $(\omega^\omega, \mathcal{N}, \in) \leq_T (\omega^\omega, \Phi_h, \in^\infty)$, where the measure on ω^ω is the product $\prod_n \mu_n$, and μ_n is defined as follows: For each $n \in \omega$, fix a series $\sum_k a^n_k = 1$ such that, for all $k \in \omega$, $0 < a^n_k < \frac{1}{2^{n+1} h(n)}$, and put $\mu_n(A) = \sum_{k \in A} a^n_k$, for $A \subseteq \omega$. Let $\varphi_- : \omega^\omega \to \omega^\omega$ be the identity and $\varphi_+(\phi) = \{x \in \omega^\omega \mid x \in^\infty \phi\}$, for $\phi \in \Phi_h$. (Note that $\varphi_+(\phi)$ is a null set) Now, the implication $x \in^\infty \phi \implies x \in \varphi_+(\phi)$ is trivial, and shows that $(\omega^\omega, \mathcal{N}, \in) \leq_T (\omega^\omega, \Phi_h, \in^\infty)$, as desired.

So assume $r \in (0,1) \cap X$. We need to define functions $\varphi_- : 2^\omega \to D_X$ and $\varphi_+ : \mathrm{Sym}(\omega) \to \mathcal{N}$ such that for all $x \in 2^\omega$ and $\pi \in \Pi_d$, if $\varphi_-(x) R_r \pi$ then $x \in \varphi_+(\pi)$. Consider the product measure μ on 2^ω of the measure m with $m(\{1\}) = r$ and $m(\{0\}) = 1 - r$. Let $X_n : 2^\omega \to 2$ be the random variable with $X_n(x) = x(n)$. Clearly the expected value of all X_n is r. By the strong law of large numbers we see

$$\mu(\{x \in 2^\omega : d(x) = r\}) = \mu\left(\lim_{n\to\infty} \frac{1}{n}(X_0 + ... + X_{n-1}) = r\right) = 1.$$

Also, if π is a permutation, then

$$\mu(\{x \in 2^\omega : d(\pi(x)) = r\}) = \mu(\pi^{-1}\{x \in 2^\omega : d(x) = r\}) = 1.$$

Now, let

$$\varphi_-(x) = x \text{ if } x \in D_{\{r\}}$$

($\varphi_-(x)$ is irrelevant if $x \notin D_r$) and

$$\varphi_+(\pi) = \{y \in 2^\omega : d(y) \neq r \text{ or } d(\pi(y)) \neq r\}.$$

By the above discussion we see that $\mu(\varphi_+(\pi)) = 0$ as required.

Let $x \in 2^\omega$ and $\pi \in \mathrm{Sym}(\omega)$, and assume $d(\pi(\varphi_-(x))) \neq r$. Then either $x \notin D_{\{r\}}$ and $d(x) \neq r$ or $\varphi_-(x) = x$ and $d(\pi(x)) \neq r$. So $x \in \varphi_+(\pi)$. This completes the proof of the theorem. □

Next recall that for functions $f, g \in \omega^\omega$, $f \leq^* g$ (f is *eventually dominated by* g) if $f(n) \leq g(n)$ holds for all but finitely many $n \in \omega$, and consider the relational system $(\omega^\omega, \omega^\omega, \not\geq^*)$. Then $\mathfrak{d}(\omega^\omega, \omega^\omega, \not\geq^*)$ is the unbounding number $\mathfrak{b}$ and $\mathfrak{b}(\omega^\omega, \omega^\omega, \not\geq^*)$ is the dominating number $\mathfrak{d}$. We show:

Theorem 20. *If* $\mathsf{osc} \in X$ *or* $\mathsf{osc} \notin Y$*, then* $(\omega^\omega, \omega^\omega, \not\geq^*) \leq_T (D_X, \mathrm{Sym}(\omega), R_Y)$*.*

We note that since $(\omega^\omega, \omega^\omega, \not\geq^*) \leq_T (\omega^\omega, \Phi_h, \in^\infty)$, this is also true if $0 \in X$ or $1 \in X$, by Theorem 16. However, it will follow from Theorem 31 that this (consistently) fails if $X \subseteq (0,1)$ and $\mathsf{osc} \in Y$.

Corollary 21. *If* $\mathsf{osc} \in X$ *or* $\mathsf{osc} \notin Y$*, then* $\mathfrak{b} \leq \mathfrak{dd}_{X,Y}$ *and* $\mathfrak{dd}^\perp_{X,Y} \leq \mathfrak{d}$*.*

Proof of Theorem 20. We need to define $\varphi_- : \omega^\omega \to D_X$ and $\varphi_+ : \mathrm{Sym}(\omega) \to \omega^\omega$ such that for all $g \in \omega^\omega$ and $\pi \in \mathrm{Sym}(\omega)$, if $\varphi_-(g) R_Y \pi$ then $g \not\geq^* \varphi_+(\pi)$. First assume $\mathsf{osc} \in X$.

Let $\varphi_+(\pi)(n) = \max\{\pi(k), \pi^{-1}(k) : k \leq n\} + 2^n$ for $\pi \in \mathrm{Sym}(\omega)$. Assume $g(n) \geq 2^n$ for almost all n. Define $\{i^g_n : n \in \omega\}$ as follows: let n_0 be minimal such that $g(n) \geq 2^n$ for all $n \geq n_0$. Let $i^g_0 = 0$ and

$$i^g_{n+1} = \begin{cases} 2^{i^g_n} & \text{if } n < n_0 \\ g(i^g_n) & \text{if } n \geq n_0 \end{cases}$$

Finally let $\varphi_-(g) = \bigcup_{n\in\omega} [i^g_{4n}, i^g_{4n+2})$. It is easy to see that $d(\varphi_-(g)) = \mathsf{osc}$; in fact, $\underline{d}(\varphi_-(g)) = 0$ and $\overline{d}(\varphi_-(g)) = 1$. (For other g the definition is irrelevant.)

Assume $g \in \omega^\omega$ and $\pi \in \mathrm{Sym}(\omega)$ are such that $g \geq^* \varphi_+(\pi)$. Then $g(n) \geq 2^n$ for almost all n so that the above definition of $\varphi_-(g)$ applies. Let n_0 be such that $g(n) \geq \varphi_+(\pi)(n)$ for all $n \geq n_0$. Fix such n. Note that if $k < i^g_{4n}$ then $\pi(k) < \varphi_+(\pi)(k) < \varphi_+(\pi)(i^g_{4n}) \leq g(i^g_{4n}) = i^g_{4n+1}$. Hence $\pi(k) \in i^g_{4n+1}$. The same argument shows that if $k < i^g_{4n+1}$ then $\pi^{-1}(k) \in i^g_{4n+2}$, that is, if $k \geq i_{4n+2}$ then

$\pi(k) \geq i^g_{4n+1}$. Therefore it follows that $|\pi[\varphi_-(g)] \cap i^g_{4n+1}| = |\varphi_-(g) \cap i^g_{4n+1}|$, and this value converges to 1. Similarly, $|\pi[\varphi_-(g)] \cap i^g_{4n+3}| = |\varphi_-(g) \cap i^g_{4n+3}|$, which converges to 0. Thus we still have $\underline{d}(\pi[\varphi_-(g)]) = 0$ and $\overline{d}(\pi[\varphi_-(g)]) = 1$, that is, $d(\pi[\varphi_-(g)]) = \mathsf{osc}$, and $\varphi_-(g) R_Y \pi$ fails, as required.

Next assume $\mathsf{osc} \notin Y$. Note that by the first half of the proof and the comment after the statement of the theorem, there is nothing to show if $\mathsf{osc} \in X$ or $0 \in X$ or $1 \in X$. Hence assume $X \subseteq (0,1)$. To illustrate the main idea, first consider the case $\frac{1}{2} \in X$.

Case 1. $r = \frac{1}{2}$. Let E be the even numbers, and O, the odd numbers. Say $\pi \in \mathrm{Sym}(\omega)$ is *big on* E if there is $k = k_\pi$ such that there are infinitely many m such that

$$\frac{|\pi[E] \cap m|}{m} > \frac{1}{2} + \frac{1}{k} \tag{3.1}$$

Similarly, $\pi \in \mathrm{Sym}(\omega)$ is *small on* E if there is $k = k_\pi$ such that there are infinitely many m such that

$$\frac{|\pi[E] \cap m|}{m} < \frac{1}{2} - \frac{1}{k} \tag{3.2}$$

If π is big on E (small on E, respectively), define

$$m^\pi_n = \min \{m \geq 2^n : m \text{ satisfies (3.1) } (m \text{ satisfies (3.2), respectively})\}$$

If π is neither big nor small on E, define

$$m^\pi_n = \min \left\{ m \geq 2^n : \frac{1}{2} - \frac{1}{2^n} \leq \frac{|\pi[E] \cap m|}{m} \leq \frac{1}{2} + \frac{1}{2^n} \right\}$$

Finally, for any $\pi \in \mathrm{Sym}(\omega)$, let

$$\varphi_+(\pi)(n) = \min \{u > m^\pi_n : \pi^{-1}[m^\pi_n] \subseteq u\}$$

Next assume $g(n) \geq 2^n$ for almost all n. Define $\{i^g_n : n \in \omega\}$ as follows: let n_0 be minimal such that $g(n) \geq 2^n$ for all $n \geq n_0$. Let $i^g_0 = 0$ and

$$i^g_{n+1} = \begin{cases} 2^{i^g_n} & \text{if } n < n_0 \\ g(i^g_n) & \text{if } n \geq n_0 \end{cases}$$

Define

$$\varphi_-(g) = \bigcup_{n \in \omega} \left([i^g_{2n}, i^g_{2n+1}) \cap E \right) \cup \bigcup_{n \in \omega} \left([i^g_{2+1}, i^g_{2n+2}) \cap O \right)$$

Clearly $d(\varphi_-(g)) = \frac{1}{2}$.

Assume $g \in \omega^\omega$ and $\pi \in \mathrm{Sym}(\omega)$ are such that $g \geq^* \varphi_+(\pi)$. We need to show that either $d(\pi[\varphi_-(g)]) = \frac{1}{2}$ or $d(\pi[\varphi_-(g)]) = \mathsf{osc} \notin Y$: the point is that $\varphi_-(g) R_Y \pi$ fails in both cases. Clearly $g(n) \geq 2^n$ for almost all n, and the above definition of $\varphi_-(g)$ applies. Let n_0 be such that $g(n) \geq \varphi_+(\pi)(n)$ for all $n \geq n_0$.

First assume π is big on E. Fix $2n \geq n_0$. Then

$$2^{i^g_{2n}} \overset{(1)}{\leq} m^\pi_{i^g_{2n}} < \varphi_+(\pi)(i^g_{2n}) \overset{(2)}{\leq} g(i^g_{2n}) = i^g_{2n+1}$$

and therefore, by (2) and (3.1),

$$\frac{\left|\pi[E \cap i^g_{2n+1}] \cap m^\pi_{i^g_{2n}}\right|}{m^\pi_{i^g_{2n}}} = \frac{\left|\pi[E] \cap m^\pi_{i^g_{2n}}\right|}{m^\pi_{i^g_{2n}}} > \frac{1}{2} + \frac{1}{k_\pi}$$

By (1) and definition of $\varphi_-(g)$, we see that for large enough n,

$$\frac{\left|\pi[\varphi_-(g)] \cap m^\pi_{i^g_{2n}}\right|}{m^\pi_{i^g_{2n}}} > \frac{1}{2} + \frac{1}{2k_\pi}$$

For the same reason, for large enough n,

$$\frac{\left|\pi[\varphi_-(g)] \cap m^\pi_{i^g_{2n+1}}\right|}{m^\pi_{i^g_{2n+1}}} < \frac{1}{2} - \frac{1}{2k_\pi}$$

Thus $d(\pi[\varphi_-(g)]) = \mathsf{osc}$, as required. The proof in case π is small on E is analogous.

Finally assume π is neither big nor small on E. In this case it is easy to see that the numbers

$$\frac{\left|\pi[E \cap i^g_{n+1}] \cap m^\pi_{i^g_n}\right|}{m^\pi_{i^g_n}} = \frac{\left|\pi[E] \cap m^\pi_{i^g_n}\right|}{m^\pi_{i^g_n}}$$

converge to $\frac{1}{2}$, and therefore the same is true for the numbers

$$\frac{\left|\pi[\varphi_-(g)] \cap m^\pi_{i^g_n}\right|}{m^\pi_{i^g_n}}$$

Hence either $d(\pi[\varphi_-(g)]) = \frac{1}{2}$ or $d(\pi[\varphi_-(g)]) = \mathsf{osc}$, as required, and we are again done.

