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Combinatorial constructions related to the separable quotient problem in Banach spaces

Abstract.

The separable quotient problem asks whether every infinite dimensional Banach space has a nontrivial separable quotient. In this survey, we review results connecting this problem to the existence of uncountable biorthogonal systems in nonseparable Banach spaces. Our discussion highlights recent advancements which apply combinatorial techniques in their proofs. Additionally, we present an old construction by Todorčević, which proves the existence of a nonseparable Banach space with no uncountable biorthogonal systems under the assumption that the bounding number is equal to $\aleph_1$.

Mathematics Subject Classification (2010): Primary: 03–02 Secondary: 03E05, 46B26, 03E17, 46B20

Keywords: quotients, biorthogonal systems, bounding number

Contents

1. Introduction

The following problem, commonly referred to as the "separable quotient problem", is one of the most significant open questions in Banach space theory.

Problem 1.1. Given an infinite dimensional Banach space X, is there a closed infinite dimensional linear subspace Y of X such that X/Y is an infinite dimensional separable Banach space?

It has likely been considered since the 1930s, alongside other important problems stemming from Banach's seminal work. It is attributed to Stanisław Mazur and Stefan Banach. However, there is no explicit mention of it in Banach's book [3], and I couldn't find any formal record of it. The earliest reference I am aware of, where the problem is explicitly stated, is Rosenthal's paper [29].

Several other important problems from Banach's book have been solved. E.g. the basis problem, which asks whether every Banach space admits a Schauder basis, was answered in the negative by Enflo in [12]; the basic sequence problem, which asks whether every Banach space admits a basic sequence, was answered in the positive by Mazur (it is stated in [3] with no proof, see also [4]).

Quotients of Banach spaces are useful for gaining insight into the structure of the space itself and vice versa. This is exemplified by the so called three space problems, which ask the following: knowing two out of the three spaces X, $Y \subseteq X$ and X/Y have a certain property, can we conclude that the third space also shares this property? There are several examples of three-space properties in the literature, see [9].

The author was partially supported by FAPESP grants (2016/25574-8 and 2023/12916-1).

During the 1960s and the 1970s, several positive results for the separable quotient problem have been obtained. Let us mention that the following classes of Banach spaces do admit separable quotients:

- Banach spaces containing c_0 (Bessaga, Pełczyński, [5]);
- Reflexive spaces (Pełczyński, [24]);
- Weakly compactly generated spaces (Amir, Lindenstrauss, [1]);
- Separable spaces (Johnson, Rosenthal, [16]);
- Spaces whose dual contains an unconditional basic sequence (Hagler, Johnson, [14]).

These are classical results and most of their original proofs have a structural flavour. In this paper, we review additional results that provide partial answers to Problem 1.1 or solve related problems. These results mostly relate the density of the Banach space to the cardinality of structures therein, and their proofs often combine classical analytic methods with combinatorial approaches. The results discussed here are only a small sample of the diverse combinatorial constructions related to separable quotients in nonseparable Banach spaces.

Section 2 contains results relating the existence of separable quotients, quotients with Schauder basis and biorthogonal systems. Section 3 focuses on the existence of biorthogonal systems in $C(K)$ spaces. The only complete proof is presented in Section 4. We prove an old result by Todorčević which is crucial to the discussion of recent results in Section 5. The reader is assumed to be familiar with classical set theory definitions, which we have not introduced unless necessary for the proofs presented. We also aimed to minimize the introduction of too many definitions from Banach space theory; however, readers can find these definitions in the references if needed. We follow the notation of [18] for set theory and [15] for Banach spaces.

2. Quotients with Schauder bases

The following result about spaces with small density, originally stated in [16, Theorem IV.1(i)] for separable spaces, was proved by Johnson and Rosenthal in the 1970s:

Theorem 2.1 (Johnson, Rosenthal, [16]). *If X is a Banach space of density strictly smaller than $\mathfrak{b}$, then X has a separable quotient.*

Basic sequences play a crucial role in its proof. Schauder bases generalize Hamel bases (for finite-dimensional vector spaces) and orthogonal bases (for Hilbert spaces) to the realm of infinite-dimensional Banach spaces. Recall that a sequence $(e_n)_n$ in a Banach space X is a Schauder basis if every vector $x \in X$ has a unique representation as a series $\sum_n \lambda_n e_n$. As noted in the introduction, it has been known since the 1970s that separable Banach spaces may not have Schauder bases, see [12]. However, it is also known that every infinite-dimensional Banach space X contains a basic sequence, meaning a sequence of vectors that forms a Schauder basis for some infinite-dimensional closed subspace of X, see [4]. In fact, Banach spaces are abundant in basic sequences.

Given a normalized weakly* null sequence $(\varphi_n)_n$ in X^* (ie. $(\varphi_n(x))_n$ converges to 0 for every $x \in X$) and $\Gamma \subseteq \omega$, let

$$Q : X \to \overline{span}\{\varphi_k : k \in \Gamma\}$$

$$Q(x) = \sum_{k\in\Gamma} \varphi_k(x)\varphi_k.$$

Notice that if Q is a well-defined continuous mapping and some nontriviality argument guarantees that it is surjective, this yields the desired separable quotient, isomorphic to $\overline{span}\{\varphi_k : k \in \Gamma\}$.

