MAE 5870 - Aula 2

MODELOS PARA SÉRIES TEMPOTAIS

1. Processos estocásticos

Definição 1. Seja T um conjunto arbitrário. Um processo estocástico é uma família $X = \{X_t, t \in T\}$, tal que, para cada $t \in T$, X_t é uma variável aleatória.

Um processo estocástico é uma família de variáveis aleatórias definidas num mesmo espaço de probabilidade (Ω, A, P).

Para $t \in T$, X_t é uma v.a. definida sobre Ω, na realidade X_t é uma função de dois argumentos, $X(t,w)$, $t \in T$, $w \in \Omega$.

Para cada t fixo, $X(t, w)$ é uma v.a. com uma distribuição de probabilidade. È possível que a função de densidade de probabilidade (fdp) no instante t_1 seja diferente da fdp no instante t_2, mas a situação usual é aquela que a fdp de $X(t, w)$ é mesma, para todos os t.

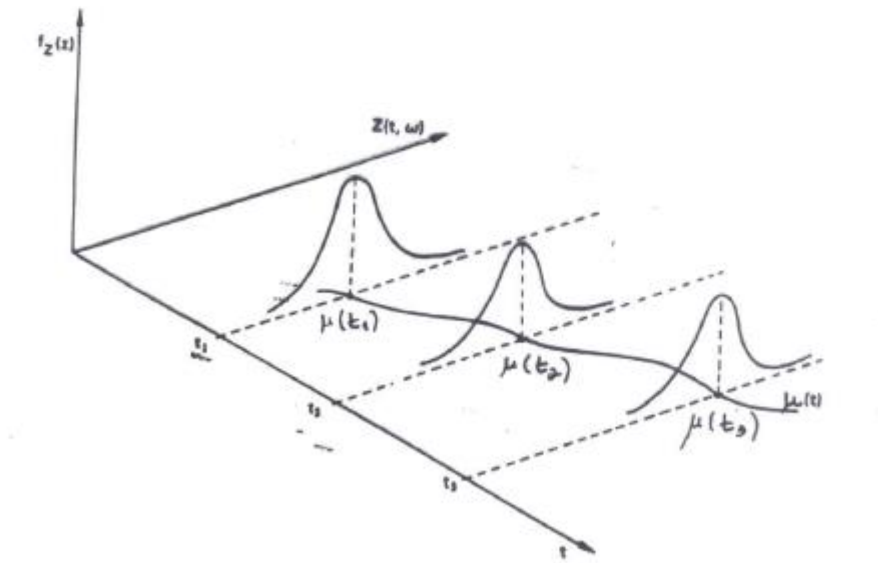

Fig. 1. – Um processo estocástico interpretado como uma família de variáveis aleatórias.

Para cada w fixo, uma função de t, ou seja, uma realização ou trajetória do processo, uma série temporal.

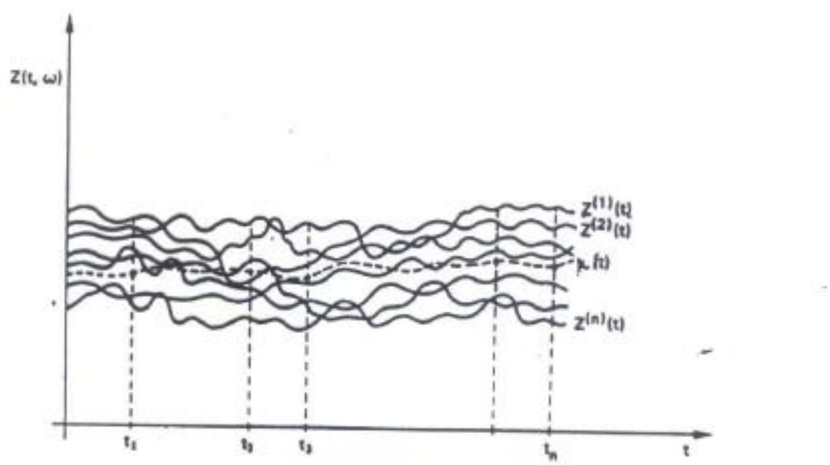

Fig. 2 – Um processo estocástico interpretado como uma família de trajetórias.