Case 2. Arbitrary $r \in (0,1)$. Let $(\ell_b : b \in \omega)$ be a sequence such that $\frac{\ell_b}{2^b}$ converges to r. We may assume that $|r - \frac{\ell_b}{2^b}| < \frac{1}{2^b}$. Recursively define $(j_b : b \in \omega)$ such that $j_0 = 0$ and $j_{b+1} = j_b + 2^{2b}$. Partition the interval $J_b = [j_b, j_{b+1})$ into 2^b many intervals I^a_b, $a \in 2^b$, of length 2^b. For $c \in \omega$, define $w_c := \binom{2^c}{\ell_c}$. Note that w_c is the number of ℓ_c-element subsets of 2^c. Let $e_c : w_c \to \mathcal{P}(2^c)$ be a bijection from w_c to the ℓ_c-element subsets of 2^c such that $e_c(0) = \ell_c$, that is, $e_c(0)$ consists of the first ℓ_c elements of 2^c. Next, for $b \geq c$ and $a \in 2^b$, partition I^a_b into 2^c many intervals $(L^{a,c}_{b,d} : d < 2^c)$ of length 2^{b-c}. For $v \in w_c$, define $E^c_v \subseteq \omega$ such that $E^c_v \cap j_c = \emptyset$ and

$$E^c_v \cap I^a_b = \bigcup\{L^{a,c}_{b,d} : d \in e_c(v)\}$$

for $b \geq c$ and $a \in 2^b$. Note that $d(E^c_v) = \frac{\ell_c}{2^c}$. Similar to Case 1, say $\pi \in \mathrm{Sym}(\omega)$ is *big* (*small*, respectively) if there is $k = k_\pi$ such that for every c_0 there is $c \geq c_0$ such that

$$\exists^\infty m \quad \frac{|\pi[E^c_0] \cap m|}{m} > \frac{\ell_c}{2^c} + \frac{1}{k} \left(< \frac{\ell_c}{2^c} - \frac{1}{k} \text{ respectively} \right) \tag{3.3}$$

Note that this means (in the big case) that for such c we can find $v(c) \in w_c$ such that for infinitely many m,

$$\frac{|\pi[E^c_0] \cap m|}{m} > \frac{\ell_c}{2^c} + \frac{1}{k} \text{ and } \frac{\left|\pi[E^c_{v(c)}] \cap m\right|}{m} < \frac{\ell_c}{2^c} \tag{3.4}$$

and similarly for the small case. If π is big (or small) and $c < n$ satisfies (3.3), define

$$m^\pi_{n,c} = \min\{m \geq 2^n : m \text{ satisfies (3.4)}\}$$

and let

$$m_n^\pi = \max\{m_{n,c}^\pi : c < n \text{ and } c \text{ satisfies (3.3)}\}$$

If π is neither big nor small, there is a sequence $(k_c : c \in \omega)$ going to ∞ such that (3.3) fails for c with $k = k_c$. Then define

$$m_n^\pi = \min\left\{m \geq 2^n : \frac{\ell_c}{2^c} - \frac{1}{k_c} \leq \frac{|\pi[E_0^c] \cap m|}{m} \leq \frac{\ell_c}{2^c} + \frac{1}{k_c}\right\} \tag{3.5}$$

Finally, for $\pi \in \mathrm{Sym}(\omega)$ and $n \in \omega$, $\varphi_+(\pi)(n) := \min\left\{u > m_n^\pi : \pi^{-1}[m_n^\pi] \subseteq u\right\}$ is defined exactly as in the special case.

Next assume that $g(n) \geq 2^n$ for almost all n. Define $\{i_n^g : n \in \omega\}$ as follows: let n_0 be minimal such that $g(n) \geq 2^n$ for all $n \geq n_0$. Let $i_0^g = 0$ and

$$i_{n+1}^g = \begin{cases} 2^{i_n^g} & \text{if } n < n_0 \\ g(i_n^g) & \text{if } n \geq n_0 \end{cases}$$

Let $(K_c : c \in \omega)$ be the interval partition of ω into intervals of length w_c. If n is the v-th element of K_c $(v < w_c)$, $i_n^g \leq b < i_{n+1}^g$, and $a \in 2^b$, define

$$\varphi_-(g) \cap I_b^a = \bigcup\{L_{b,d}^{a,c} : d \in e_c(v)\}$$

Clearly $d(\varphi_-(g)) = r$. Also note that $\varphi_-(g) \cap [i_n^g, i_{n+1}^g) = E_v^c \cap [i_n^g, i_{n+1}^g)$ $(\star)$.

Given $g \in \omega^\omega$ and $\pi \in \mathrm{Sym}(\omega)$ with $g \geq^* \varphi_+(\pi)$, we need to show again that either $d(\pi[\varphi_-(g)]) = r$ or $d(\pi[\varphi_-(g)]) = \mathsf{osc} \notin Y$. Let n_0 be such that $g(n) \geq \varphi_+(\pi)(n)$ for all $n \geq n_0$. Let $n(c)$ be the 0-th element of K_c for $c \in \omega$. Assume $n(c) \geq n_0$. In case π is neither big nor small, we see by (3.5) that the numbers

$$\frac{\left|\pi[E_0^c \cap i_{n(c)+1}^g] \cap m_{i_{n(c)}^g}^\pi\right|}{m_{i_{n(c)}^g}^\pi} = \frac{\left|\pi[E_0^c] \cap m_{i_{n(c)}^g}^\pi\right|}{m_{i_{n(c)}^g}^\pi}$$

converge to r as c goes to infinity, and therefore, by $(\star)$, so do the numbers

$$\frac{\left|\pi[\varphi_-(g)] \cap m_{i_{n(c)}^g}^\pi\right|}{m_{i_{n(c)}^g}^\pi}$$

Thus either $d(\pi[\varphi_-(g)]) = r$ or $d(\pi[\varphi_-(g)]) = \mathsf{osc}$ as required. So assume without loss of generality that π is big. (The proof in case π is small is analogous.) For $n(c) \geq n_0$, we see by (3.4) that

$$\frac{\left|\pi[E_0^c \cap i_{n(c)+1}^g] \cap m_{i_{n(c)}^g}^\pi\right|}{m_{i_{n(c)}^g}^\pi} = \frac{\left|\pi[E_0^c] \cap m_{i_{n(c)}^g}^\pi\right|}{m_{i_{n(c)}^g}^\pi} > \frac{\ell_c}{2^c} + \frac{1}{k_\pi}$$

and, using additionally $(\star)$, we infer that $\overline{d}(\pi[\varphi_-(g)]) \geq r + \frac{1}{k_\pi}$. On the other hand, letting $n'(c)$ be the $v(c)$-th element of K_c for $c \in \omega$, (3.4) also implies that

$$\frac{\left|\pi[E_{v(c)}^c \cap i_{n'(c)+1}^g] \cap m_{i_{n'(c)}^g}^\pi\right|}{m_{i_{n'(c)}^g}^\pi} = \frac{\left|\pi[E_{v(c)}^c] \cap m_{i_{n'(c)}^g}^\pi\right|}{m_{i_{n'(c)}^g}^\pi} < \frac{\ell_c}{2^c}$$

so that, by $(\star)$, $\underline{d}(\pi[\varphi_-(g)]) \leq r$. Thus $d(\pi[\varphi_-(g)]) = \mathsf{osc}$, and the proof of the theorem is complete. □

Next consider the relational system $(2^\omega, \mathcal{M}, \in)$. Note that this is the dual of the system considered in Section 2, $(2^\omega, \mathcal{M}, \in) = (\mathcal{M}, 2^\omega, \not\ni)^\perp$. In particular, $\mathfrak{d}(2^\omega, \mathcal{M}, \in) = \mathsf{cov}(\mathcal{M})$ and $\mathfrak{b}(2^\omega, \mathcal{M}, \in) = \mathsf{non}(\mathcal{M})$.

Theorem 22. *If* $\mathsf{osc} \in X$ *or* $\mathsf{osc} \notin Y$*, then* $(2^\omega, \mathcal{M}, \in) \leq_T (D_X, \mathrm{Sym}(\omega), R_Y)$.

Note that – as for Theorem 20 – the assumption is optimal because if $\mathsf{osc} \notin X$ and $\mathsf{osc} \in Y$ we are in the situation of the first half of Theorem 16 and $(D_X, \mathrm{Sym}(\omega), R_Y) \leq_T (\mathcal{M}, 2^\omega, \not\ni)$ holds.

Corollary 23. *If* $\mathsf{osc} \in X$ *or* $\mathsf{osc} \notin Y$*, then* $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{dd}_{X,Y}$ *and* $\mathfrak{dd}^\perp_{X,Y} \leq \mathsf{non}(\mathcal{M})$.

Proof of Theorem 22. We need to define $\varphi_- : 2^\omega \to D_X$ and $\varphi_+ : \mathrm{Sym}(\omega) \to \mathcal{M}$ such that for all $x \in 2^\omega$ and $\pi \in \mathrm{Sym}(\omega)$, if $\varphi_-(x) R_Y \pi$ then $x \in \varphi_+(\pi)$.

First consider the case $\mathsf{osc} \in X$. Fix arbitrary $y \in D_{\{\mathsf{osc}\}}$. Let

$$\varphi_-(x) = \begin{cases} x & \text{if } x \in D_{\{\mathsf{osc}\}} \\ y & \text{if } x \notin D_{\{\mathsf{osc}\}} \end{cases}$$

for $x \in 2^\omega$ and

$$\varphi_+(\pi) = \{x \in 2^\omega : d(x) \neq \mathsf{osc} \text{ or } d(\pi[x]) \neq \mathsf{osc}\}$$

for $\pi \in \mathrm{Sym}(\omega)$. Then $\varphi_+(\pi) \in \mathcal{M}$ because $D_{\{\mathsf{osc}\}}$ is comeager in 2^ω and so is $\pi^{-1}[D_{\{\mathsf{osc}\}}]$.

Let $x \in 2^\omega$ and $\pi \in \mathrm{Sym}(\omega)$. Then either $d(x) \neq \mathsf{osc}$ and $x \in \varphi_+(\pi)$ follows, or $d(x) = \mathsf{osc}$, $\varphi_-(x) = x$, and if $x R_Y \pi$ then $d(\pi[x]) \neq \mathsf{osc}$ and $x \in \varphi_+(\pi)$ follows again.

We now come to the case $\mathsf{osc} \notin Y$. If $\mathsf{osc} \in X$, we are done by the above argument, and if $0 \in X$ or $1 \in X$, apply Theorem 24 below (using the trivial $(2^\omega, \mathcal{M}, \in) \leq_T (\mathfrak{c}, [\mathfrak{c}]^{\aleph_0}, \in)$). Hence it suffices to consider the case $(0,1) \cap X \neq \emptyset$. To illustrate the basic idea, we first deal with the special case $\frac{1}{2} \in X$.

<u>Case 1. $r = \frac{1}{2}$.</u> (Special case) Define φ_- such that for $x \in 2^\omega$,

$$\varphi_-(x)(2n) = \begin{cases} 0 & \text{if } x(n) = 0 \\ 1 & \text{if } x(n) = 1 \end{cases} \quad \text{and} \quad \varphi_-(x)(2n+1) = \begin{cases} 1 & \text{if } x(n) = 0 \\ 0 & \text{if } x(n) = 1 \end{cases}$$

It is clear that $d(\varphi_-(x)) = \frac{1}{2}$. Next let

$$\varphi_+(\pi) = \left\{ x : d(\pi[\varphi_-(x)]) \text{ is distinct from } \frac{1}{2} \text{ and } \mathsf{osc} \right\}$$

for $\pi \in \mathrm{Sym}(\omega)$. The conclusion then holds obviously, and we need to show that $\varphi_+(\pi) \in \mathcal{M}$. Clearly it suffices to prove that the set

$$\left\{ x : \overline{d}(\pi[\varphi_-(x)]) \geq \frac{1}{2} \text{ and } \underline{d}(\pi[\varphi_-(x)]) \leq \frac{1}{2} \right\}$$

is comeager.

Given $s \in 2^{<\omega}$ we define $\varphi_-(s)$ as above with $|\varphi_-(s)| = 2|s|$. (So this means that for any $x \in 2^\omega$ with $s \subseteq x$, $\varphi_-(s) = \varphi_-(x) \restriction 2|s|$.) We claim that given $s \in 2^{<\omega}$ and $n_0 \in \omega$ there are $n \geq n_0$ and $t \supseteq s$ with $\pi^{-1}[2n] \subseteq 2|t|$ and such that

$$\frac{|\pi[\varphi_-(t)] \cap 2n|}{2n} \geq \frac{1}{2}$$

and similarly with $\geq$ replaced by $\leq$. This claim clearly finishes the proof.

To see the claim (for $\geq$), assume without loss of generality that $|s| = n_0$. Choose $n \geq n_0$ such that $\pi[2n_0] \subseteq 2n$, and then choose $m \geq n$ such that $\pi^{-1}[2n] \subseteq 2m$. Extend s to t with $|t| = m$ as follows. Recursively construct s_j for j with $n_0 \leq j \leq m$ such that, letting

$$c_j = n_0 + |\{2n_0 \leq i < 2j : \pi(i) \in 2n \text{ and } \varphi_-(s_j)(i) = 1\}|$$

and

$$d_j = 2n_0 + |\{2n_0 \leq i < 2j : \pi(i) \in 2n\}|$$

we have

- $|s_j| = j$,
- $s = s_{n_0} \subset s_{n_0+1} \subset ... \subset s_j \subset ... \subset s_m = t$,
- $\frac{c_j}{d_j} \geq \frac{1}{2}$.

Suppose s_j has been constructed. If either (case 1) both $\pi(2j)$ and $\pi(2j+1)$ belong to $2n$ or (case 2) both do not belong to $2n$, we extend s_j arbitrarily to s_{j+1} (i.e. it does not matter whether $s_{j+1}(j)$ is 0 or 1). If (case 3) $\pi(2j) \in 2n$ and $\pi(2j+1) \notin 2n$, we let $s_{j+1}(j) = 1$, and if (case 4) $\pi(2j) \notin 2n$ and $\pi(2j+1) \in 2n$, we let $s_{j+1}(j) = 0$.

We need to check the last item, that is, $\frac{c_{j+1}}{d_{j+1}} \geq \frac{1}{2}$. By induction hypothesis, we know $\frac{c_j}{d_j} \geq \frac{1}{2}$. Let $c_j = \frac{d_j}{2} + k_j$ with $k_j \geq 0$. In case 1 we have $c_{j+1} = c_j + 1$ and $d_{j+1} = d_j + 2$. Thus

$$\frac{c_{j+1}}{d_{j+1}} = \frac{\frac{d_j+2}{2} + k_j}{d_j + 2} = \frac{1}{2} + \frac{k_j}{d_j+2} \geq \frac{1}{2}$$

as required. In case 2, $c_{j+1} = c_j$ and $d_{j+1} = d_j$ and the conclusion is trivial. In cases 3 and 4, $c_{j+1} = c_j + 1$ and $d_{j+1} = d_j + 1$ and we get

$$\frac{c_{j+1}}{d_{j+1}} = \frac{\frac{d_j+1}{2} + k_j + \frac{1}{2}}{d_j + 1} = \frac{1}{2} + \frac{k_j + \frac{1}{2}}{d_j+1} \geq \frac{1}{2}.$$

This completes the construction.