Let $D \subseteq S_X$ be such that $|D| < \mathfrak{b}$ and the linear span of D is norm-dense in X. $\mathfrak{b}$ is called the bounding number and it states for the smallest cardinality of a subset of ω^ω which is unbounded with respect to $\leqslant^*$, where $f \leqslant^* g$ means that $f(n) \leqslant g(n)$ fails for finitely many n's. For each $x \in D$, since $(\varphi_n)_n$ is weakly* null, let $f_x \in \omega^\omega$ be such that

$$k \geqslant f_x(n) \Rightarrow |\varphi_k(x)| < \frac{1}{2^n}.$$

From $|D| < \mathfrak{b}$, we get that there is a $\leqslant^*$-dominating $f \in \omega^\omega$ for $\{f_x : x \in D\}$.

Without loss of generality, f can be assumed to be strictly increasing. Let $\Gamma = \{f(n) : n \in \omega\}$. For each $x \in D$, there is $n_0 \in \omega$ such that $n \geqslant n_0$ implies $f_x(n) \leqslant f(n)$. Hence,

$$\sum_{k\in\Gamma} |\varphi_k(x)| = \sum_{n\in\omega} |\varphi_{f(n)}(x)| = \sum_{n<n_0} \varphi_{f(n)}(x)| + \sum_{n\geqslant n_0} |\varphi_{f(n)}(x)|$$

$$\leqslant \sum_{n<n_0} |\varphi_{f(n)}(x)| + \sum_{n\geqslant n_0} \frac{1}{2^n},$$

so that the series $\sum_{k\in\Gamma} |\varphi_k^*(x)|$ converges.

Hence, if $(\varphi_{f(n)})_n$ forms a basic sequence in X^*, then we get Q well-defined. The job in [16] is to refine the sequence $(\varphi_n)_n$ in order to get this and ensure the surjectivity.

In this argument, the basic sequence plays a crucial role in identifying a natural candidate for a quotient space, and it allows us to obtain a quotient with a Schauder basis at no additional cost. The following consistency result gets separable quotients with Schauder basis in spaces of large density:

Theorem 2.2 (Dodos, Lopez-Abad, Todorčević, [11])**.** *It is consistent with the usual axioms of ZFC that every Banach space with density at least $\aleph_\omega$ has a separable quotient with an unconditional basis.*

The proof extracts a partition property of some cardinal κ which ensures the existence of an unconditional basic sequence in the dual of every Banach space of density at least κ. A result from [14] and the fact that $\aleph_\omega$ consistently satisfies this partition property imply the previous result. Another result combining combinatorial methods and Hagler and Johnson's result to guarantee the existene of a separable quotient is the following:

Theorem 2.3 (Argyros, Dodos, Kanellopoulos, [2])**.** *Every dual Banach space has a separable quotient.*

Pelczyński asked in [24] whether the following problem is equivalent to the original separable quotient problem:

Problem 2.1. Does every Banach space have a nontrivial quotient with Schauder basis?

Theorem 2.1 implies that this is equivalent to the original separable quotient problem, since a separable quotient would itself have a separable quotient with Schauder basis. A natural stronger version of Problem 2.1 was originally posed by Plichko in [26]:

Problem 2.2. Does every Banach space have a quotient with Schauder basis of the length of its density?

A negative answer was given by Plichko himself. First, he gave a negative answer to the following question, posed by Davis and Johnson in [10].

Problem 2.3. Does every Banach space have a bounded fundamental biorthogonal system?

Recall that a family of pairs $(x_\alpha, \varphi_\alpha)_{\alpha\in\kappa}$ in $X \times X^*$ is a biorthogonal system if $\varphi_\alpha(x_\alpha) = 1$ and $\varphi_\alpha(x_\beta) = 0$ if $\alpha \neq \beta$. It is a fundamental biorthogonal system if moreover $span\{x_\alpha : \alpha \in \kappa\}$ is norm dense in X. If we could start with a biorthogonal system in the argument presented after Theorem 2.1, it would guarantee that the map is surjective, if well-defined. In [25], Plichko showed the following result.

Theorem 2.4. *If* Γ *is an index set of cardinality greater than* $\mathfrak{c}$*, then* $\ell_\infty^{\mathfrak{c}}(\Gamma)$ *admits no bounded fundamental biorthogonal system.*

A few years later Plichko proved in [26] that Problems 2.2 and 2.3 are equivalent. Finally, Godefroy and Louveau posed in [13] the following more general question:

Problem 2.4. Does every nonseparable Banach space have an uncountable biorthogonal system?

The next section discusses consistent negative solutions to this problem in the context of spaces of continuous functions.