2. Especificação de um processo estocástico

$$F_t(x) = P\{x_t \leq x\}$$

or the corresponding marginal density functions

$$f_t(x) = \frac{\partial F_t(x)}{\partial x},$$

when they exist, are often informative for examining the marginal behavior of a series.[2] Another informative marginal descriptive measure is the mean function.

Definition 1.1 *The* **mean function** *is defined as*

$$\mu_{xt} = E(x_t) = \int_{-\infty}^{\infty} x f_t(x)\, dx, \qquad (1.9)$$

provided it exists, where E denotes the usual expected value operator. When no confusion exists about which time series we are referring to, we will drop a subscript and write μ_{xt} as μ_t.

A função média:

$$\mu(t) = E(X_t)$$

A função de autocovariancia (facv) de X é:

$$\gamma(t_1, t_2) = E\{X_{t1} X_{t2}\} - E\{X_{t1}\}E\{X_{t2}\}$$

se $t_1 = t_2 = t$,

$$\gamma(t, t) = var\{X_t\} = E\{X_t^2\} - E^2\{X_t\}$$

A função de auto-correlação (fac) de X é:

$$\rho(t_1,t_2) = \gamma(t_1, t_2)/(\gamma(t_1, t_1)\,\gamma(t_2, t_2))^{1/2}$$

A função de covariancia cruzada entre X e Y é:

$$\gamma_{xy}(t_1, t_2) = E\{X_{t1}Y_{t2}\} - E\{X_{t1}\}E\{Y_{t2}\}$$

A função de correlação cruzada entre X e Y é:

$$\rho_{xy}(t_1,t_2) = \gamma_{xy}(t_1, t_2)/(\gamma_x(t_1, t_1)\ \gamma_y(t_2, t_2))^{1/2}$$

Again, we have the result $-1 \leq \rho_{xy}(h) \leq 1$ which enables comparison with the extreme values -1 and 1 when looking at the relation between x_{t+h} and y_t. The cross-correlation function is not generally symmetric about zero [i.e., typically $\rho_{xy}(h) \neq \rho_{xy}(-h)$]; however, it is the case that

$$\rho_{xy}(h) = \rho_{yx}(-h), \tag{1.28}$$

[2] If x_t is Gaussian with mean μ_t and variance σ_t^2, abbreviated as $x_t \sim \mathrm{N}(\mu_t, \sigma_t^2)$, the marginal density is given by $f_t(x) = \frac{1}{\sigma_t\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma_t^2}(x-\mu_t)^2\right\}$.

Example 1.13 Mean Function of a Moving Average Series

If w_t denotes a white noise series, then $\mu_{wt} = E(w_t) = 0$ for all t. The top series in Figure 1.8 reflects this, as the series clearly fluctuates around a mean value of zero. Smoothing the series as in Example 1.9 does not change the mean because we can write

$$\mu_{vt} = E(v_t) = \tfrac{1}{3}[E(w_{t-1}) + E(w_t) + E(w_{t+1})] = 0.$$

Example 1.14 Mean Function of a Random Walk with Drift

Consider the random walk with drift model given in (1.4),

$$x_t = \delta\, t + \sum_{j=1}^{t} w_j, \qquad t = 1, 2, \ldots\ .$$

Because $E(w_t) = 0$ for all t, and δ is a constant, we have

$$\mu_{xt} = E(x_t) = \delta\, t + \sum_{j=1}^{t} E(w_j) = \delta\, t$$

which is a straight line with slope δ. A realization of a random walk with drift can be compared to its mean function in Figure 1.10.

Example 1.15 Mean Function of Signal Plus Noise

A great many practical applications depend on assuming the observed data have been generated by a fixed signal waveform superimposed on a zero-mean noise process, leading to an additive signal model of the form (1.5). It is clear, because the signal in (1.5) is a fixed function of time, we will have

$$\begin{aligned}\mu_{xt} &= E(x_t) = E\big[2\cos(2\pi t/50 + .6\pi) + w_t\big]\\ &= 2\cos(2\pi t/50 + .6\pi) + E(w_t)\\ &= 2\cos(2\pi t/50 + .6\pi),\end{aligned}$$

and the mean function is just the cosine wave.

3. Processos estacionários

Séries estacionárias são aquelas que desenvolvem no tempo ao redor de uma média constante.