Now simply note that for $j = m$, the assumptions $\pi[2n_0] \subseteq 2n$ and $\pi^{-1}[2n] \subseteq 2m$ imply that $d_m = 2n$ and $c_m = |\pi[\varphi_-(t)] \cap 2n|$. Hence we obtain

$$\frac{|\pi[\varphi_-(t)] \cap 2n|}{2n} \geq \frac{1}{2}$$

as required.

Case 2. Arbitrary $r \in (0,1)$. (General case) As in the proof of Theorem 20, let $(\ell_b : b \in \omega)$ be a sequence such that $\frac{\ell_b}{2^b}$ converges to r and $|r - \frac{\ell_b}{2^b}| < \frac{1}{2^b}$. Let $(i_b : b \in \omega)$ be such that $i_0 = 0$ and $i_{b+1} = i_b + 2^b$, and let $(j_b : b \in \omega)$ such that $j_0 = 0$ and $j_{b+1} = j_b + 2^{2b}$. Partition the interval $J_b = [j_b, j_{b+1})$ into 2^b many intervals I_b^a, $a \in 2^b$, of length 2^b. Again let $w_b := \binom{2^b}{\ell_b}$, the number of ℓ_b-element subsets of 2^b, and let $e_b : w_b \to \mathcal{P}(2^b)$ be a bijection from w_b to the ℓ_b-element subsets of 2^b. Let $f \in \omega^\omega$ be the function with $f(n) = w_b$ whenever $i_b \leq n < i_{b+1}$. Work with the space of functions below f, $\prod f := \prod_{n \in \omega} f(n)$, instead of 2^ω (this is possible because $(2^\omega, \mathcal{M}, \in) \equiv_T (\prod f, \mathcal{M}, \in)$). For $x \in \prod f$ define $\varphi_-(x) \in [\omega]^\omega$ such that for all $b \in \omega$ and all $a \in 2^b$,

$$\varphi_-(x) \cap I_b^a = g_b^a[e_b(x(i_b + a))]$$

where g_b^a is the canonical bijection between 2^b and I_b^a sending $k \in 2^b$ to the k-th element of I_b^a. (Note here that $x(i_b + a) \in f(i_b + a) = w_b$.) It is clear that $d(\varphi_-(x)) = r$. Let

$$\varphi_+(\pi) = \{x : d(\pi[\varphi_-(x)]) \text{ is distinct from } r \text{ and } \mathsf{osc}\}$$

for $\pi \in \mathrm{Sym}(\omega)$. Again, we only need to show that $\varphi_+(\pi) \in \mathcal{M}$ and, again, this is done by proving that the set

$$\left\{x : \overline{d}(\pi[\varphi_-(x)]) \geq r \text{ and } \underline{d}(\pi[\varphi_-(x)]) \leq r\right\}$$

is comeager.

Given $s \in \prod^{\leq} f := \bigcup_{n\in\omega} \prod_{k<n} f(k)$ define $\varphi_-(s)$ as above. We claim that given $s \in \prod^{\leq} f$, $b_0 \in \omega$ and $\epsilon > 0$ there are $b_2 \geq b_1 \geq b_0$ and $t \supseteq s$ with $t \in \prod^{\leq} f$ and $|t| = i_{b_2}$ such that $\pi^{-1}[j_{b_1}] \subseteq j_{b_2}$ and

$$\frac{|\pi[\varphi_-(t)] \cap j_{b_1}|}{j_{b_1}} \geq r - \epsilon$$

and similarly with $\geq r - \epsilon$ replaced by $\leq r + \epsilon$. This will finish the proof.

To see the claim (for $\geq r - \epsilon$), assume without loss of generality that $|s| = i_{b_0}$. Also assume b_0 is so large that

$$\frac{|\varphi_-(s) \cap j_{b_0}|}{j_{b_0}} \geq r - \epsilon \text{ and } \frac{\ell_b}{2^b} \geq r - \epsilon \text{ for all } b \geq b_0$$

Choose $b_1 \geq b_0$ such that $\pi[j_{b_0}] \subseteq j_{b_1}$, and then choose $b_2 \geq b_1$ such that $\pi^{-1}[j_{b_1}] \subseteq j_{b_2}$. Extend s to t with $|t| = i_{b_2}$ as follows. Recursively construct s_k for k with $i_{b_0} \leq k \leq i_{b_2}$ such that if $k = i_b + a$ for some b with $b_0 \leq b \leq b_2$ and $a \in 2^b$, letting

$$c_k = |\varphi_-(s) \cap j_{b_0}| + |\{j_{b_0} \leq \ell < j_b + a \cdot 2^b : \pi(\ell) \in j_{b_1} \text{ and } \ell \in \varphi_-(s_k)\}|$$

and

$$d_k = j_{b_0} + |\{j_{b_0} \leq \ell < j_b + a \cdot 2^b : \pi(\ell) \in j_{b_1}\}|$$

we have

- $|s_k| = k$,
- $s = s_{i_{b_0}} \subset s_{i_{b_0}+1} \subset \ldots \subset s_k \subset \ldots \subset s_{i_{b_2}} = t$,
- $\frac{c_k}{d_k} \geq r - \epsilon$.

Note that the last item holds for $k = i_{b_0}$ by assumption. Suppose s_k has been constructed for some $k = i_b + a$ with $b_0 \leq b < b_2$ and $a \in 2^b$. We need to define $s_{k+1}(k) = s_{k+1}(i_b + a) \in f(i_b + a) = w_b$. Consider the set $z_b^a = \pi^{-1}[j_{b_1}] \cap I_b^a$. If this set has at least ℓ_b elements choose any ℓ_b-element subset y_b^a of z_b^a. Otherwise let y_b^a any ℓ_b-element subset of I_b^a containing z_b^a. Let

$$s_{k+1}(i_b + a) := e_b^{-1}\left((g_b^a)^{-1}[y_b^a]\right) \in w_b$$

So $\varphi_-(s_{k+1}) \cap I_b^a = y_b^a$.

We need to check the last item, that is, $\frac{c_{k+1}}{d_{k+1}} \geq r - \epsilon$. Note that, in terms of $\frac{c_k}{d_k}$, the smallest possible values for this quotient are obtained <u>either</u> if $z_b^a = I_b^a$ in which case $c_{k+1} = c_k + \ell_b$ and $d_{k+1} = d_k + 2^b$ <u>or</u> if $z_b^a = \emptyset$ in which case $c_{k+1} = c_k$ and $d_{k+1} = d_k$. Therefore we obtain

$$\frac{c_{k+1}}{d_{k+1}} \geq \min\left\{\frac{c_k}{d_k}, \frac{c_k + \ell_b}{d_k + 2^b}\right\}$$

Since the latter value is between $\frac{c_k}{d_k}$ and $\frac{\ell_b}{2^b}$, and both of these are $\geq r - \epsilon$, the first by inductive assumption and the second by choice of b_0, we see that $\frac{c_{k+1}}{d_{k+1}} \geq r - \epsilon$, as required. This completes the construction.

Now simply note that for $k = i_{b_2}$, the assumptions $\pi[j_{b_0}] \subseteq j_{b_1}$ and $\pi^{-1}[j_{b_1}] \subseteq j_{b_2}$ imply that $d_{i_{b_2}} = j_{b_1}$ and $c_{i_{b_2}} = |\pi[\varphi_-(t)] \cap j_{b_1}|$. Therefore we obtain

$$\frac{|\pi[\varphi_-(t)] \cap j_{b_1}|}{j_{b_1}} \geq r - \epsilon$$

as required. This finishes the proof of the theorem. □

The simple relational system $(\mathfrak{c}, [\mathfrak{c}]^{\aleph_0}, \in)$ satisfies $\mathfrak{d}(\mathfrak{c}, [\mathfrak{c}]^{\aleph_0}, \in) = \mathfrak{c}$ and $\mathfrak{b}(\mathfrak{c}, [\mathfrak{c}]^{\aleph_0}, \in) = \aleph_1$.

Theorem 24. *If $0 \in X$ or $1 \in X$ and $\mathsf{osc} \notin Y$, or if $\mathsf{osc} \in X$ and $0 \notin Y$ or $1 \notin Y$, then $(\mathfrak{c}, [\mathfrak{c}]^{\aleph_0}, \in) \leq_T (D_X, \mathrm{Sym}(\omega), R_Y)$.*

The assumptions in this theorem are mostly necessary: by Theorems 16 (1) and 26, if $\mathsf{osc} \in Y$ and $\mathsf{osc} \notin X$, then $\mathfrak{dd}_{X,Y} \leq \mathsf{non}(\mathcal{M})$, if $\mathsf{osc} \in Y$ and $0 \in Y$ and $1 \in Y$, then $\mathfrak{dd}_{X,Y} \leq \max\{\mathfrak{r}, \mathsf{non}(\mathcal{M})\}$, and if $0 \notin X$ and $1 \notin X$ and $0 \in Y$ and $1 \in Y$, then $\mathfrak{dd}_{X,Y} \leq \mathfrak{r}$. However, we do not know whether we can have $\mathfrak{dd}_{X,Y} = \mathfrak{c}$ in case $0 \notin X$ and $1 \notin X$ and $\mathsf{osc} \notin X$, i.e., in case $X \subseteq (0,1)$, though $\mathfrak{dd}_{X,Y} \leq \mathfrak{r}$ will follow in some cases from Theorem 26.

Corollary 25. *Under the assumptions of Theorem 24, $\mathfrak{dd}_{X,Y} = \mathfrak{c}$ and $\mathfrak{dd}^{\perp}_{X,Y} \leq \aleph_1$.*

Note that in general we can only prove $\mathfrak{dd}^{\perp}_{X,Y} \leq \aleph_1$ even though in natural cases $\mathfrak{dd}^{\perp}_{X,Y} = \aleph_1$ will hold. By definition $\mathfrak{dd}^{\perp}_{X,Y} \geq 2$, and if for example $X = \{0\}$ and $Y = (\frac{1}{2}, 1]$ then $\mathfrak{dd}^{\perp}_{X,Y} = 2$, as witnessed by two disjoint sets of density 0. Similarly, it is easy to see that if $X = \{0\}$ and $Y = (\frac{1}{3}, 1]$, then $\mathfrak{dd}^{\perp}_{X,Y} = 3$, etc. So all finite numbers ≥ 2 can be realized as $\mathfrak{dd}^{\perp}_{X,Y}$. We do not know whether $\mathfrak{dd}^{\perp}_{X,Y} = \aleph_0$ for some choices of X and Y.

Proof of Theorem 24. We need to define $\varphi_- : \mathfrak{c} \to D_X$ and $\varphi_+ : \mathrm{Sym}(\omega) \to [\mathfrak{c}]^{\aleph_0}$ such that for all $\alpha \in \mathfrak{c}$ and $\pi \in \mathrm{Sym}(\omega)$, if $\varphi_-(\alpha) R_Y \pi$ then $\alpha \in \varphi_+(\pi)$.

First consider the first case. Assume without loss of generality that $0 \in X$. (The case $1 \in X$ is analogous: work with complements!) Let A be such that $d(A) = 0$. Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be an almost disjoint family of subsets of A. So $d(A_\alpha) = 0$ for all $\alpha < \mathfrak{c}$. Let $\varphi_-(\alpha) = A_\alpha$. Fix a permutation π. Note that $\varphi_+(\pi) := \{\alpha : d(\pi[\varphi_-(\alpha)])$ is defined and $> 0\}$ can be at most countable. Since $\mathsf{osc} \notin Y$, $\varphi_-(\alpha) R_Y \pi$ implies $\alpha \in \varphi_+(\pi)$.

Next consider the second case. Again assume without loss of generality that $0 \notin Y$. Let $\{\varphi_-(\alpha) : \alpha < \mathfrak{c}\}$ be an almost disjoint family of subsets of ω without density (i.e., $d(\varphi_-(\alpha)) = \mathsf{osc}$). Again, $\varphi_+(\pi) := \{\alpha : d(\pi[\varphi_-(\alpha)])$ is defined and $> 0\}$ is at most countable. Since $0 \notin Y$, $\varphi_-(\alpha) R_Y \pi$ again implies $\alpha \in \varphi_+(\pi)$, and the theorem is proved. □

Let UR be the unreaping (or, unsplitting) relation on $[\omega]^\omega$, that is, $A \; UR \; B$ if $B \subseteq^* A$ or $B \cap A$ is finite. This means that A does not split B. $\mathfrak{d}([\omega]^\omega, [\omega]^\omega, UR)$ is the reaping number $\mathfrak{r}$ and $\mathfrak{b}([\omega]^\omega, [\omega]^\omega, UR)$ is the splitting number $\mathfrak{s}$.

Theorem 26. *Assume $0 \notin X$ and $1 \notin X$. Also assume*

(1) <u>either</u> $0 \in Y$ and $1 \in Y$,

(2) *or* $\mathsf{osc} \notin X$ *and there is a number* $r \in (0,1]$ *such that for all* $x \in X$*, both* $r + x(1-r)$ *and* $x(1-r)$ *belong to* Y*.*

Then $(D_X, \mathrm{Sym}(\omega), R_Y) \leq_T ([\omega]^\omega, [\omega]^\omega, UR)$*.*

Note that for $r = 1$, (2) is a special case of (1). By Theorem 16 (2) and the consistency of $\mathfrak{r} < \mathsf{non}(\mathcal{M})$, the assumption $0, 1 \notin X$ is necessary. If $\mathsf{osc} \in X$, then $0, 1 \in Y$ is also necessary by Theorem 24. However, if $\mathsf{osc} \notin X$, condition (2) can be relaxed: e.g., we know that $\mathfrak{dd}_{(0,1),\{\mathsf{osc}\}} \leq \mathfrak{r}$, see Proposition 29 and Corollary 30.