3. $C(K)$ spaces without biorthogonal systems

Given a compact Hausdorff space K, let $C(K)$ be the space of continuous real-valued functions on K with the supremum norm. The class of $C(K)$ spaces has been of great importance in Banach space theory, particularly in the context of nonseparable spaces, see [28]. It is a great source of interesting examples and their structure can be analysed from the properties of the topological space K. Moreover, every Banach space X is isometrically isomorphic to a subspace of $C(B_{X^*})$, where the dual ball is equipped with the weak* topology.

The classical Stone-Weierstrass Theorem guarantees that the density of $C(K)$ equals the weight of K and Riesz Representation Theorem identifies each linear continuous functional on $C(K)$ with a regular Borel measure on K.

There is a natural way to get biorthogonal systems in $C(K)$ from discrete subsets of K: if $\{x_\alpha : \alpha \in \Gamma\}$ is a discrete subset of K, Urysohn's Lemma guarantees the existence, for each $\alpha < \kappa$, of $\varphi_\alpha \in C(K)$ such that $\varphi_\alpha(x_\alpha) = 1$ and $\varphi_\alpha(x_\beta) = 0$ for $\beta \neq \alpha$. Taking the point-evaluating functional $\delta_\alpha \in C(K)^*$ defined by $\delta_\alpha(\varphi) = \varphi(x_\alpha)$, we get that $(\varphi_\alpha, \delta_\alpha)_{\alpha\in\Gamma}$ is a biorthogonal system in $C(K)$.

This argument can be improved to show the following result:

Theorem 3.1 (Todorčević, [32])**.** *If a compact Hausdorff space K has a nonseparable subspace, then $C(K)$ contains an uncountable biorthogonal system.*

On the other hand, the following result is some sort of contrapositive for scattered spaces:

Theorem 3.2 (folklore)**.** *Let K be a compact Hausdorff scattered space. If K^n is hereditarily separable for every $n \in \omega$, then $C(K)$ has no uncountable biortogonal systems.*

Proof. Suppose $(f_\alpha, \varphi_\alpha)_{\alpha\in\omega_1}$ is an uncountable biorthogonal system in $C(K)$ and we may assume without loss of generality that $\|f_\alpha\| \leqslant 1$ for every $\alpha \in \omega_1$. From Riesz Representation Theorem, each φ_α is a regular Borel measure on K and, in case of a scattered space, these measures are atomic, i.e. for each $\alpha \in \omega_1$, $\varphi_\alpha = \sum_{n\in\omega} \lambda_n^\alpha \delta_{x_n^\alpha}$ for some sequence of scalars $(\lambda_n^\alpha)_n$ with $\sum_{n\in\omega} |\lambda_n| < \infty$ and some sequence of points $(x_n^\alpha)_n$ in K.

Given a sufficiently small $\varepsilon > 0$ and using the fact that

$$\{\sum_{i<n} \lambda_i \delta_{x_i} : \lambda_i \in \mathbb{Q} \text{ and } x_i \in K\}$$

is norm-dense in $C(K)^*$, counting and approximation arguments give us an uncountable $\Gamma \subseteq \omega_1$, $n \in \omega$ and $(\lambda_i)_{i<n}$ in $\mathbb{Q}$ such that for every $\alpha \in \Gamma$,

$$\|\varphi_\alpha - \sum_{i<n} \lambda_i \delta_{x_i^\alpha}\| < \varepsilon.$$

Let $S = \{(x_1^\alpha, \ldots, x_n^\alpha) : \alpha \in \Gamma\} \subseteq K^n$. From the hypothesis, there is $I \in [\Gamma]^\omega$ such that $S \subseteq \overline{\{(x_1^\alpha, \ldots, x_n^\alpha) : \alpha \in I\}}$. Given any $\gamma \in \Gamma \setminus I$ and $\delta = \frac{\varepsilon}{n\cdot\max\{|\lambda_i|:1\leqslant i\leqslant n\}}$, let

$$U = \Pi_{i=1}^n f^{-1}[(f_\gamma(x_i^\gamma) - \delta, f_\gamma(x_i^\gamma) + \delta)]$$

and notice that $U \cap S$ is an open set in S such that $(x_1^\gamma, \ldots, x_n^\gamma) \in U \cap S$. Hence, there is $\alpha \in I$ such that $(x_1^\alpha, \ldots, x_n^\alpha) \in U$. This implies that

$$\begin{aligned}|\varphi_\gamma(f_\gamma) - \varphi_\alpha(f_\gamma)| &\leqslant \|\varphi_\gamma - \sum_{i=1}^n \lambda_i \delta_{x_i^\gamma}\| + |\sum_{i=1}^n \lambda_i (f_\gamma(x_i^\gamma) - f_\gamma(x_i^\alpha))| + \|\sum_{i=1}^n \lambda_i \delta_{x_i^\alpha} - \varphi_\alpha\| \\ &< \varepsilon + \sum_{i=1}^n |\lambda_i|\delta + \varepsilon < 3\varepsilon,\end{aligned}$$

contradicting the fact that $\varphi_\gamma(f_\gamma) = 1$ and $\varphi_\alpha(f_\gamma) = 0$. $\square$

The previous result derives from arguments from the 1980s related to pointwise convergence and the weak topology in $C(K)$, which can be found in [22, 34, 35]. It identifies a class of Banach spaces where one might seek counterexamples to Question 2.4. Notably, there are several consistent constructions of nonmetrizable compact scattered spaces with hereditarily separable finite powers. The most known is likely Kunen's construction, presented in [22, Theorem 7.1] and achieved under the continuum hypothesis. Additionally, Shelah built such a space under $\diamondsuit$ (see [30]), and a variation of Ostaszewski's construction under $\clubsuit$ (see [23]) was described in [15, Theorem 4.36].