Definição 2: Um processo estocástico $X = \{X_t,\ t \in T\}$ diz-se estritamente estacionário se todas as distribuições finito-dimensionais permanecem as mesmas sob translações no tempo, ou seja,

$$F(X_1, \ldots, X_n; t_{1+\tau}, \ldots, t_{n+\tau}) = F(X_1, \ldots, X_n; t_1, \ldots, t_n)$$

Definição 3: Um processo estocástico $X = \{X_t,\ t \in T\}$ diz-se fracamente estacionário ou estacionário de segunda ordem se e somente se

a) $E\{X_t\} = \mu(t) = \mu$, constante;

b) $E\{X_t^2\}$ finita

c) $\gamma(t_1, t_2) = Cov\{X_{t1}X_{t2}\}$ é uma função de $|t_1 - t_2|$.

Definição 4: um processo estocástico $X = \{X_t, t \in T\}$ diz-se Gaussiano se, para qualquer conjunto $t_1, t_2, \ldots, t_n$ de T, as v.a. $X_{t1}, \ldots, X_{tn}$ têm distribuição normal n-variada.

Definição 5: As séries $\{X_t, Y_t, t \in T\}$ são conjuntamente estacionárias se a covariância cruzada depende apenas de $|t_1 - t_2|$, isto é, $\gamma_{xy}(t_1, t_2) = \gamma_{xy}(|t_1 - t_2|)$.

4. Função de autocovariância

Definição 6: Uma função real K é não negativa definida se e somente se

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j K(\tau_i - \tau_j) \geq 0$$

para qualquer números reais $a_1, \ldots, a_n$ e $\tau_1, \ldots, \tau_n \in R$.

Seja $\{X(t), t \in T\}$ um processo estacionário real discreto, de média zero e facv $\gamma_\tau = E\{X(t)X(t+\tau)\}$.

Proposição 1: A facv γ_τ satisfaz as seguintes propriedades:

a) $\gamma_0 > 0$;

b) $\gamma_\tau = \gamma_{-\tau}$;

c) $|\gamma_\tau| \leq \gamma_0$

d) γ_τ é não negativa definida, no sentido que

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \gamma_{\tau_i - \tau_j} \geq 0$$

para qualquer números reais $a_1, \ldots, a_n$ e $\tau_1, \ldots, \tau_n$ €R.

Observação: a recíproca da propriedade d) também é verdadeira, isto é, dada uma função γ_τ, tendo a propriedade d), existe um processo estocástico X(t), tendo γ_τ como facv.

Tipicamente, a facv de um processo estacionário tende a zero, para $|\gamma_\tau| \rightarrow \infty$.

Teorema: uma função real definida sobre Z é a função de auto-covariância de um processo estacionário se e somente se ela for não **negativa definida e par**.

Exemplos:

Autocovariance of White Noise

The white noise series w_t has $E(w_t) = 0$ and

$$\gamma_w(s,t) = \operatorname{cov}(w_s, w_t) = \begin{cases} \sigma_w^2 & s = t, \\ 0 & s \neq t. \end{cases} \tag{1.12}$$