Corollary 27. *Under the assumptions of Theorem 26,* $\mathfrak{dd}_{X,Y} \leq \mathfrak{r}$ *and* $\mathfrak{s} \leq \mathfrak{dd}^\perp_{X,Y}$*.*

Proof of Theorem 26. We need to define $\varphi_- : D_X \to [\omega]^\omega$ and $\varphi_+ : [\omega]^\omega \to \mathrm{Sym}(\omega)$ such that for all $x \in D_X$ and $y \in [\omega]^\omega$, if $y \subseteq^* \varphi_-(x)$ or $y \cap \varphi_-(x)$ is finite, then $x R_Y \varphi_+(y)$.

Assume first $0, 1 \in Y$. Let $\varphi_-(x) = x$ for $x \in D_X$ and let $\varphi_+(y)$ be such that $d(\varphi_+(y)[y]) = 1$ for $y \in [\omega]^\omega$. If $y \subseteq^* x$ then $d(\varphi_+(y)[x]) = 1$ and if $y \cap x$ is finite then $d(\varphi_+(y)[x]) = 0$. In either case $d(\varphi_+(y)[x]) \neq d(x)$, and $x R_Y \varphi_+(y)$ holds.

Next assume $X \subseteq (0,1)$, and let r be as required in (2). Again let $\varphi_-(x) = x$ for $x \in D_X$. Fix a set z of density r. For $y \in [\omega]^\omega$, let $y' \subseteq y$ be a subset of density 0 and let $\varphi_+(y)$ be the unique permutation of ω that maps y' to z and $\omega \setminus y'$ to $\omega \setminus z$, both in an order-preserving fashion. Note that $d(\omega \setminus z) = 1 - r$.

Now assume $y \subseteq^* x$ with $d(x) \in X$. Since $d(y') = 0$, the relative density of $x \setminus y'$ in $\omega \setminus y'$ is equal to $d(x)$ and, by definition of $\varphi_+(y)$, the relative density of $\varphi_+(y)[x \setminus y']$ in $\varphi_+(y)[\omega \setminus y'] = \omega \setminus z$ is $d(x)$ as well. Thus, as $\varphi_+(y)[y'] \subseteq^* \varphi_+(y)[x]$, the density of $\varphi_+(y)[x]$ is $r + d(x)(1-r) > d(x)$, which belongs to Y by assumption. If, on the other hand, $y \cap x$ is finite, the same reasoning shows the density of $\varphi_+(y)[x]$ is $d(x)(1-r) < d(x)$. In either case, $x R_Y \varphi_+(y)$ holds, and the theorem is proved. □

As for the rearrangement numbers in [7, Theorem 5], for fixed $X \subseteq [0,1]$, the value of $\mathfrak{dd}_{X,Y}$ does not change as long as $\{\mathsf{osc}\} \subseteq Y \subseteq \mathsf{all}$. This is based on the following lemma:

Lemma 28 ([7, Lemma 4])**.** *For any permutation* $\pi \in \mathrm{Sym}(\omega)$ *there exists a permutation* $\sigma_\pi \in \mathrm{Sym}(\omega)$ *such that* $\sigma_\pi[n] = n$ *for infinitely many* n *and* $\sigma_\pi[n] = \pi[n]$ *for infinitely many* n*.*

Proposition 29. $\mathfrak{dd}_{X,\{\mathsf{osc}\}} = \mathfrak{dd}_{X,\mathsf{all}}$ *for all choices of* $X \subseteq [0,1]$*.*

Proof. Only $\mathfrak{dd}_{X,\{\mathsf{osc}\}} \leq \mathfrak{dd}_{X,\mathsf{all}}$ needs proof. Let Π be a witness for $\mathfrak{dd}_{X,\mathsf{all}}$ and show that $\Pi \cup \{\sigma_\pi : \pi \in \Pi\}$ is a witness for $\mathfrak{dd}_{X,\{\mathsf{osc}\}}$. Let $x \in D_X$, and let $\pi \in \Pi$ be such that $d(\pi[x]) \neq d(x)$. If $d(\pi[x]) = \mathsf{osc}$, we are done. On the other hand, if $d(\pi[x]) \in [0,1]$, then, by the properties of σ_π, $d(\sigma_\pi[x]) = \mathsf{osc}$, and the proof is complete. □

Lemma 28 is not helpful for proving the dual equality, and we do not know whether it holds. For rearrangement numbers it does, as shown by van der Vlugt using an argument involving sequential composition of relations (see [20, Theorem 3.3.5]).

Let us summarize the results of this section about $\mathfrak{dd}_{X,Y}$ for the most interesting choices of X and Y, namely, when either set is any of all, $[0,1]$, $(0,1)$, $\{0,1\}$, or $\{\mathsf{osc}\}$. This will subsume Corollary 17.

Corollary 30. *(1) If* $\mathsf{osc} \in Y$*, then* $\mathfrak{dd}_{[0,1],Y} = \mathfrak{dd}_{\{0,1\},Y} = \mathsf{non}(\mathcal{M})$*.*

(2) *If* $\mathsf{osc} \notin Y$, *then* $\mathfrak{dd}_{\mathsf{all},Y} = \mathfrak{dd}_{[0,1],Y} = \mathfrak{dd}_{\{0,1\},Y} = \mathfrak{c}$; *also* $\mathfrak{dd}_{\{\mathsf{osc}\},(0,1)} = \mathfrak{c}$.
(3) $\mathsf{cov}(\mathcal{N}) \leq \mathfrak{dd}_{(0,1),\mathsf{all}} = \mathfrak{dd}_{(0,1),\{\mathsf{osc}\}} \leq \min\{\mathfrak{r}, \mathsf{non}(\mathcal{M})\}$.
(4) $\max\{\mathsf{cov}(\mathcal{N}), \mathsf{cov}(\mathcal{M}), \mathfrak{b}\} \leq \mathfrak{dd}_{(0,1),[0,1]} \leq \mathfrak{dd}_{(0,1),(0,1)}, \mathfrak{dd}_{(0,1),\{0,1\}} \leq \mathfrak{r}$.
(5) $\max\{\mathfrak{b}, \mathsf{cov}(\mathcal{M})\} \leq \mathfrak{dd}_{\{\mathsf{osc}\},\mathsf{all}} = \mathfrak{dd}_{\{\mathsf{osc}\},[0,1]} \leq \mathfrak{dd}_{\{\mathsf{osc}\},\{0,1\}} \leq \mathfrak{r}$.
(6) $\max\{\mathsf{non}(\mathcal{M}), \mathsf{cov}(\mathcal{M})\} \leq \mathfrak{dd}_{\mathsf{all},\mathsf{all}} = \max\{\mathsf{non}(\mathcal{M}), \mathfrak{dd}_{\{\mathsf{osc}\},\mathsf{all}}\} \leq \max\{\mathsf{non}(\mathcal{M}), \mathfrak{r}\}$.

Proof. By Corollaries 17, 19, 21, 23, 25, and 27.

Note that for $\mathfrak{dd}_{(0,1),\mathsf{all}} = \mathfrak{dd}_{(0,1),\{\mathsf{osc}\}}$ in (3) we use Proposition 29. For (6) use $\mathfrak{dd}_{\mathsf{all},\mathsf{all}} = \max\{\mathfrak{dd}_{[0,1],\mathsf{all}}, \mathfrak{dd}_{\{\mathsf{osc}\},\mathsf{all}}\}$, (1), and (5). □

Table 2 and Diagram 1 below show the relationship between the cardinals $\mathfrak{dd}_{X,Y}$ for the above five choices of X and Y and some of the classical cardinal invariants of the continuum.

$X \backslash Y$	$(0,1)$	$\{0,1\}$ (or $[0,1]$)	all	$\{\text{osc}\}$
$(0,1)$	$\mathfrak{b}, \text{cov}(\mathcal{M}), \text{cov}(\mathcal{N})$ $\mathfrak{r}$	$\mathfrak{b}, \text{cov}(\mathcal{M}), \text{cov}(\mathcal{N})$ $\mathfrak{r}$	$\text{cov}(\mathcal{N})$ $\mathfrak{r}, \text{non}(\mathcal{M})$	$\text{cov}(\mathcal{N})$ $\mathfrak{r}, \text{non}(\mathcal{M})$
$\{0,1\}$	$\mathfrak{c}$	$\mathfrak{c}$	$\text{non}(\mathcal{M})$	$\text{non}(\mathcal{M})$
$[0,1]$	$\mathfrak{c}$	$\mathfrak{c}$	$\text{non}(\mathcal{M})$	$\text{non}(\mathcal{M})$
all	$\mathfrak{c}$	$\mathfrak{c}$	$\text{non}(\mathcal{M}), \text{cov}(\mathcal{M})$ $\max\{\text{non}(\mathcal{M}), \mathfrak{r}\}$	-
$\{\text{osc}\}$	$\mathfrak{c}$	$\mathfrak{b}, \text{cov}(\mathcal{M})$ $\mathfrak{r}$	$\mathfrak{b}, \text{cov}(\mathcal{M})$ $\mathfrak{r}$	-

TABLE 2. Upper and lower bounds for the cardinals in Corollary 30

X is in the left column and Y in the top row; the upper row of cells containing two rows gives lower bounds, and the lower row, upper bounds. The columns for $Y = \{0,1\}$ and $Y = [0,1]$ coincide, so we merged them into one. This means the corresponding cardinals have the same bounds, but it is unclear whether they coincide.

We shall prove two independence results (Theorems 31 and 35) in the next section showing that some of the inequalities in the corollary are consistently strict, but a number of questions remain open (see in particular Questions 38 and 39 in Section 6). In particular, we do not know whether any of the (X,Y)-density numbers is distinct from all classical cardinal invariants.

4. Variations on the density number: consistency results

Our first consistency result says that the unbounding number $\mathfrak{b}$ is not a lower bound of $\mathfrak{dd}_{(0,1),\mathsf{all}}$; it is based on the fact that σ-centered forcing does not increase the latter cardinal.

Theorem 31. *$\mathfrak{dd}_{X,\mathsf{all}} < \mathfrak{p}$ is consistent for all $X \subseteq (0,1)$. In particular, $\mathfrak{dd}_{(0,1),\mathsf{all}} < \mathfrak{b}$ and $\mathfrak{dd}_{(0,1),\mathsf{all}} < \mathfrak{dd}$ are consistent.*

Proof. Let $(I_n : n \in \omega)$ be an interval partition of ω such that $|I_{n+1}| \geq n \cdot \sum_{i \leq n} |I_i|$ for all n. Let $k \in \omega$ and j such that $0 < j < k-1$. Say that $A \subseteq \omega$ is a *(j,k)-set* if

$$\frac{|A \cap I_n|}{|I_n|} \in \left[\frac{j}{k}, \frac{j+1}{k}\right]$$

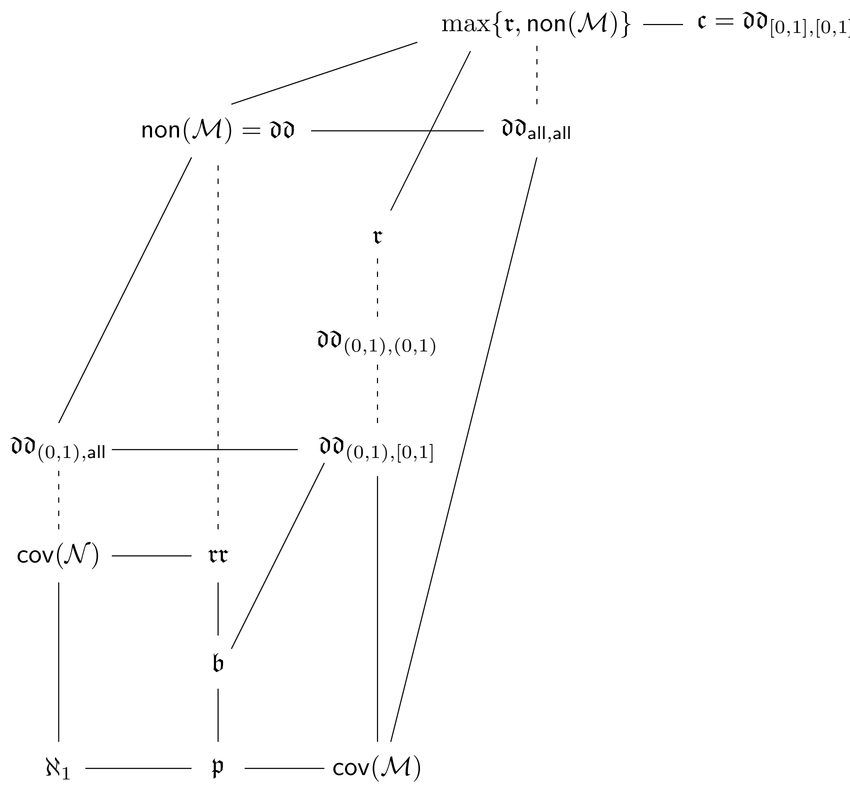

Figure 1. Some cardinals

For simplicity, we consider here only the cases $X, Y = \mathsf{all}, (0,1), [0,1]$: cardinals grow as one moves right and up; straight lines signify that strict inequality is consistent, dotted lines mean that we do not know whether equality holds or strict inequality is consistent. Note that $\mathfrak{dd} = \mathfrak{dd}_{[0,1],\mathsf{all}}$ and that $\mathfrak{c}$ is also equal to each of $\mathfrak{dd}_{[0,1],(0,1)}$, $\mathfrak{dd}_{\mathsf{all},[0,1]}$, and $\mathfrak{dd}_{\mathsf{all},(0,1)}$.

for almost all $n \in \omega$. It is easy to see that if A is a (j,k)-set then $\underline{d}(A) \leq \frac{j+1}{k}$ and $\overline{d}(A) \geq \frac{j}{k}$. In particular, if A has asymptotic density then $d(A) \in \left[\frac{j}{k}, \frac{j+1}{k}\right]$. A permutation π *disrupts* the (j,k)-set A if π fixes all I_n and for infinitely many n, $\pi[A \cap I_n]$ is an initial segment of I_n. Note that $\pi[A]$ is still a (j,k)-set.