We have chosen to present here the following construction by Todorčević, as it will be crucial in the subsequent discussion:

Theorem 3.3 (Todorčević, Theorem 2.4, [31])**.** *Assuming that $\mathfrak{b} = \aleph_1$, there exists a nonmetrizable compact scattered Hausdorff space K such that K^n is hereditarily separable for every $n \in \omega$. In particular, $C(K)$ is a nonseparable Asplund space with no uncountable biorthogonal systems.*

A complete proof of this result is provided in the next section. Several other consistent examples of nonseparable Banach spaces without uncountable biorthogonal systems have been obtained by forcing. The versatility of the forcing method has enabled the construction of examples with a wide range of properties. We will highlight two constructions that illustrate this diversity by pursuing different directions.

The first construction proves the consistency of a gap between the density of a Banach space and the maximal cardinality of a biorthogonal system.

Theorem 3.4 (Brech, Koszmider, [7])**.** *It is consistent with the usual axioms of set theory ZFC that there exists a compact scattered Hausdorff space K of weight $\aleph_2$ such that K^n is hereditarily separable for every $n \in \omega$. In particular, $C(K)$ is a Banach space of density $\aleph_2$ with no uncountable biorthogonal systems.*

All $C(K)$ constructions discussed so far have the property of being Asplund spaces: a Banach space X is Asplund if every separable subspace has a separable dual. Namioka and Phelps proved in [21] that $C(K)$ is Asplund if and only if K is scattered. This was important in the proof of Theorem 3.2, as it ensured that the functionals on $C(K)$ are atomic measures. Indeed, Asplund spaces can be considered "small", which might explain why the nonseparable examples which do not admit uncountable biorthogonal systems were found in this class, even with density $\aleph_2$. This will be relevant in Section 5. For now, let us turn to the second construction, which demonstrates that being Asplund is consistently not a necessary condition for the nonexistence of uncountable biorthogonal systems in $C(K)$ spaces:

Theorem 3.5 (Koszmider, [17])**.** *It is consistent with the usual axioms of set theory ZFC that there exists a compact Hausdorff space K of weight $\aleph_1$ such that $C(K)$ is a space with no uncountable semi-biorthogonal systems, i.e. there is no sequence of pairs $(x_\alpha, \varphi_\alpha)_{\alpha\in\kappa}$ in $X \times X^*$ such that $\varphi_\alpha(x_\alpha) = 1$, $\varphi_\alpha(x_\beta) = 0$ if $\alpha > \beta$ and $\varphi_\alpha(x_\beta) \geqslant 0$ if $\alpha < \beta$.*

On one hand, it follows from a result of [19] that K is not scattered, hence $C(K)$ is not Asplund. On the other hand, by results from [6] and [32], the example from Theorem 3.5 has an uncountable semi-biorthogonal system.

4. A construction by Todorčević

In this section we provide a proof of Theorem 3.3, obtained by Todorčević, see [31]. Recall that the theorem says that assuming that $\mathfrak{b} = \aleph_1$, there exists a nonmetrizable compact scattered Hausdorff space K such that K^n is hereditarily separable for every $n \in \omega$. In particular, $C(K)$ is a nonseparable Asplund space with no uncountable biorthogonal systems.

Let $(f_\alpha)_{\alpha<\omega_1}$ be an unbounded family in $(\omega^\omega, \leqslant^*)$ and without loss of generality we may assume that $f_\alpha <^* f_\beta$ for every $\alpha < \beta < \omega_1$.

Fix $e : [\omega_1]^2 \to \omega$ a function with the following properties:

- For every $\beta \in \omega_1$, $e_\beta := e(\{\cdot, \beta\}) : \beta \to \omega$ is injective.
- For every $\alpha \in \omega_1$, $\{e_\beta \restriction_\alpha : \beta < \omega_1\}$ is a countable set.

The existence of such e is a consequence of the existence of an Aronszajn tree, see e.g. [18].