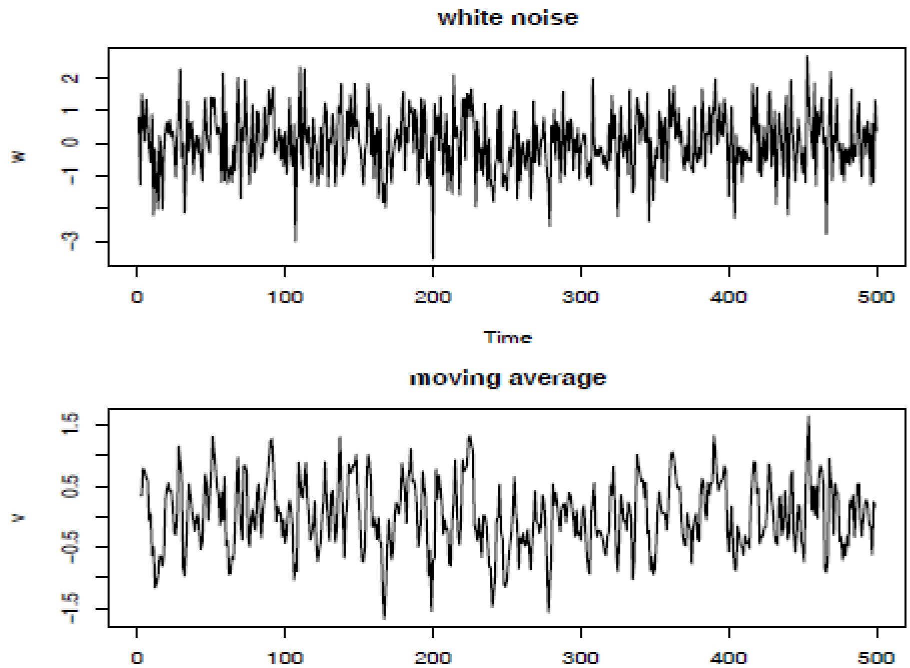

Fig. 1.8. Gaussian white noise series (top) and three-point moving average of the Gaussian white noise series (bottom).

Autocovariance of a Moving Average

Consider applying a three-point moving average to the white noise series w_t of the previous example as in Example 1.9. In this case,

$$\gamma_v(s,t) = \text{cov}(v_s, v_t) = \text{cov}\left\{\tfrac{1}{3}\left(w_{s-1} + w_s + w_{s+1}\right), \tfrac{1}{3}\left(w_{t-1} + w_t + w_{t+1}\right)\right\}.$$

When $s = t$ we have[3]

$$\begin{aligned}\gamma_v(t,t) &= \tfrac{1}{9}\text{cov}\{(w_{t-1} + w_t + w_{t+1}), (w_{t-1} + w_t + w_{t+1})\}\\ &= \tfrac{1}{9}[\text{cov}(w_{t-1}, w_{t-1}) + \text{cov}(w_t, w_t) + \text{cov}(w_{t+1}, w_{t+1})]\\ &= \tfrac{3}{9}\sigma_w^2.\end{aligned}$$

When $s = t + 1$,

$$\begin{aligned}\gamma_v(t+1,t) &= \tfrac{1}{9}\text{cov}\{(w_t + w_{t+1} + w_{t+2}), (w_{t-1} + w_t + w_{t+1})\}\\ &= \tfrac{1}{9}[\text{cov}(w_t, w_t) + \text{cov}(w_{t+1}, w_{t+1})]\\ &= \tfrac{2}{9}\sigma_w^2,\end{aligned}$$

using (1.12). Similar computations give $\gamma_v(t-1,t) = 2\sigma_w^2/9$, $\gamma_v(t+2,t) = \gamma_v(t-2,t) = \sigma_w^2/9$, and 0 when $|t-s| > 2$. We summarize the values for all s and t as

$$\gamma_v(s,t) = \begin{cases} \frac{3}{9}\sigma_w^2 & s = t,\\ \frac{2}{9}\sigma_w^2 & |s-t| = 1,\\ \frac{1}{9}\sigma_w^2 & |s-t| = 2,\\ 0 & |s-t| > 2.\end{cases} \tag{1.13}$$

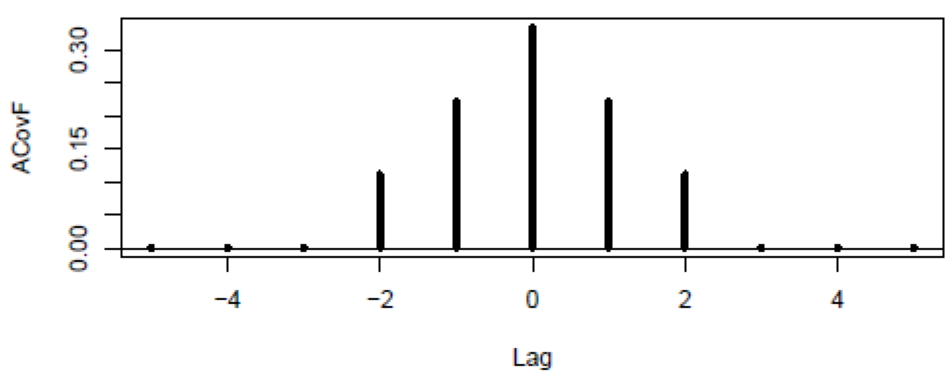

Fig. 1.12. Autocovariance function of a three-point moving average.