Lemma 32. *If A is a (j,k)-set and π disrupts A then $\overline{d}(\pi[A]) = 1$. In particular, $\pi[A]$ does not have asymptotic density.*

Proof. Let $\varepsilon > 0$. Let $n > \frac{k}{\varepsilon}$ such that $\pi[A \cap I_{n+1}]$ is an initial segment of I_{n+1}. Let $\ell = \max(\pi[A \cap I_n]) + 1$. Then

$$\frac{|\pi[A] \cap \ell|}{\ell} \geq \frac{\frac{j}{k}|I_{n+1}|}{\sum_{i \leq n} |I_i| + \frac{j}{k}|I_{n+1}|} \geq \frac{\frac{j}{k}|I_{n+1}|}{\left(\frac{1}{n} + \frac{j}{k}\right)|I_{n+1}|} \geq \frac{j}{\varepsilon + j} = \frac{j(j-\varepsilon)}{j^2 - \varepsilon^2} \geq 1 - \frac{\varepsilon}{j}$$

as required. $\square$

Lemma 33. *Let $\mathbb{P}$ be a σ-centered forcing notion. Let $\dot{A}$ be a $\mathbb{P}$-name for a (j,k)-set. Then there are (j,k)-sets $(A_m : m \in \omega)$ such that if π disrupts all A_m, then π is forced to disrupt $\dot{A}$.*

Proof. Assume $\mathbb{P} = \bigcup_m P_m$ where all P_m are centered. For each m and n let $A_m \cap I_n$ be such that no condition in P_m forces that $\dot{A} \cap I_n \neq A_m \cap I_n$. Since there are only finitely many possibilities for this intersection the centeredness of P_m implies that such a $A_m \cap I_n$ can indeed be found. Now assume that π disrupts all A_m. Let $n_0 \in \omega$ and $p \in \mathbb{P}$. There is m such that $p \in P_m$. Find $n \geq n_0$ such that $\pi[A_m \cap I_n]$ is an initial segment of I_n. There is $q \leq p$ such that $q \Vdash \dot{A} \cap I_n = A_m \cap I_n$. So q forces that $\pi[\dot{A} \cap I_n]$ is an initial segment of I_n. □

The following is a standard argument for finite support iterations of ccc forcing. We include the proof for the sake of completeness.

Lemma 34. *Let δ be a limit ordinal, and let $(\mathbb{P}_\alpha : \alpha \leq \delta)$ be a finite support iteration of ccc forcing such that for any $\alpha < \delta$ and any $\mathbb{P}_\alpha$-name $\dot{A}$ for a (j,k)-set there are (j,k)-sets $(A_m : m \in \omega)$ such that if π disrupts all A_m, then π is forced to disrupt $\dot{A}$. Then for any $\mathbb{P}_\delta$-name $\dot{A}$ for a (j,k)-set there are (j,k)-sets $(A_m : m \in \omega)$ such that if π disrupts all A_m, then π is forced to disrupt $\dot{A}$.*

Proof. If $cf(\delta) > \omega$ there is nothing to show. So assume without loss of generality that $\delta = \omega$, and let $\dot{A}$ be a $\mathbb{P}_\omega$-name for a (j,k)-set. Fix $m \in \omega$. In $V[G_m]$, where G_m is $\mathbb{P}_m$-generic over V, we may find a decreasing sequence of conditions $(p^\ell_m : \ell \in \omega)$ in the remainder forcing $\mathbb{P}_{[m,\omega)}$ deciding $\dot{A}$, i.e., there is a (j,k)-set A_m such that $p^\ell_m \Vdash \dot{A} \cap I_\ell = A_m \cap I_\ell$ for all $\ell \in \omega$. (We may assume p^0_m decides from which I_n onwards $\dot{A}$ satisfies the condition of being a (j,k)-set, and then A_m will automatically be a (j,k)-set as well.) Back in the ground model V, let $(\dot{p}^\ell_m : \ell \in \omega)$ and $\dot{A}_m$ be $\mathbb{P}_m$-names for $(p^\ell_m : \ell \in \omega)$ and A_m, respectively.

By assumption, for each m there are (j,k)-sets $(A^n_m : n \in \omega)$ such that if π disrupts all A^n_m, $n \in \omega$, then π is forced to disrupt $\dot{A}_m$. We show that the family $(A^n_m : n, m \in \omega)$ is as required for $\dot{A}$. Assume π disrupts all A^n_m, $n, m \in \omega$. Let $p \in \mathbb{P}_\omega$ and $\ell_0 \in \omega$ be arbitrary. We need to find $q \leq p$ and $\ell \geq \ell_0$ such that $\pi[\dot{A} \cap I_\ell]$ is forced to be an initial segment of I_ℓ. Find m such that $p \in \mathbb{P}_m$. Find $p' \leq p$ in $\mathbb{P}_m$ and $\ell \geq \ell_0$ such that $p' \Vdash$ "$\pi[\dot{A}_m \cap I_\ell]$ is an initial segment of I_ℓ". We may assume p' decides $\dot{p}^\ell_m$ as well, $p' \Vdash \dot{p}^\ell_m = p^\ell_m$. Let $q = p' \cup p^\ell_m$. Then $q \Vdash \dot{A} \cap I_\ell = \dot{A}_m \cap I_\ell$ and thus q also forces that $\pi[\dot{A} \cap I_\ell]$ is an initial segment of I_ℓ, as required. □

We are ready to complete the proof of the theorem. Assume CH, let $\kappa > \aleph_1$ be a regular cardinal, and let $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \kappa)$ be a finite support iteration of σ-centered partial orders forcing $\mathfrak{p} = \kappa$. It suffices to show that in the generic extension $V[G_\kappa]$, $\mathfrak{dd}_{(0,1),\mathsf{all}} = \aleph_1$, and that this is witnessed by the ground model permutations. Let $r \in (0,1)$ in $V[G_\kappa]$. Let $\dot{r}$ be a $\mathbb{P}_\kappa$-name for this real. We can find a condition $p \in \mathbb{P}_\kappa$ and numbers k and j with $0 < j < k-1$ such that p forces $\dot{r} \in \left(\frac{j}{k}, \frac{j+1}{k}\right)$. By strengthening p, if necessary, we may assume that for some $m \geq 3$,

$$p \Vdash \dot{r} \in \left[\frac{mj+1}{mk}, \frac{mj+m-1}{mk}\right]$$

Work below p. Let $\dot{A}$ be a $\mathbb{P}_\kappa$-name for a set of density $\dot{r}$. We first check that p forces that $\dot{A}$ is a (j,k)-set.

To see this work in the generic extension with p belonging to the generic filter. Suppose this fails and assume (without loss of generality) that there are infinitely many n with $\frac{|A\cap I_n|}{|I_n|} > \frac{j+1}{k}$. Let n be such that $\frac{|A\cap I_{n+1}|}{|I_{n+1}|} > \frac{j+1}{k}$. Then

$$\frac{|A\cap(\max(I_{n+1})+1)|}{\max(I_{n+1})+1} \geq \frac{|A\cap I_{n+1}|}{\sum_{i\leq n}|I_i|+|i_{n+1}|} \geq \frac{|A\cap I_{n+1}|}{\left(1+\frac{1}{n}\right)|I_{n+1}|} > \frac{j+1}{k}\cdot\frac{1}{1+\frac{1}{n}}$$

which is larger than $\frac{mj+m-1}{mk}$ for large enough n, a contradiction.

By the two previous lemmas, there are (j,k)-sets $(A_m : m\in\omega)$ in the ground model such that if π disrupts all A_m then p forces that π disrupts $\dot{A}$. Let π be a permutation satisfying this assumption. Thus p forces that $\pi[\dot{A}]$ does not have asymptotic density (by the above it forces $\underline{d}(\pi[\dot{A}]) \leq \frac{j+1}{k}$ and $\overline{d}(\pi[\dot{A}]) = 1$). This completes the argument showing that the ground model permutations witness $\mathfrak{dd}_{(0,1),\mathsf{all}} = \aleph_1$. □

Theorem 35. *$\mathfrak{dd}_{(0,1),[0,1]} > \mathsf{cof}(\mathcal{N})$ is consistent.*

Note that this establishes in particular that $\mathfrak{dd}_{(0,1),[0,1]} > \max\{\mathfrak{b},\mathsf{cov}(\mathcal{N}),\mathsf{cov}(\mathcal{M})\}$ is consistent (this max is a lower bound of $\mathfrak{dd}_{(0,1),[0,1]}$ by item (4) of Corollary 30).

Proof. This holds in the Silver model, that is, the model obtained by the ω_2-stage countable support iteration of Silver forcing $\mathbb{S}$. $\mathsf{cof}(\mathcal{N}) = \aleph_1$ is well-known (this is so because the countable support product of Silver forcing has the Sacks property [15, Lemma 24.2 and Related Result 132], and the Sacks property implies that bases of the null ideal are preserved [5, Lemma 6.3.39]). We show $\mathfrak{dd}_{\{\frac{1}{2}\},[0,1]} = \aleph_2 = \mathfrak{c}$.

Let $(I_n : n\in\omega)$ be an interval partition of ω such that $|I_n| = 2^n$. Let $s_n \in 2^{I_n}$ be an alternating sequence of zeros and ones, i.e.,

$$s_n(\min(I_n)) = 0, s_n(\min(I_n)+1) = 1, s_n(\min(I_n)+2) = 0, ...$$

Let $\bar{s}_n = 1 - s_n$.

Recall that Silver forcing $\mathbb{S}$ consists of functions $f:\omega\to 2$ with infinite codomain. The order is given by $g\leq f$ if $g\supseteq f$. If G is $\mathbb{S}$-generic over V, $x = \bigcup\{f\in\mathbb{S} : f\in G\}$ is the generic Silver real. Fix $n\in\omega$. If $|(x\restriction n)^{-1}(\{1\})|$ is even let $a_n = s_n$, if it is odd let $a_n = \bar{s}_n$. Put $a = \bigcup_n a_n \in 2^\omega$ and think of a as a subset of ω. By construction $d(a) = \frac{1}{2}$. Let $\dot{a}$ be an $\mathbb{S}$-name for a.

Let $f\in\mathbb{S}$ and let π be a permutation in V. Suppose that for some $r>0$, $f\Vdash \overline{d}(\pi[\dot{a}]) \geq \frac{1}{2}+r$. We show that $f\Vdash \underline{d}(\pi[\dot{a}]) \leq \frac{1}{2}-r$. In particular, the ground model permutations are not a witness for $\mathfrak{dd}_{\{\frac{1}{2}\},[0,1]}$, and $\mathfrak{dd}_{\{\frac{1}{2}\},[0,1]} = \aleph_2$ in the Silver model follows.

Let $g\leq f$ and $\varepsilon>0$ be arbitrary. Let $n_0 = \min(\omega\setminus\mathrm{dom}(g))$. Let k_0 be so large that $\pi[\bigcup_{i\leq n_0} I_i]\subseteq k_0$ and $\pi^{-1}[\bigcup_{i\leq n_0} I_i]\subseteq k_0$ and such that for some $g'\leq g$,

$$g'\Vdash \frac{|\pi[\dot{a}]\cap k_0|}{k_0} \geq \frac{1}{2}+r-\varepsilon$$

Let $b_0 = \pi^{-1}[k_0]\cap(\omega\setminus k_0)$ and $b_1 = \pi^{-1}[\omega\setminus k_0]\cap k_0$. Clearly $|b_0| = |b_1|$ and $\pi[b_0]\subseteq k_0\setminus\bigcup_{i\leq n_0} I_i$ and $b_1\subset k_0\setminus\bigcup_{i\leq n_0} I_i$. Let n_1 be such that $b_0,\pi[b_1]\subseteq\bigcup_{i\leq n_1} I_i$. We may assume $n_1\subseteq\mathrm{dom}(g')$. This means that g' decides $\bigcup_{i\leq n_1}\dot{a}_i = \dot{a}\restriction\bigcup_{i\leq n_1} I_i$. Now let g'' be such that $g''\leq g$, $\mathrm{dom}(g'') = \mathrm{dom}(g')$, $g''(n_0) = 1-g'(n_0)$, and $g''(k) = g'(k)$ for $k\in\mathrm{dom}(g')\setminus\{n_0\}$. Letting c_0 and c_1 be the values forced to $\dot{a}\restriction b_0$ and $\dot{a}\restriction b_1$ by g', respectively, we see immediately that $g''\Vdash$ "$\dot{a}\restriction b_0 = 1-c_0$ and $\dot{a}\restriction b_1 = 1-c_1$". Since the trivial condition forces $\pi[\dot{a}]\cap k_0 = [(\dot{a}\cap k_0)\cup\dot{a}\restriction b_0]\setminus\dot{a}\restriction b_1$, we see that

$$g'' \Vdash \frac{|\pi[\dot{a}] \cap k_0|}{k_0} \leq \frac{1}{2} - r + \varepsilon$$

as required. □

Presumably a modification of this argument shows that $\mathfrak{dd}_{\{r\},[0,1]} = \mathfrak{c}$ holds for any r and not just $\frac{1}{2}$ in the Silver model.

We can simultaneously separate four of the density numbers:

Theorem 36. *It is consistent that* $\aleph_1 = \mathfrak{dd}_{(0,1),\mathsf{all}} < \mathfrak{dd} = \mathsf{non}(\mathcal{M}) < \mathsf{cov}(\mathcal{M}) = \mathfrak{dd}_{(0,1),[0,1]} = \mathfrak{r} < \mathfrak{dd}_{[0,1],[0,1]} = \mathfrak{c}$.

Proof sketch. This is a standard argument, which we will only sketch. Assume for simplicity GCH. Let $\aleph_1 < \kappa < \lambda < \mu$ be cardinals with κ and λ being regular and μ of uncountable cofinality. Let $f : \lambda \to \kappa$ be a function such that $f^{-1}(\{\alpha\})$ is cofinal in λ for each $\alpha < \kappa$. First add μ Cohen reals to force $\mathfrak{c} = \mu$.