Let $\Delta(\alpha, \beta) = \min\{n \in \omega : f_\alpha(n) \neq f_\beta(n)\}$ if $\alpha \neq \beta$, $\Delta(\alpha, \alpha) = \infty$,

$$H(\beta) = \{\alpha < \beta : e(\alpha, \beta) \leqslant f_\beta(\Delta(\alpha, \beta))\}$$

and recursively define

$$V(\beta) = \{\beta\} \cup \bigcup_{\eta \in H(\beta)} \{\alpha \in V(\eta) : \forall \xi \in H(\beta) \cup \{\beta\} \ (\xi \neq \eta \Rightarrow \Delta(\alpha, \xi) < \Delta(\alpha, \eta))\}.$$

Let us denote by $\varphi(\alpha, \eta, \beta)$ the sentence

$$\forall \xi \in H(\beta) \cup \{\beta\} \ (\xi \neq \eta \Rightarrow \Delta(\alpha, \xi) < \Delta(\alpha, \eta)),$$

so that

$$V(\beta) = \{\beta\} \cup \bigcup_{\eta \in H(\beta)} \{\alpha \in V(\eta) : \varphi(\alpha, \eta, \beta) \text{ holds}\}.$$

Finally, let $V_n(\beta) = \{\alpha \in V(\beta) : \Delta(\alpha, \beta) \geqslant n\}$ and we claim that there is a topology τ in ω_1 such that $\{V_n(\beta) : n \in \omega\}$ forms a local topological basis at β. The desired space K will be the one-point compactification of $L := (\omega_1, \tau)$.

Claim 1. If $\alpha \in V_n(\beta)$, then there is $k \in \omega$ such that $V_k(\alpha) \subseteq V_n(\beta)$.

Proof. We prove this by induction on β. Given $\alpha \in V_n(\beta)$, $\alpha \neq \beta$, let $\eta \in H(\beta)$ be such that $\alpha \in V(\eta)$ and $\varphi(\alpha, \eta, \beta)$ holds. In particular, $n \leqslant \Delta(\alpha, \beta) < \Delta(\alpha, \eta)$, so that $\alpha \in V_n(\eta)$. By the inductive hypothesis, there is $k \in \omega$ such that $V_k(\alpha) \subseteq V_n(\eta)$. We may assume without loss of generality $k \geqslant \max\{n, \Delta(\alpha, \eta)\}$ and let us check that $V_k(\alpha) \subseteq V_n(\beta)$. Fix $\alpha' \in V_k(\alpha)$. First, since $\Delta(\alpha', \alpha) \geqslant k \geqslant n$ and $\Delta(\alpha, \beta) \geqslant n$, we get that $\Delta(\alpha', \beta) \geqslant n$. Second, $\Delta(\alpha', \alpha) \geqslant \Delta(\alpha, \eta)$ implies that $\varphi(\alpha', \eta, \beta)$ holds, so that $\alpha' \in V(\beta)$. Hence, $V_k(\alpha) \subseteq V_n(\beta)$. □

Let $L = (\omega_1, \tau)$ and notice that L is Hausdorff since $V_{\Delta(\alpha,\beta)}(\alpha) \cap V_{\Delta(\alpha,\beta)}(\beta)$ are disjoint for $\alpha \neq \beta$. Let us prove that it is locally compact. For each $\beta \in \omega_1$ and $n \in \omega$, let

$$F_{\beta,n} = \{\eta \in H(\beta) : e(\eta, \beta) \leqslant f_\beta(n)\}$$

and notice that the first property of e guarantees that $F_{\beta,n}$ is finite.

Claim 2. For every $\beta \in \omega_1$ and every $n \in \omega$, if $\alpha \in V_n(\beta) \setminus V_{n+1}(\beta)$, then there is $\eta \in F_{\beta,n}$ such that $\alpha \in V_{\Delta(\eta,\beta)}(\eta)$ and $\varphi(\alpha, \eta, \beta)$ holds.

Proof. If $\alpha \in V_n(\beta) \setminus V_{n+1}(\beta)$, then $\Delta(\alpha, \beta) = n$ and there is $\eta \in H(\beta)$ such that $\varphi(\alpha, \eta, \beta)$ holds. In particular, $\Delta(\alpha, \beta) < \Delta(\alpha, \eta)$. Hence, $\Delta(\eta, \beta) = \Delta(\alpha, \beta) = n$, and since $\eta \in H(\beta)$, we get that $e(\eta, \beta) \leqslant f_\beta(n)$, which ensures that $\eta \in F_{\beta,n}$ and concludes the proof of the claim. □

Claim 3. For every $\beta \in \omega_1$ and every $m \in \omega$, $V_m(\beta)$ is compact.

Proof. We prove it by induction on β. Let $X \subseteq V_m(\beta)$ be an infinite set and notice that one of the following alternatives holds:

(1) For every $n \geqslant m$, $X \cap V_n(\beta)$ is infinite.
(2) There exists $n \geqslant m$ such that $X \cap (V_n(\beta) \setminus V_{n+1}(\beta))$ is infinite.

If (1) holds, then β is an accumulation point of X in $V_m(\beta)$ and we are done. If (2) holds, it follows from Claim 2 that there is $\eta \in F_{\beta,n}$ such that $V_{\Delta(\eta,\beta)}(\eta) \cap X$ is infinite and $\varphi(\alpha, \eta, \beta)$ holds for every $\alpha \in V_{\Delta(\eta,\beta)}(\eta) \cap X$. By the inductive hypothesis, there is $\gamma \in V_{\Delta(\eta,\beta)}(\eta)$ an accumulation point of $V_{\Delta(\eta,\beta)}(\eta) \cap X$. Let us show that $\gamma \in V_m(\beta)$.