Example 1.11 Random Walk with Drift

A model for analyzing trend such as seen in the global temperature data in Figure 1.2, is the random walk with drift model given by

$$x_t = \delta + x_{t-1} + w_t \tag{1.3}$$

for $t = 1, 2, \ldots$, with initial condition $x_0 = 0$, and where w_t is white noise. The constant δ is called the drift, and when $\delta = 0$, (1.3) is called simply a random walk. The term random walk comes from the fact that, when $\delta = 0$, the value of the time series at time t is the value of the series at time $t-1$ plus a completely random movement determined by w_t. Note that we may rewrite (1.3) as a cumulative sum of white noise variates. That is,

$$x_t = \delta\, t + \sum_{j=1}^{t} w_j \tag{1.4}$$

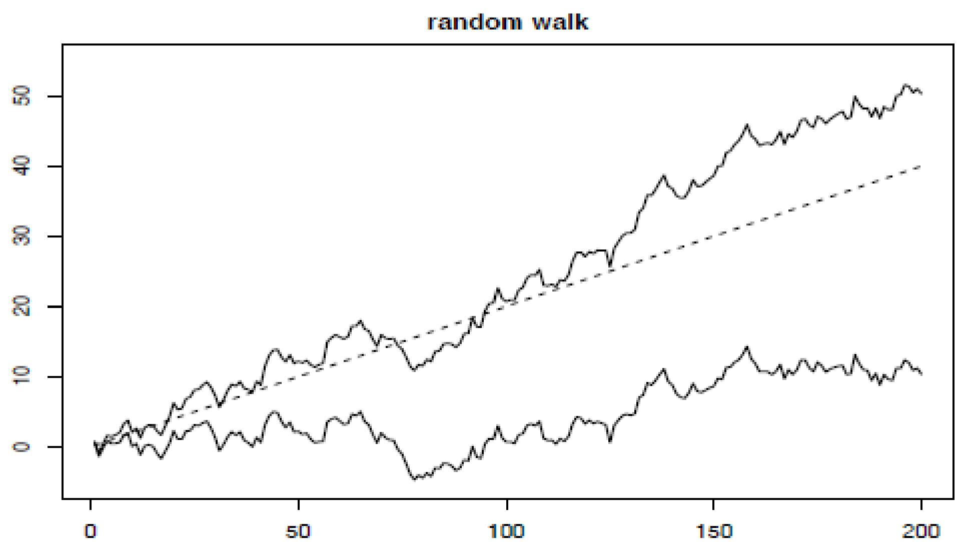

Fig. 1.10. Random walk, $\sigma_w = 1$, with drift $\delta = .2$ (upper jagged line), without drift, $\delta = 0$ (lower jagged line), and a straight line with slope .2 (dashed line).

Mean Function of a Random Walk with Drift

Consider the random walk with drift model given in (1.4),

$$x_t = \delta\, t + \sum_{j=1}^{t} w_j, \qquad t = 1, 2, \dots \;.$$

Because $E(w_t) = 0$ for all t, and δ is a constant, we have

$$\mu_{xt} = E(x_t) = \delta\, t + \sum_{j=1}^{t} E(w_j) = \delta\, t$$

Autocovariance of a Random Walk

For the random walk model, $x_t = \sum_{j=1}^{t} w_j$, we have

$$\gamma_x(s,t) = \operatorname{cov}(x_s, x_t) = \operatorname{cov}\left(\sum_{j=1}^{s} w_j, \sum_{k=1}^{t} w_k\right) = \min\{s,t\}\, \sigma_w^2,$$

Example 1.12 Signal in Noise

Many realistic models for generating time series assume an underlying signal with some consistent periodic variation, contaminated by adding a random noise. For example, it is easy to detect the regular cycle fMRI series displayed on the top of Figure 1.6. Consider the model

$$x_t = 2\cos(2\pi t/50 + .6\pi) + w_t \tag{1.5}$$

for $t = 1, 2, \dots, 500$, where the first term is regarded as the signal, shown in the upper panel of Figure 1.11. We note that a sinusoidal waveform can be written as

$$A\cos(2\pi\omega t + \phi), \tag{1.6}$$

where A is the amplitude, ω is the frequency of oscillation, and ϕ is a phase shift. In (1.5), $A = 2$, $\omega = 1/50$ (one cycle every 50 time points), and $\phi = .6\pi$.