Then perform a *matrix iteration*, that is, a two-dimensional system of finite support iterations $\langle \mathbb{P}_{\alpha,\gamma}, \dot{\mathbb{Q}}_{\alpha,\gamma} : \gamma < \lambda \rangle$, $\alpha < \kappa$, such that

1. $\mathbb{P}_{\alpha,0}$ is the forcing adding α Cohen reals $(c_\beta : \beta < \alpha)$,
2. $\mathbb{P}_{\alpha,\gamma+1}$ is the two-step iteration $\mathbb{P}_{\alpha,\gamma} \star \dot{\mathbb{Q}}_{\alpha,\gamma}$, and if $\alpha < f(\gamma)$, then $\dot{\mathbb{Q}}_{\alpha,\gamma}$ is a $\mathbb{P}_{\alpha,\gamma}$-name for the trivial forcing, while if $\alpha \geq f(\gamma)$, then $\dot{\mathbb{Q}}_{\alpha,\gamma}$ is a $\mathbb{P}_{\alpha,\gamma}$-name for Mathias forcing $\mathbb{M}_{\dot{\mathcal{U}}_\gamma}$ where $\dot{\mathcal{U}}_\gamma$ is a $\mathbb{P}_{f(\gamma),\gamma}$-name for an ultrafilter in $V_{f(\gamma),\gamma}$,
3. if γ is a limit ordinal, then $\mathbb{P}_{\alpha,\gamma}$ is the direct limit of the $\mathbb{P}_{\alpha,\delta}$, $\delta < \gamma$,
4. $\mathbb{P}_{\alpha,\gamma} \lessdot \mathbb{P}_{\beta,\delta}$ for $\alpha \leq \beta$ and $\gamma \leq \delta$.

As usual $V_{\alpha,\gamma}$ denotes the intermediate extension via $\mathbb{P}_{\alpha,\gamma}$. It is clear that this construction can be carried out. For item (4) note that $\mathbb{P}_{\alpha,\gamma} \lessdot \mathbb{P}_{\alpha,\delta}$ is trivial because we are dealing with a standard iteration while $\mathbb{P}_{\alpha,\delta} \lessdot \mathbb{P}_{\beta,\delta}$ is proved by induction on $\delta < \lambda$. The basic step is obvious by (1), and so is the successor step by (2). For the limit step use e.g. [10, Lemma 10].

We need to see that the $\mathbb{P}_{\kappa,\lambda}$-generic extension $V_{\kappa,\lambda}$ is the required model. Clearly $\mathfrak{dd}_{[0,1],[0,1]} = \mathfrak{c} = \mu$ is preserved (see Corollary 30 (2)), and $\mathfrak{dd}_{(0,1),\mathsf{all}} = \aleph_1$ follows from the fact that all iterands are σ-centered and the techniques of the proof of Theorem 31 (in particular Lemmas 33 and 34).

Next, since $\langle \mathbb{P}_{\kappa,\gamma} : \gamma < \lambda \rangle$ is a finite support iteration of length λ, $\mathsf{cov}(\mathcal{M}) \geq \lambda$ follows. On the other hand, the Cohen reals $\{c_\alpha : \alpha < \kappa\}$ form a non-meager set of size κ in the final extension so that $\mathsf{non}(\mathcal{M}) \leq \kappa$ holds.

To see the latter, it suffices to prove by induction on $\gamma \leq \lambda$ that

$$\mathbb{P}_{\alpha,\gamma} \times \mathbb{C}_\alpha \lessdot \mathbb{P}_{\alpha+1,\gamma} \quad (\star)$$

for all $\alpha < \kappa$, where $\mathbb{C}_\alpha$ adds the α-th Cohen real c_α in the initial step, that is, $\mathbb{P}_{\alpha+1,0} = \mathbb{P}_{\alpha,0} \times \mathbb{C}_\alpha$. In particular, for $\gamma = 0$, $(\star)$ is obvious. Next assume $\gamma = \delta + 1$ is successor. If $\alpha < f(\delta)$, then $\mathbb{P}_{\alpha,\gamma} = \mathbb{P}_{\alpha,\delta}$ and $(\star)$ is immediate by induction hypothesis. If $\alpha \geq f(\delta)$, then $\mathbb{P}_{\beta,\gamma} = \mathbb{P}_{\beta,\delta} \star \mathbb{M}_{\dot{\mathcal{U}}_\delta}$ for $\beta \in \{\alpha, \alpha+1\}$. Therefore by induction hypothesis and the product lemma, we obtain

$$\mathbb{P}_{\alpha,\gamma} \times \mathbb{C}_\alpha = (\mathbb{P}_{\alpha,\delta} \star \mathbb{M}_{\dot{\mathcal{U}}_\delta}) \times \mathbb{C}_\alpha = (\mathbb{P}_{\alpha,\delta} \times \mathbb{C}_\alpha) \star \mathbb{M}_{\dot{\mathcal{U}}_\delta} \lessdot \mathbb{P}_{\alpha+1,\delta} \star \mathbb{M}_{\dot{\mathcal{U}}_\delta} = \mathbb{P}_{\alpha+1,\gamma}$$

as required. Finally, if γ is a limit ordinal, a slight modification of the argument of [10, Lemma 10] works: let $\{(p_n, c_n) : n \in \omega\}$ be a maximal antichain in $\mathbb{P}_{\alpha,\gamma} \times \mathbb{C}_\alpha$, and let p be a condition in $\mathbb{P}_{\alpha+1,\gamma}$. There is $\delta < \gamma$ such that $p \in \mathbb{P}_{\alpha+1,\delta}$.

The projection $\{(p_n{\restriction}\delta, c_n) : n \in \omega\}$ of the antichain to $\mathbb{P}_{\alpha,\delta} \times \mathbb{C}_\alpha$ is a predense set. Therefore, by induction hypothesis, there is n such that p and $(p_n{\restriction}\delta, c_n)$ are compatible with common extension $q \in \mathbb{P}_{\alpha+1,\delta}$. It is now obvious that q is also compatible with (p_n, c_n) in $\mathbb{P}_{\alpha+1,\gamma}$, and $(\star)$ follows.

For each $\gamma < \lambda$ let m_γ be the Mathias-generic added by $\mathbb{M}_{\mathcal{U}_\gamma}$ over $V_{f(\gamma),\gamma}$, and note that $\{m_\gamma : \gamma < \lambda\}$ is an unreaped family so that $\mathfrak{r} \leq \lambda$ follows. $\mathsf{cov}(\mathcal{M}) = \mathfrak{d}_{(0,1),[0,1]} = \mathfrak{r} = \lambda$ is now immediate by Corollary 30 (4). Finally, if $A \subseteq [\omega]^\omega$ has size less than κ, then there are $\gamma < \lambda$ and $\alpha < \kappa$ such that $A \in V_{\alpha,\gamma}$. Choosing $\delta \geq \gamma$ such that $f(\delta) \geq \alpha$ we see that the Mathis generic m_δ is not split by any member of A. Thus $\mathfrak{s} \geq \kappa$. Since $\mathfrak{s} \leq \mathsf{non}(\mathcal{M})$ holds in ZFC, $\mathfrak{s} = \mathsf{non}(\mathcal{M}) = \kappa$ follows, and the proof is complete. □

5. Questions and final remarks

As mentioned after Theorem 18, one of the questions we could not answer is the following:

Question 37. *Is $\mathfrak{d}\mathfrak{d}_{\{\mathsf{osc}\},\mathsf{all}} \geq \mathsf{cov}(\mathcal{N})$?*

This is really a problem about the random model: if random forcing adds a set without density such that for all ground model permutations, the permuted set still has no density, then the answer is yes. If random forcing does not add such a set, $\mathfrak{d}\mathfrak{d}_{\{\mathsf{osc}\},\mathsf{all}} < \mathsf{cov}(\mathcal{N})$ is consistent (and holds in the random model).

We know by item (3) of Corollary 30 that $\mathsf{cov}(\mathcal{N}) \leq \mathfrak{d}\mathfrak{d}_{(0,1),\mathsf{all}} \leq \min\{\mathfrak{r}, \mathsf{non}(\mathcal{M})\}$ and by Theorem 31 the second inequality can be consistently strict. However, we do not know whether one can separate $\mathfrak{d}\mathfrak{d}_{(0,1),\mathsf{all}}$ from $\mathsf{cov}(\mathcal{N})$.

Question 38. *Is $\mathsf{cov}(\mathcal{N}) < \mathfrak{d}\mathfrak{d}_{(0,1),\mathsf{all}}$ consistent? Or are the cardinals equal?*

Similarly, by (4) of Corollary 30, we know $\max\{\mathfrak{b}, \mathsf{cov}(\mathcal{N}), \mathsf{cov}(\mathcal{M})\} \leq \mathfrak{d}\mathfrak{d}_{(0,1),[0,1]} \leq \mathfrak{r}$ and by Theorem 35 the first inequality is consistently strict, but we do not know the answer to the following:

Question 39. *Is $\mathfrak{d}\mathfrak{d}_{(0,1),[0,1]} < \mathfrak{r}$ consistent? Or are the cardinals equal?*

Note that by Theorem 16, $\mathfrak{d}\mathfrak{d}_{\{0\},\mathsf{all}} = \mathfrak{d}\mathfrak{d}_{\{1\},\mathsf{all}} = \mathsf{non}(\mathcal{M})$. Also, it is easy to see that $\mathfrak{d}\mathfrak{d}_{\{r\},\mathsf{all}} = \mathfrak{d}\mathfrak{d}_{\{1-r\},\mathsf{all}}$ for any $r \in (0,1)$ (and similarly with all replaced by other natural choices like $[0,1]$, $\{0,1\}$, or $(0,1)$), though these cardinals may be strictly smaller than $\mathsf{non}(\mathcal{M})$. We do not know:

Conjecture 40. *$\mathfrak{d}\mathfrak{d}_{\{r\},\mathsf{all}} = \mathfrak{d}\mathfrak{d}_{\{\frac{1}{2}\},\mathsf{all}}$ for all $r \in (0,1)$ (and similarly with all replaced by $[0,1]$, $\{0,1\}$, or $(0,1)$).*

A more sweeping conjecture would be:

Conjecture 41. *Assume $X, X' \subseteq \mathsf{all}$ are such that $X \setminus (0,1) = X' \setminus (0,1)$, and let Y be arbitrary. Then $\mathfrak{d}\mathfrak{d}_{X,Y} = \mathfrak{d}\mathfrak{d}_{X',Y}$.*

This would mean $\mathfrak{d}\mathfrak{d}_{X,Y}$ is completely independent of the intersection $X \cap (0,1)$. It probably depends on $Y \cap (0,1)$, at least on its size, though we have not pursued this (see Theorem 55 for a related result).

A related question is how many of the density numbers can be (consistently) simultaneously distinct. We do have models with four values (Theorem 36).

Question 42. *Can five or more density numbers of the form $\mathfrak{d}\mathfrak{d}_{X,Y}$ be simultaneously distinct? Can infinitely many be simultaneously distinct?*

As for the density number, there are several natural variants of the rearrangement number, and some of them have been considered in the literature [7]. Let us introduce a general framework similar to Definition 15. We use again the symbol osc, and write $\sum_n a_n = \mathsf{osc}$ if the series $\sum_n a_n$ diverges by oscillation. Let $\mathsf{all} = \mathbb{R} \cup \{+\infty, -\infty, \mathsf{osc}\}$.

Definition 43. Assume $X, Y \subseteq \mathsf{all}$ are such that $X \neq Y$ and for all $x \in X$ there is $y \in Y$ with $y \neq x$. The (X,Y)-*rearrangement number* $\mathfrak{rr}_{X,Y}$ is the smallest cardinality of a family $\Pi \subseteq \mathrm{Sym}(\omega)$ such that for every p.c.c. series $\sum_n a_n$ with $\sum_n a_n \in X$ there is $\pi \in \Pi$ such that $\sum_n a_{\pi(n)} \in Y$ and $\sum_n a_{\pi(n)} \neq \sum_n a_n$.

Again, $X' \subseteq X$ and $Y' \supseteq Y$ obviously imply $\mathfrak{rr}_{X',Y'} \leq \mathfrak{rr}_{X,Y}$. Also $\mathfrak{rr} = \mathfrak{rr}_{\mathbb{R},\mathsf{all}}$ and $\mathfrak{rr}' = \mathfrak{rr}_{\mathbb{R}\cup\{\pm\infty\},\mathsf{all}} (= \mathsf{non}(\mathcal{M}))$, where $\mathsf{all} = \mathbb{R} \cup \{\pm\infty\} \cup \{\mathsf{osc}\}$. In [7], the two cardinals $\mathfrak{rr}_i = \mathfrak{rr}_{\mathbb{R},\{\pm\infty\}}$ and $\mathfrak{rr}_f = \mathfrak{rr}_{\mathbb{R},\mathbb{R}}$ were thoroughly investigated, and it was proved that both are above the dominating number $\mathfrak{d}$ and consistently strictly below the continuum $\mathfrak{c}$ [7, Theorems 12, 29, and 38]. With respect to the analogy of the table in Section 2, $\mathfrak{rr}$ would correspond to $\mathfrak{dd}_{(0,1),\mathsf{all}}$, $\mathfrak{rr}_i$ to $\mathfrak{dd}_{(0,1),\{0,1\}}$, and $\mathfrak{rr}_f$ to $\mathfrak{dd}_{(0,1),(0,1)}$, respectively. However, $\mathfrak{rr}$ is above $\mathfrak{b}$ while $\mathfrak{dd}_{(0,1),\mathsf{all}}$ is consistently below $\mathfrak{b}$ (Theorem 31); also, as mentioned above, $\mathfrak{rr}_i$ and $\mathfrak{rr}_f$ are above $\mathfrak{d}$ while $\mathfrak{dd}_{(0,1),\{0,1\}}$ and $\mathfrak{dd}_{(0,1),(0,1)}$ are below $\mathfrak{r}$ (Corollary 30 (4)) with $\mathfrak{r} < \mathfrak{d}$ being consistent (this holds in the Miller and the Blass-Shelah models, see [6, 11.9] and [8]). Why does the analogy break down?