Given $k \geqslant \max\{n, \Delta(\gamma, \eta)\}$, there is $\alpha \in V_k(\gamma) \cap V_{\Delta(\eta,\beta)}(\eta) \cap X$. In particular, $\alpha \in V_n(\beta) \setminus V_{n+1}(\beta)$. Hence, $\Delta(\gamma, \alpha) \geqslant k \geqslant n$ and $\Delta(\alpha, \beta) = n$, so that $\Delta(\gamma, \beta) = n$. Moreover, $\Delta(\gamma, \alpha) \geqslant k \geqslant \Delta(\gamma, \eta)$, so that

$$\forall \xi \in H(\beta) \cup \{\beta\},\ \xi \neq \eta \Rightarrow \Delta(\gamma, \xi) = \Delta(\alpha, \xi) < \Delta(\alpha, \eta) = \Delta(\gamma, \eta).$$

This proves that $\varphi(\gamma, \eta, \beta)$ holds and, therefore, $\gamma \in V_n(\beta) \subseteq V_m(\beta)$. □

L is clearly a scattered space, since for any nonempty $X \subseteq \omega_1$, $\min X$ is isolated in X. Let K be the one point compactification of L.

The proof that K^n is hereditarily separable requires some extra work. Given $\Gamma \subseteq \omega_1$, we say that a family $(\beta_1^\xi, \ldots, \beta_n^\xi)_{\xi \in \Gamma} \subseteq \omega_1^n$ is cofinal if for every $\alpha \in \omega_1$, there is $\eta \in \Gamma$ such that $\alpha < \beta_i^\xi$ for every $\xi \geqslant \eta$ in Γ and every $1 \leqslant i \leqslant n$.

Claim 4. ([31, Lemma 2.0]) If $(\beta_1^\xi, \ldots, \beta_n^\xi)_{\xi \in \omega_1} \subseteq \omega_1^n$ is cofinal, then there are $\delta < \xi < \omega_1$ such that $\beta_i^\delta \in H(\beta_i^\xi)$ for every $1 \leqslant i \leqslant n$.

Before proving the claim, let us finish the proof of the theorem. We want to prove that if K is the one-point compactification of L, then K^n is hereditarily separable for every $n \in \omega$. From [27, Theorem 3.1], K^n is hereditarily separable if and only if no uncountable sequence is left-separated, that is, for every uncountable

$(\bar{\beta}_\xi)_{\xi<\omega_1} \subseteq K^n$, there is $\eta < \omega_1$ such that $\bar{\beta}_\eta \in \overline{\{\bar{\beta}_\xi : \xi < \eta\}}$. For each $\xi < \omega_1$, let $\bar{\beta}_\xi = (\beta_1^\xi, \dots, \beta_n^\xi) \in K^n$.

We prove this by induction on n (take $K^0 = \{\omega_1\}$). Suppose that there is $1 \leqslant j \leqslant n$ and $\Gamma \in [\omega_1]^{\omega_1}$ such that $(\beta_j^\xi)_{\xi\in\Gamma}$ is constant. Then, we can omit the jth coordinate to get an uncountable sequence in K^{n-1} which cannot be left-separated by the inductive hypothesis. This immediately yields that $(\bar{\beta}_\xi)_{\xi<\omega_1}$ is not left-separated either.

Otherwise, we may assume withuot loss of generality that each $(\beta_i^\xi)_{\xi\in\omega_1}$ is strictly increasing (and does not include ω_1). By contradiction, suppose that, for each $\xi < \omega_1$, there is $(m_1^\xi, \dots, m_n^\xi) \in \omega^n$ such that

$$\forall \xi < \omega_1 \quad \forall 1 \leqslant i \leqslant n \quad \beta_i^\xi \in V_{m_i^\xi}(\beta_i^\xi)$$

and

$$\forall \xi < \eta < \omega_1 \quad \exists 1 \leqslant i \leqslant n \quad \beta_i^\xi \notin V_{m_i^\eta}(\beta_i^\eta).$$

Passing to an uncountable subset $\Gamma \subseteq \omega_1$, we may assume that, for each $1 \leqslant i \leqslant n$, there is $m_i \in \omega$ such that $m_i^\xi = m_i$ for every $\xi \in \Gamma$. Also, refining Γ to a further uncountable subset, we may assume without loss of generality that $\Delta(\beta_i^\xi, \beta_i^\eta) \geqslant m_i$ for every $\xi < \eta$ in Γ.

Since $(\beta_1^\xi, \dots, \beta_n^\xi)_{\xi\in\Gamma} \subseteq \omega_1^n$ is cofinal, it follows from Claim 4 that there are $\xi < \eta$ in Γ such that $\beta_i^\xi \in H(\beta_i^\eta)$ for every $1 \leqslant i \leqslant n$. Since $H(\beta) \subseteq V(\beta)$ and $\Delta(\beta_i^\xi, \beta_i^\eta) \geqslant m_i$ for every $\xi < \eta$ in Γ, we conclude that $\beta_i^\xi \in V_{m_i}(\beta_i^\eta)$ for every $1 \leqslant i \leqslant n$, which contradicts our assumption and concludes the proof of the theorem.