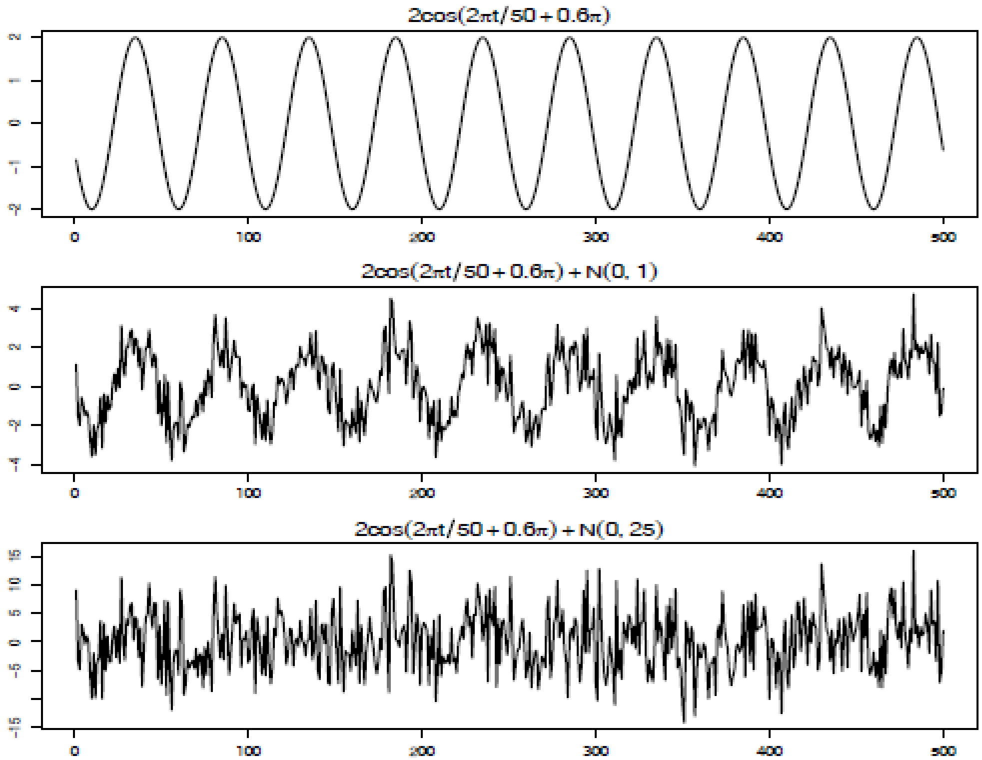

Fig. 1.11. Cosine wave with period 50 points (top panel) compared with the cosine wave contaminated with additive white Gaussian noise, $\sigma_w = 1$ (middle panel) and $\sigma_w = 5$ (bottom panel); see (1.5).

Example 1.15 Mean Function of Signal Plus Noise

A great many practical applications depend on assuming the observed data have been generated by a fixed signal waveform superimposed on a zero-mean noise process, leading to an additive signal model of the form (1.5). It is clear, because the signal in (1.5) is a fixed function of time, we will have

$$\begin{aligned}\mu_{xt} \;=\; E(x_t) &= E\big[2\cos(2\pi t/50 + .6\pi) + w_t\big] \\ &= 2\cos(2\pi t/50 + .6\pi) + E(w_t) \\ &= 2\cos(2\pi t/50 + .6\pi),\end{aligned}$$

and the mean function is just the cosine wave.

Example 1.10 Autoregressions

Suppose we consider the white noise series w_t of Example 1.8 as input and calculate the output using the second-order equation

$$x_t = x_{t-1} - .9x_{t-2} + w_t \tag{1.2}$$

successively for $t = 1, 2, \ldots, 500$. Equation (1.2) represents a regression or prediction of the current value x_t of a time series as a function of the past two values of the series, and, hence, the term autoregression is suggested