The rough answer is that while we can replace a series converging to a value r by a series with the same positive and negative terms in the same order, and thus converging to the same value r, but containing long intervals of zeroes (this is called "padding with zeroes" in [7]), there is no corresponding operation for infinite-coinfinite sets A. Padding with zeroes should correspond to introducing elements belonging neither to A nor to its complement. This suggests we should consider the relative density of A in some larger set B, with $\omega \setminus B$ playing the role of the set of padded zeroes.

Let $A \subseteq B \subseteq \omega$. Define the *lower relative density of A in B*

$$\underline{d}_B(A) = \lim\inf_{n\to\infty} \frac{|A \cap n|}{|B \cap n|} = \lim\inf_{n\to\infty} \frac{d_n(A)}{d_n(B)}$$

and the *upper relative density of A in B*

$$\overline{d}_B(A) = \lim\sup_{n\to\infty} \frac{|A \cap n|}{|B \cap n|} = \lim\sup_{n\to\infty} \frac{d_n(A)}{d_n(B)}$$

If $\underline{d}_B(A) = \overline{d}_B(A)$, the common value $d_B(A)$ is the *relative density of A in B*. As for density, the interesting case is when both A and $B \setminus A$ are infinite. As in Section 3, say $d_B(A) = \mathsf{osc}$ if $\underline{d}_B(A) < \overline{d}_B(A)$, and let $\mathsf{all} = [0,1] \cup \{\mathsf{osc}\}$.

Definition 44. Assume $X, Y \subseteq \mathsf{all}$ are such that $X \neq \emptyset$ and for all $x \in X$ there is $y \in Y$ with $y \neq x$. The (X,Y)-*relative density number* $\mathfrak{dd}^{\mathsf{rel}}_{X,Y}$ is the smallest cardinality of a family $\Pi \subseteq \mathrm{Sym}(\omega)$ such that for every $A \subseteq B \subseteq \omega$ with A and $B \setminus A$ both infinite and with $d_B(A) \in X$ there is $\pi \in \Pi$ such that $d_{\pi[B]}(\pi[A]) \in Y$ and $d_{\pi[B]}(\pi[A]) \neq d_B(A)$.

Let us provide the Tukey framework for these cardinals: for X, Y as in the definition, consider triples $(D^{\mathsf{rel}}_X, \mathrm{Sym}(\omega), R^{\mathsf{rel}}_Y)$ where D^{rel}_X is the collection of pairs (A, B) such that $A \subseteq B \subseteq \omega$ with both A and $B \setminus A$ infinite and $d_B(A) \in X$, and the relation R^{rel}_Y is given by $(A,B)R^{\mathsf{rel}}_Y\pi$ if $d_{\pi[B]}(\pi[A]) \in Y$ and $d_{\pi[B]}(\pi[A]) \neq d_B(A)$, for $(A,B) \in D^{\mathsf{rel}}_X$ and $\pi \in \mathrm{Sym}(\omega)$. Then $\mathfrak{dd}^{\mathsf{rel}}_{X,Y} = \mathfrak{d}(D^{\mathsf{rel}}_X, \mathrm{Sym}(\omega), R^{\mathsf{rel}}_Y)$, and we let $\mathfrak{dd}^{\mathsf{rel},\perp}_{X,Y} = \mathfrak{b}(D^{\mathsf{rel}}_X, \mathrm{Sym}(\omega), R^{\mathsf{rel}}_Y)$.

Proposition 45. *(1) $(D_X, \mathrm{Sym}(\omega), R_Y) \leq_T (D^{\mathsf{rel}}_X, \mathrm{Sym}(\omega), R^{\mathsf{rel}}_Y)$ for any X, Y.*
(2) $(D^{\mathsf{rel}}_X, \mathrm{Sym}(\omega), R^{\mathsf{rel}}_Y) \leq_T (\mathcal{M}, 2^\omega, \not\ni)$ if $\mathsf{osc} \in Y$ and $\mathsf{osc} \notin X$.

In particular, $\mathfrak{dd}_{X,Y} \leq \mathfrak{dd}^{\mathsf{rel}}_{X,Y}$ always holds.

Corollary 46. *Suppose X, Y are such that $\mathsf{osc} \notin X$, $X \cap \{0,1\} \neq \emptyset$ and $\mathsf{osc} \in Y$. Then $\mathfrak{dd}^{\mathsf{rel}}_{X,Y} = \mathsf{non}(\mathcal{M})$.*

Proof of Proposition 45. For (1), take φ_+ to be the identity function, $\varphi_+(\pi) = \pi$, and $\varphi_-(A) = (A, \omega)$, for $A \in D_X$.

The proof of (2) is analogous to the first part of the proof of Theorem 8. □

The next two theorems (and their corollaries) should be seen as analogues of the results $\mathfrak{b} \leq \mathfrak{rr}$ and $\mathfrak{d} \leq \mathfrak{rr}_i, \mathfrak{rr}_f$ about rearrangement numbers [7, Theorems 11 and 12].

Theorem 47. *$(\omega^\omega, \omega^\omega, \not\geq^*) \leq_T (D^{\mathsf{rel}}_X, \mathrm{Sym}(\omega), R^{\mathsf{rel}}_Y)$ for any X, Y.*

Corollary 48. *$\mathfrak{b} \leq \mathfrak{dd}^{\mathsf{rel}}_{X,Y}$ for any X, Y.*

Proof of Theorem 47. From [7], recall the definition of the cardinal $\mathfrak{j}$ and the proof that $\mathfrak{b} = \mathfrak{j}$. There, the argument for $\mathfrak{b} \leq \mathfrak{j}$ can be easily modified to give a reduction $(\omega^\omega, \omega^\omega, \not\geq^*) \leq_T ([\omega]^\omega, \mathrm{Sym}(\omega), \not P)$, where $AP\pi$ means that π does not change the relative order of members of A except possibly for finitely many elements (in this case, we say that π *preserves* A). Indeed, for each $\pi \in \mathrm{Sym}(\omega)$, let $\psi_+(\pi) = f_\pi \in \omega^\omega$ be such that $n < f_\pi(n)$ and $\pi(x) < \pi(y)$ for all $x \leq n < f_\pi(n) \leq y$. Also, for each strictly increasing $g \in \omega^\omega$, put $\psi_-(g) = A_g = \{g(0), g(g(0)), \ldots, g^k(0), \ldots\}$. If g is not of this form, choose some strictly increasing $g' > g$ and put $\psi_-(g) = \psi_-(g')$. Then the implication $g \geq^* f_\pi \implies A_g P \pi$ holds, so (ψ_-, ψ_+) gives the desired Tukey reduction.

Now, it suffices to show that $([\omega]^\omega, \mathrm{Sym}(\omega), \not P) \leq_T (D^{\mathsf{rel}}_X, \mathrm{Sym}(\omega), R^{\mathsf{rel}}_Y)$. To this end, first let φ_+ be the identity on $\mathrm{Sym}(\omega)$. To define $\varphi_- : [\omega]^\omega \to D^{\mathsf{rel}}_X$, fix, for each $B \in [\omega]^\omega$, a set $A^B \subseteq B$ such that $d_B(A^B) \in X$, and put $\varphi_-(B) = (A^B, B)$. Suppose $BP\pi$. Then π preserves B, so $d_{\pi(B)}(\pi(A^B)) = d_B(A^B)$, and $\varphi_-(B)R^{\mathsf{rel}}_Y\pi$ fails. This completes the proof of the theorem. □

Theorem 49. *If $\mathsf{osc} \notin Y$ and $X \cap [0,1] \neq \emptyset$, then $(\omega^\omega, \omega^\omega, \leq^*) \leq_T (D^{\mathsf{rel}}_X, \mathrm{Sym}(\omega), R^{\mathsf{rel}}_Y)$.*

If $X = \{\mathsf{osc}\}$ and $0 \notin Y$ or $1 \notin Y$, this also holds by Theorem 24. We do not know whether this is still true if $X = \{\mathsf{osc}\}$ and $\{0,1\} \subseteq Y$.

Corollary 50. *If $\mathsf{osc} \notin Y$ and $X \cap [0,1] \neq \emptyset$, then $\mathfrak{d} \leq \mathfrak{dd}^{\mathsf{rel}}_{X,Y}$.*

Proof of Theorem 49. The case $\mathsf{osc} \notin X$ clearly suffices, and an argument analogous to the one in [7] for $\mathfrak{d} \leq \mathfrak{rr}_{fi}$ works. We provide the necessary adaptations to get a Tukey reduction.

So, suppose $\mathsf{osc} \notin X$. For $g \in \omega^\omega$ and $\pi \in \mathrm{Sym}(\omega)$, let $\varphi_-(g) = (A_g, B_g)$ and $\varphi_+(\pi) = f_\pi$, where B_g and f_π are as in the last proof, and $d_{B_g}(A_g) \in X$. Now, to get the implication $(A_g, B_g)R^{\mathsf{rel}}_Y\pi \implies g \leq^* f_\pi$, note that if $g \not\leq^* f_\pi$ then the sequence $\left(\frac{|\pi[A_g] \cap n|}{|\pi[B_g] \cap n|} : n \in \omega\right)$ has a subsequence converging to $d_{B_g}(A_g)$. So $(A_g, B_g)R^{\mathsf{rel}}_Y\pi$ fails, and this completes the proof. □

The above results suggest the question of how far the analogy between $\mathfrak{rr}_{X,Y}$ and $\mathfrak{dd}^{\mathsf{rel}}_{X,Y}$ goes. In particular, one could ask:

Question 51. *Is* $\mathfrak{rr} = \mathfrak{dd}^{\mathsf{rel}}_{(0,1),\mathsf{all}}$*?*

The point is that both cardinals have the same lower bounds $\mathfrak{b}$ and $\mathsf{cov}(\mathcal{N})$ and the same upper bound $\mathsf{non}(\mathcal{M})$ (see [7] and Theorem 18, Proposition 45, and Theorem 47). Analogously one may ask whether $\mathfrak{rr}_i = \mathfrak{dd}^{\mathsf{rel}}_{(0,1),\{0,1\}}$ or $\mathfrak{rr}_f = \mathfrak{dd}^{\mathsf{rel}}_{(0,1),(0,1)}$.

We may also look at the similarity or nonsimilarity between rearrangement numbers and relative density numbers by considering the set of permutations that leave all conditionally convergent series (all relative densities, resp.) fixed. The former have been studied in a number of papers (e.g. [2] or [14]), while a connection between the two has been established by Garibay, Greenberg, Resendis, and Rivaud [14]. But, alas, they have a different notion of relative density! Let $\{b_i : i \in \omega\}$ be the increasing enumeration of B. Say that $A \subseteq B$ has *strong relative density* r in B, $sd_B(A) = r$, if given any $\varepsilon > 0$ there is N such that if $m - n > N$, then

$$\left| \frac{|A \cap \{b_n, b_{n+1}, ..., b_{m-1}\}|}{m-n} - r \right| < \varepsilon$$

Note that $sd_B(A) = r$ implies $d_B(A) = r$ but not vice-versa. Say that $\pi \in \mathrm{Sym}(\omega)$ *preserves c.c. series* (*preserves (strong) density*, resp.) if given any conditionally convergent series $\sum a_n$ (any sets $A \subseteq B$ such that the (strong) relative density $(s)d_B(A)$ exists, resp.), $\sum_n a_{\pi(n)} = \sum a_n$ ($(s)d_{\pi[B]}(\pi[A]) = (s)d_B(A)$, resp.) holds. We need some more combinatorial notions:

Definition 52.
(1) For finite subsets $M, N \subseteq \omega$ write $M < N$ if $\max(M) < \min(N)$.
(2) [14, Definition 1.2] Two sets $M = \{m_0 < ... < m_k\}$ and $N = \{n_0 < ... < n_k\}$ of natural numbers of the same size are *collated* if $m_0 < n_0 < m_1 < ... < m_k < n_k$.
(3) [14, Definition 1.3] $\pi \in \mathrm{Sym}(\omega)$ *satisfies condition* A if there exists $k \in \omega$ such that whenever M and N are collated and $\pi[M] < \pi[N]$ then $|N| = |M| < k$.
(4) $\pi \in \mathrm{Sym}(\omega)$ *satisfies condition* B if there exists $k \in \omega$ such that for any M, N with $|M| = |N|$, $M < N$, and $\pi[N] < \pi[M]$, we have $|M| = |N| < k$.

It is easy to see that the permutations satisfying condition B form a subgroup of $\mathrm{Sym}(\omega)$. On the other hand, inverses of permutations with condition A do not necessarily have condition A [14, Example 1.8]. Furthermore, if π satisfies condition B it also satisfies condition A, while it is easy to see there are π such that both π and π^{-1} satisfy condition A but not condition B. The main result of [14] is:

Theorem 53 (Garibay, Greenberg, Resendis, and Rivaud)**.** *For* $\pi \in \mathrm{Sym}(\omega)$*, the following are equivalent:*

(1) π *satisfies condition* A*,*
(2) there exists $k \in \omega$ *such that for every* n*,* $\pi^{-1}[n]$ *is a union of at most* k *intervals,*
(3) π^{-1} *preserves c.c. series,*
(4) π *preserves strong density.*

The equivalence of (2) and (3) is originally due to Agnew [2], and the equivalence of the two combinatorial conditions (1) and (2) is relatively easy to see (see also [14, Proposition 2.2]).

Theorem 54. *For $\pi \in \mathrm{Sym}(\omega)$, the following are equivalent:*

(1) π satisfies condition B*,*
(2) π preserves density.