Let us finally prove Claim 4.

Proof of Claim 4. Since ω^ω with the usual Baire topology is a second countable space, there is $I \in [\omega_1]^\omega$ such that for every $k \in \omega$ and every $\xi \in \omega_1$, there is $\delta \in I$ such that $\Delta(\beta_i^\delta, \beta_i^\xi) \geqslant k$ for every $1 \leqslant i \leqslant n$. Fix $\gamma \in \omega_1$ such that $\beta_i^\delta < \gamma$ for every $\delta \in I$ and every $1 \leqslant i \leqslant n$. Let $\Gamma \in [\omega_1]^{\omega_1}$ be such that $(\beta_1^\xi, \dots, \beta_n^\xi)_{\xi\in\Gamma}$ is still cofinal and if $\xi < \eta$ in Γ, then $\beta_i^\xi < \beta_j^\eta$ for all $1 \leqslant i, j \leqslant n$. We may assume, without loss of generality, that $\gamma < \beta_i^\xi$ for every $\xi \in \Gamma$ and every $1 \leqslant i \leqslant n$.

We will proceed by refining the cofinal family several times to some cofinal subfamily with better properties. To simplify the notation, we will keep calling Γ the uncountable subset obtained after each further refinement.

We use the second property of the function e to refine Γ to an uncountable subset such that for each $1 \leqslant i \leqslant n$, there is $e^i : \gamma \to \omega$ such that

$$\forall \xi \in \Gamma \quad e_{\beta_i^\xi} \restriction_\gamma = e^i.$$

We claim that we can refine Γ to some uncountable subset to ensure that for each $1 \leqslant i \leqslant n$, there is $m_i \in \omega$ such that

$$\forall \xi, \eta \in \Gamma \quad f_{\beta_i^\xi} \restriction_{m_i} = f_{\beta_i^\eta} \restriction_{m_i}$$

and

$$\forall k \in \omega \quad \exists \xi \in \Gamma \quad \forall 1 \leqslant i \leqslant n \quad f_{\beta_i^\xi}(m_i) > k.$$

We prove it for $n = 1$. Suppose by contradiction that for each $m \in \omega$ and $s \in \omega^m$ such that

$$\Gamma_s = \{\xi \in \Gamma : f_{\beta_1^\xi} \restriction_m = s\}$$

is uncountable, there is $k_s \in \omega$ such that

$$\forall \xi \in \Gamma_s \quad f_{\beta_1^\xi}(m) \leqslant k_s.$$

Let $f \in \omega^\omega$ be defined by

$$f(m) = \max\{k_s : s \in \omega^{m+1} \text{ and } \forall j \in m+1,\ s(j) \leqslant k_{s\restriction_j}\}.$$

For each $m \in \omega$, let

$$\Gamma_m = \bigcup\{\Gamma_s : s \in \omega^m \text{ and } \Gamma_s \text{ is uncountable}\}.$$

Clearly $\Gamma_{m+1} \subseteq \Gamma_m$ and by induction one proves that each Γ_m is cocountable in Γ, so that $\bigcap_{m\in\omega} \Gamma_m$ is an uncountable set. It remains to notice that

$$\forall \xi \in \bigcap_{m\in\omega} \Gamma_m \quad f_{\beta_1^\xi} \leqslant^* f,$$

which contradicts the fact that $(f_{\beta_1^\xi})_{\xi\in\Gamma}$ is unbounded since $(\beta_1^\xi)_{\xi\in\Gamma}$ is cofinal in ω_1. This holds because if $\xi \in \bigcap_{m\in\omega} \Gamma_m$, then for all $m \in \omega$, $s_m = f_{\beta_1^\xi} \restriction_m$ is such that Γ_s is uncountable. Therefore, $f_{\beta_1^\xi}(m) \leqslant k_s \leqslant f(m)$. The general case requires a multi-dimensional version of the preceding argument.

Now, we use an auxiliary and arbitrary $\xi_0 \in \Gamma$ to choose $\delta \in I$ such that $\Delta(\beta_i^\delta, \beta_i^{\xi_0}) \geqslant m_i$, so that $f_{\beta_i^\delta} \restriction_{m_i} = f_{\beta_i^{\xi_0}} \restriction_{m_i}$. Since $f_{\beta_i^\delta} <^* f_\gamma$, let $m_0 \in \omega$ be such that $f_{\beta_i^\delta}(k) < f_\gamma(k)$ for all $k \geqslant m_0$.