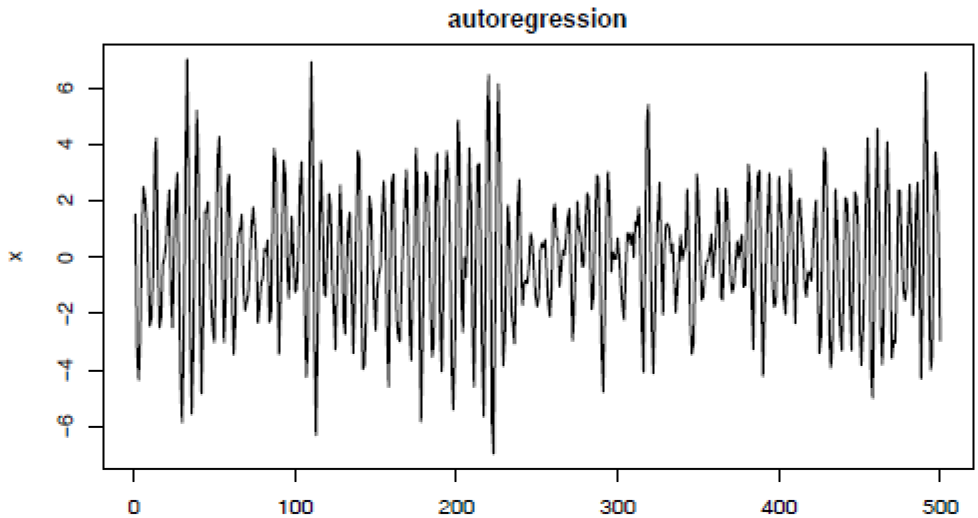

Fig. 1.9. Autoregressive series generated from model (1.2).

Example 1.21 Joint Stationarity

Consider the two series, x_t and y_t, formed from the sum and difference of two successive values of a white noise process, say,

$$x_t = w_t + w_{t-1}$$

and

$$y_t = w_t - w_{t-1},$$

where w_t are independent random variables with zero means and variance σ_w^2. It is easy to show that $\gamma_x(0) = \gamma_y(0) = 2\sigma_w^2$ and $\gamma_x(1) = \gamma_x(-1) = \sigma_w^2, \gamma_y(1) = \gamma_y(-1) = -\sigma_w^2$. Also,

$$\gamma_{xy}(1) = \text{cov}(x_{t+1}, y_t) = \text{cov}(w_{t+1} + w_t, w_t - w_{t-1}) = \sigma_w^2$$

because only one term is nonzero (recall footnote 3 on page 20). Similarly, $\gamma_{xy}(0) = 0, \gamma_{xy}(-1) = -\sigma_w^2$. We obtain, using (1.27),

$$\rho_{xy}(h) = \begin{cases} 0 & h = 0, \\ 1/2 & h = 1, \\ -1/2 & h = -1, \\ 0 & |h| \geq 2. \end{cases}$$

Clearly, the autocovariance and cross-covariance functions depend only on the lag separation, h, so the series are jointly stationary.

Propriedades de função de auto-covariância de um processo estacionário:

The autocovariance function of a stationary process has several useful properties (also, see Problem 1.25). First, the value at $h = 0$, namely

$$\gamma(0) = E[(x_t - \mu)^2] \tag{1.24}$$

is the variance of the time series; note that the Cauchy–Schwarz inequality implies

$$|\gamma(h)| \leq \gamma(0).$$

A final useful property, noted in the previous example, is that the autocovariance function of a stationary series is symmetric around the origin; that is,

$$\gamma(h) = \gamma(-h) \tag{1.25}$$

for all h. This property follows because shifting the series by h means that

$$\begin{aligned} \gamma(h) &= \gamma(t + h - t) \\ &= E[(x_{t+h} - \mu)(x_t - \mu)] \\ &= E[(x_t - \mu)(x_{t+h} - \mu)] \\ &= \gamma(t - (t + h)) \\ &= \gamma(-h), \end{aligned}$$

5. Tipos de modelos

a) modelos paramétricos: a análise é feita no domínio do tempo;

- os modelos de regressão
- os modelos de auto-regressivos e de médias moveis(ARMA)
- os modelos auto-regressivo integrados e de médias mpveis(ARIMA)
- modelos de memória longa (ARFIMA)
- modelos estruturais e modelos não lineares

b) modelos não paramétricos: a análise é feita no domínio de frequência. A função de autocovariância e sua transformada de Fourier, o espectro.