Proof. (1) $\implies$ (2): Assume π does not preserve density. So there are $A \subseteq B \subseteq \omega$ such that $r := d_B(A)$ is defined and $d_{\pi[B]}(\pi[A])$ is distinct (possibly undefined). Without loss of generality $\overline{d}_{\pi[B]}(\pi[A]) > d_B(A) = r$. Let $s > r$ be such that for infinitely many k, $\frac{|\pi[A]\cap k|}{|\pi[B]\cap k|} \geq s$. Let $\varepsilon := \frac{s-r}{2}$, and choose ℓ^* such that for all $\ell \geq \ell^*$, $\frac{|A\cap \ell|}{|B\cap \ell|} < r + \varepsilon$. Let $(k_i : i \in \omega)$ be an increasing enumeration of k with $\frac{|\pi[A]\cap k|}{|\pi[B]\cap k|} \geq s$ and such that $|\pi[B] \cap k_0| \geq |B \cap \ell^*|$. Let $(\ell_i : i \in \omega)$ and $(m_i : i \in \omega)$ be such that $m_i = |B \cap \ell_i| = |\pi[B] \cap k_i|$. Since

$$\frac{|\pi[A] \cap k_i|}{m_i} \geq s > \frac{r+s}{2} > \frac{|A \cap \ell_i|}{m_i}$$

there must be $A_i \subseteq A \setminus \ell_i$ of size $> \varepsilon \cdot m_i$ such that $\pi[A_i] \subseteq k_i$ and $B_i \subseteq B \cap \ell_i$ of the same size such that $\pi[B_i] \subseteq \omega \setminus k_i$. Hence $B_i < A_i$ and $\pi[A_i] < \pi[B_i]$ and condition B fails.

(2) $\implies$ (1): Recursively define $(k_i : i \in \omega)$ such that $k_0 = 2$ and $k_{i+1} = 2^{i+1} \sum_{j\leq i} k_j$. Let $\ell_i = \sum_{j\leq i} k_j$; so $\ell_0 = 0$ and $k_{i+1} = 2^{i+1} \cdot \ell_{i+1}$. Assume π does not satisfy condition B. Then there are finite subsets $N_i \subseteq \omega$ and $M_i \subseteq \omega$ such that $|N_i| = |M_i| = k_i$, $M_i < N_i < M_{i+1}$, and $\pi[N_i] < \pi[M_i] < \pi[N_{i+1}]$, $i \in \omega$. Let $A_i \subseteq N_i$ of size ℓ_i such that $\pi[A_i]$ is an initial segment of $\pi[N_i]$. Let $B_i \subseteq N_i \setminus A_i$ be arbitrary of size ℓ_i. Let $k'_0 = k_0 = 2$ and $k'_i = (2^i - 2)\ell_i = k_i - 2\ell_i$ for $i > 0$. Let $(n_{i,j} : j < k'_i)$ be the increasing enumeration of $N_i \setminus (A_i \cup B_i)$ and let $(m_{i,j} : j < k_i)$ be the increasing enumeration of M_i. Let $B = \bigcup_{i\in\omega}(N_i \cup M_i)$ and

$$A = \bigcup_{i\in\omega} \left(A_i \cup \left\{ n_{i,2j} : j < \frac{k'_i}{2} \right\} \cup \left\{ m_{i,2j} : j < \frac{k_i}{2} \right\} \right)$$

We first check that $d_B(A) = \frac{1}{2}$. Indeed, let $m \in \omega$. Let i be minimal such that $M_i < m$. If also $N_i < m$, we easily see that

$$\frac{1}{2} \leq \frac{|A \cap m|}{|B \cap m|} \leq \frac{|B \cap m| + 2}{2|B \cap m|}$$

Otherwise let $\bar{m}_i = \min N_i$ and note that $|B \cap \bar{m}_i| = 2\ell_i + k_i = (2^i + 2)\ell_i$ and, similarly, $|A \cap \bar{m}_i| = (2^{i-1} + 1)\ell_i$. Then we have:

$$\frac{(2^{i-1}+1)\ell_i + \frac{|N_i\cap m| - \ell_i}{2}}{(2^i+2)\ell_i + |N_i \cap m|} \leq \frac{|A \cap m|}{|B \cap m|} \leq \frac{(2^{i-1}+1)\ell_i + \frac{|N_i\cap m| + \ell_i + 2}{2}}{(2^i+2)\ell_i + |N_i \cap m|}$$

Clearly, the upper and lower bounds converge in both cases to $\frac{1}{2}$ as i goes to ∞.

On the other hand, it is easy to see that, letting $\bar{a}_i = \max \pi[A_i] + 1$ and $\bar{n}_i = \min \pi[M_i]$,

$$\frac{|\pi[A] \cap \bar{a}_i|}{|\pi[B] \cap \bar{a}_i|} = \frac{2\ell_i}{3\ell_i} = \frac{2}{3}$$

and

$$\frac{|\pi[A] \cap \bar{n}_i|}{|\pi[B] \cap \bar{n}_i|} = \frac{\ell_i + \frac{k_i}{2}}{2\ell_i + k_i} = \frac{1}{2}$$

so that $\overline{d}_{\pi[B]}(\pi[A]) \geq \frac{2}{3}$ and $\underline{d}_{\pi[B]}(\pi[A]) \leq \frac{1}{2}$. Thus, π does not preserve density. □

These results show that the set of permutations preserving c.c. series and density are actually distinct. We have no idea whether this means that we can also distinguish the two concepts, rearrangement of c.c. series and of relative density, on the level of cardinal invariants (see Question 51).

Still regarding the analogy between $\mathfrak{rr}_{X,Y}$ and $\mathfrak{dd}^{\mathsf{rel}}_{X,Y}$, in both cases it is natural to expect these cardinals to be big if Y is small in some sense. In the case of the former, one has:

Theorem 55. *If $Y \subseteq \mathbb{R}$ and $|Y| < \mathfrak{c}$, then $\mathfrak{rr}_{\mathbb{R},Y} = \mathfrak{c}$.*

Proof. Fix $Y \subseteq \mathbb{R}$ and let $\mathcal{F}$ be a family witnessing $\mathfrak{rr}_{\mathbb{R},Y}$. Also, let $\sum_n a_n$ be a c.c. series, say converging to a.

For each $t \in \mathbb{R}$, let $\sum_n c_n^t$ be a series converging absolutely to $t - a$. Putting $a_n^t = a_n + c_n^t$, we get that $\sum_n a_n^t$ is a c.c. series converging to t. For each $t \in \mathbb{R}$, we can choose $\pi^t \in \mathcal{F}$ and $x^t \in Y$, $x^t \neq t$, such that $\sum_n a^t_{\pi^t(n)}$ converges to x^t.

Now, $\mathbb{R} = \bigcup_{x \in Y} \{t \in \mathbb{R} : x^t = x\}$. Since $|Y| < \mathfrak{c}$, for every $\kappa < \mathfrak{c}$ there is $s \in Y$ such that the set $S_s := \{t \in \mathbb{R} : x^t = s\}$ has cardinality strictly larger than κ.

Let us show that the function $t \mapsto \pi^t$ is injective when restricted to any S_s, which guarantees that $\mathcal{F}$ has cardinality $\mathfrak{c}$.

Indeed, let $t_1, t_2 \in S$ (i.e., $x^{t_i} = s$) and suppose $\pi^{t_1} = \pi^{t_2} =: \pi$. Then

$$\left(\sum_n a_{\pi(n)}\right) + t_1 - a = \sum_n a^{t_1}_{\pi(n)} = s = \sum_n a^{t_2}_{\pi(n)} = \left(\sum_n a_{\pi(n)}\right) + t_2 - a$$

so $t_1 = t_2$. □

It does not appear obvious how to adapt the above line of reasoning to $\mathfrak{dd}^{\mathsf{rel}}_{(0,1),Y}$. Indeed, the fact that one can add an absolutely convergent series to an arbitrary series without essentially changing any information related to the convergence of the latter, even after rearranging the terms, is essential in the above proof. The set of infinite-coinfinite subsets of ω having asymptotic density lacks this structure.

Question 56. *Suppose $Y \subseteq (0,1)$ is such that $|Y| < \mathfrak{c}$. Is $\mathfrak{dd}^{\mathsf{rel}}_{(0,1),Y} = \mathfrak{c}$?*

Still, one could reestablish the analogy by considering the following: for a real sequence $(a_0, a_1, a_2, \ldots)$, let

$$\mu(a_0, a_1, a_2, \ldots) = \lim_{n\to\infty} \frac{a_0 + \ldots + a_{n-1}}{n}$$

be its *asymptotic mean*, if the limit exists. The sequences that have the same asymptotic mean regardless of the order of the terms are the ones that converge. The following is analogous to Riemann's Rearrangement Theorem:

Theorem 57. *Let $r = (r_0, r_1, r_2, \ldots)$ be a sequence of real numbers having asymptotic mean. Then, there is $\pi \in S_\omega$ such that $\mu(r_{\pi(0)}, r_{\pi(1)}, r_{\pi(2)}, \ldots) = m$ if, and only if*

$$\liminf(r) \leq m \leq \limsup(r).$$

We use the fact that if a real sequence has asymptotic mean and the terms of another real sequence are put in a sufficiently sparse set of indices, then the mean does not change:

Lemma 58. *Let* $(x_0, x_1, \ldots)$ *and* $(y_0, y_1, \ldots)$ *be sequences of real numbers such that* $\mu(x_0, x_1, \ldots) = m$*. Then there is a set* $A = \{a_0 < a_1 < \ldots\} \subseteq \omega$ *such that, if* $\omega \setminus A = \{b_0 < b_1 < \ldots\}$ *and* $(z_0, z_1, \ldots)$ *is defined by*

$$z_k = \begin{cases} x_n & \text{if } k = b_n \\ y_n & \text{if } k = a_n \end{cases}$$

then $\mu(z_0, z_1, \ldots) = m$*.*

Proof. Let the N-th partial sum of $(z_0, z_1, \ldots)$ be

$$\sum_{k=0}^{N-1} z_k = \sum_{k=0}^{n-1} x_k + \sum_{k=0}^{\ell-1} y_k.$$

Note that

$$\frac{1}{N}\sum_{k=0}^{N-1} z_k - \frac{1}{n}\sum_{k=0}^{n-1} x_k = \frac{1}{N}\sum_{k=0}^{\ell-1} y_k - \frac{\ell}{N}\left(\frac{1}{n}\sum_{k=0}^{n-1} x_k\right).$$

Choosing the indices of A amounts to choosing $\ell = \ell(N)$ for each $N \in \omega$. The above equality shows that the conclusion of the Lemma will hold if $\ell(N)$ is such that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{\ell-1} y_k = \lim_{N\to\infty} \frac{\ell}{N} = 0.$$

It is clearly possible to choose $\ell(N)$ to grow so slowly that the above is true. □

Proof of Theorem 57. The "only if" part is clear.

Now, suppose $s = \liminf(r)$, $S = \limsup(r)$, and $s \leq m \leq S$.

If $m = s$, fix a subsequence $(x_0, x_1 \ldots)$ of r such that $\lim_{n\to\infty} x_n = s$ and let the remaining terms of r be $(y_0, y_1 \ldots)$. The sequence $(z_0, z_1 \ldots)$ given by the above Lemma is a rearrangement of r with mean equal to s. The case $m = S$ is treated similarly.

Now, suppose $s < m < S$. Let $u = (u_0, u_1 \ldots)$ and $v = (v_0, v_1, \ldots)$ be subsequences of r converging to s and S respectively. Without loss of generality, suppose the indices of u and v are disjoint and the remaining terms of r form the subsequence $z = (z_0, z_1 \ldots)$.

First, we form a sequence $(x_0, x_1 \ldots)$ using only terms from u and v, in a way that is similar to the traditional proof of Riemann's Theorem: Always respecting the order, keep adding terms from v until the first time $\frac{x_0+\ldots+x_{n-1}}{n} > m$. This is possible, since v converges to $S > m$, Then keep adding terms from u until the first time $\frac{x_0+\ldots+x_{l-1}}{l} < m$, which is possible, since x converges to $s < m$. Proceeding in this way, clearly $\lim_{n\to\infty} \frac{x_0+\ldots+x_{n-1}}{n} = m$.

Now, put the remaining terms $z_1, z_2 \ldots$ in the sparse set of indices A given by the Lemma to get a permuted sequence $c = (c_0, c_1 \ldots)$ such that $\mu(c) = m$. □

Definition 59. Let $\mathcal{D}$ be the set of sequences of real numbers having an asymptotic mean. $\mathfrak{mm}$ is the minimal cardinality of a family $\mathcal{F} \subseteq \mathrm{Sym}(\omega)$ such that, for every $a \in \mathcal{D}$, there is $\pi \in \mathcal{F}$ such that $\mu(\pi[a]) \neq \mu(a)$. Analogously, one can define $\mathfrak{mm}_{X,Y}$ as above.

The above definition clearly implies $\mathfrak{mm} \geq \mathfrak{dd}$. An argument similar to the first (easier) part of the proof of Theorem 8 gives $\mathfrak{mm} \leq \mathsf{non}(\mathcal{M})$, so $\mathfrak{mm} = \mathfrak{dd} = \mathsf{non}(\mathcal{M})$. Still, one can define the variants $\mathfrak{mm}_{X,Y}$, using the obvious definitions, and ask how these cardinal behave in comparison to $\mathfrak{dd}_{X,Y}$. For instance, in direct analogy to Theorem 55, one has:

Theorem 60. *If $Y \subseteq \mathbb{R}$ and $|Y| < \mathfrak{c}$, then $\mathfrak{mm}_{\mathbb{R},Y} = \mathfrak{c}$.*

Proof. The argument is analogous to the one in the proof of Theorem 55. We point out the necessary adaptations.

Fix $Y \subseteq \mathbb{R}$ and let $\mathcal{F}$ be a family witnessing $\mathfrak{mm}_{\mathbb{R},Y}$. Also, let $(a_0, a_1 \ldots)$ be a sequence of real numbers, with asymptotic mean equal to a.

For each $t \in \mathbb{R}$, let $(c_0^t, c_1^t \ldots)$ be sequence of real numbers converging to $t - a$. Putting $a_n^t = a_n + c_n^t$, we get that $(a_0^t, a_1^t \ldots)$ has asymptotic mean equal to t. For each $t \in \mathbb{R}$, we can choose $\pi^t \in \mathcal{F}$ and $x^t \in Y$, $x^t \neq t$, such that $(a^t_{\pi^t(0)}, a^t_{\pi^t(1)} \ldots)$ has mean equal to x^t.

Now, the proof proceeds exactly like in the proof of Theorem 55. $\square$

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