Finally, choose $\xi \in \Gamma$ such that $f_{\beta_i^\xi}(m_i) \geqslant \max\{e^i(\beta_i^\delta), f_{\beta_i^\delta}(m_i) + 1\}$ for all $1 \leqslant i \leqslant n$. We have that $\delta < \xi$ are such that $e(\beta_i^\delta, \beta_i^\xi) = e_{\beta_i^\xi}(\beta_i^\delta) = e^i(\beta_i^\delta) \leqslant f_{\beta_i^\xi}(m_i)$. To conclude that $\delta \in H(\xi)$, it remains to see that $m_i = \Delta(\beta_i^\delta, \beta_i^\xi)$. From the choice of δ and the fact that $\Delta(\beta_i^\xi, \beta_i^{\xi_0}) \geqslant m_i$, we know that $m_i \leqslant \Delta(\beta_i^\delta, \beta_i^{\xi_0}) = \Delta(\beta_i^\delta, \beta_i^\xi)$. On the other hand, $f_{\beta_i^\xi}(m_i) > f_{\beta_i^\delta}(m_i)$, so that $\Delta(\beta_i^\delta, \beta_i^\xi) \leqslant m_i$, which concludes the proof. □

5. Biorthogonal systems in nonseparable spaces

In the previous sections we focused our attention in $C(K)$ spaces. But there are also forcing constructions of other sorts of nonseparable Banach spaces without uncountable biorthogonal systems, see e.g. [20]. In this section we would like to present results ensuring the consistency of the existence of uncountable biorthogonal systems in every nonseparable Banach space. We start with the following important result:

Theorem 5.1 (Todorčević, 2006, [32]). *Martin's maximum implies that every nonseparable Banach space has a quotient with a monotone Schauder basis of length* $\aleph_1$.

The proof of this result involves an improvement to the uncountable context of the argument presented after Theorem 2.1. We discuss below a variation of that argument, which proves the following equivalence result:

Theorem 5.2 (Brech, Todorčević, 2023, [8])**.** *Under the P-ideal dichotomy, the following are equivalent:*

(1) $\mathfrak{b} > \aleph_1$
(2) *Asplund spaces of density $\aleph_1$ have a quotient with a monotone Schauder basis of length $\aleph_1$.*
(3) *Nonseparable Asplund spaces have a biorthogonal system of length $\aleph_1$.*

The contrapositive implication from $\neg(1) \Rightarrow \neg(3)$ follows immediately from [31, Theorem 2.4] (Theorem 3.3 above), with no use of the P-ideal dichotomy. (2)$\Rightarrow$(3) holds in ZFC because if X is a nonseparable Asplund space and Y is a subspace of X o density $\aleph_1$, (2) implies that Y has a quotient with a monotone Schauder basis of length $\aleph_1$. The associated biorthogonal system in this quotient can be lifted to a biorthogonal system in Y using the quotient mapping. And the functionals of this biorthogonal system can be lifted to the whole space using Hahn-Banach Theorem.

The real work is to prove (1)$\Rightarrow$(2). Here, instead of getting the quotient mapping from a sequence in X as in the proof of Theorem 2.1, the idea is to construct an uncountable transfinite basic sequence $(\varphi_\alpha)_{\alpha\in\Gamma}$ in X^* such that the quotient mapping is defined as follows:

$$Q : X \to \overline{span}\{\varphi^*_\alpha : \alpha \in \Gamma\}$$

$$Q(x) = \sum_{\alpha\in\Gamma} \varphi_\alpha(x)\varphi^*_\alpha.$$

To get such a basic sequence $(\varphi_\alpha)_{\alpha\in\Gamma}$ in X^*, we start from a suitable normalized sequence $(\varphi_\alpha)_{\alpha\in\omega_1}$ in X^* such that for every $x \in X$, $(\varphi_\alpha(x))_{\alpha\in\omega_1}$ has countable support.

In the original countable setting, we had a weakly*-null convergent sequence for free. Here, the P-ideal dichotomy is used to select the uncountable version of such a sequence: an uncountable $\Gamma \subseteq \omega_1$ such that

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \{\alpha \in \Gamma : |\varphi_\alpha(x)| \geqslant \varepsilon\} \text{ is finite.}$$

The argument is indeed a bit more involving as the job is not only to refine, but also to modify it. The ideals used in this argument contain countable pieces of the desired uncountable Γ. They are ensured to be P-ideals using the fact that X has density $\aleph_1 < \mathfrak{b}$.

The sequence is again refined using PID to obtain an uncountable $\Gamma_0 \subseteq \Gamma$

$$\forall x \in D \quad \sum_{\alpha\in\Gamma} |\varphi_\alpha(x)| < \infty.$$

for a suitable dense subspace D of X. The proof that the ideal containing countable sets where this happens is indeed a P-ideal is similar to the proof presented in Section 2.

Theorem 5.1 had already been reformulated in [33], where Martin's maximum was replaced by the P-ideal dichotomy and the cardinal assumption $\mathfrak{p} > \aleph_1$. It is worth recalling that the conclusion of Theorem 5.2 holds for Asplund spaces, while both in Theorem 5.1 (and in its modification in) [33], the conclusion holds for Banach spaces. The point is that the cardinal assumption $\mathfrak{p} > \aleph_1$ allows stronger diagonalisation arguments than the weaker $\mathfrak{b} > \aleph_1$. Asplund spaces have weak* sequentially compact dual balls and this helps in finding convergent sequences and replaces the diagonalisation arguments at some point. In both cases, convergent sequences are used to kill one of the alternatives of the PID.

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