MAE 5871 - Aula 2

Análise de Fourier Clássica

O objetivo básico é de aproximar uma função f(t) por uma combinação linear de componentes senoidais, cada uma com dada frequência.

O conjunto $\{w_n(t) = e^{int}, n = 0, \pm 1, \ldots\}$ de funções ortogonais, de período 2π, forma a base para a análise de Fourier.

Na realidade, esse conjunto é gerado por dilatações de uma única função $w(t)=e^{it}$, ou seja, $w_n(t) = w(nt)$ para qualquer n inteiro.

O fato básico é que toda função periódica, de período 2π, de quadrado integrável, é gerada por uma superposição de dilatações inteiras da função w(t).

A formula de Euler

$$e^{int} = \cos(nt) + i\,\text{sen}(nt).$$

Relaciona o sistema das exponenciais complexas com o sistema de senos e cossenos,

$$\{\cos(nt), \text{sen}(nt), n = 0, \pm 1, \ldots\}.$$

Função periódica de período p:

$$\mathbf{f(t) = f(t + kp), t \in R,}\ k = 0, \pm 1, \ldots$$

Função harmônica de frequência angular λ e amplitude A, λ e A positivos:

$$\mathbf{f(t) = A.\cos(\lambda t) \text{ ou } f(t) = A.\text{sen}(\lambda t)}$$

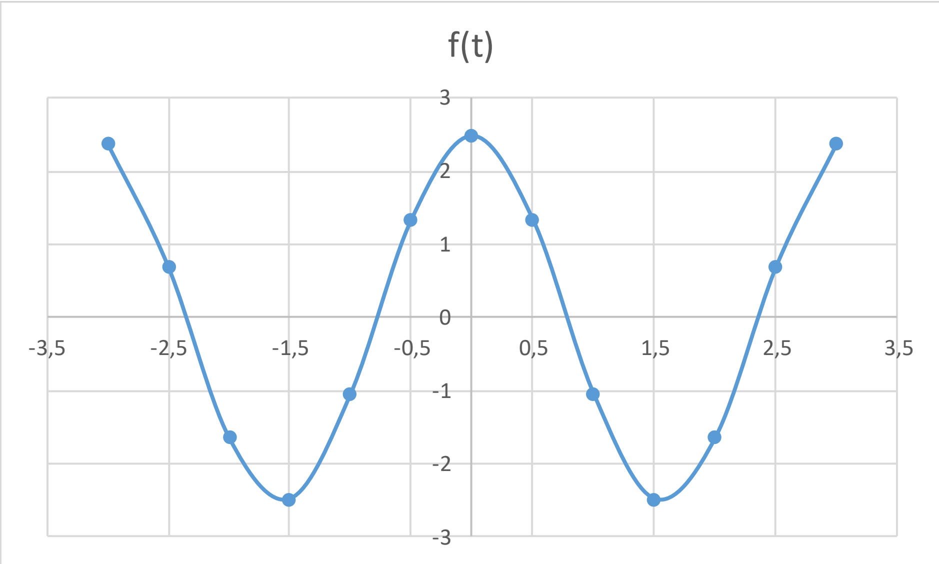
Período: p = 2π/λ, λ = número de ciclos completos em 2π unidades de tempo.

Frequência em ciclos por unidade de tempo: ν = λ/2π

P = 1/ν

Exemplo: um harmônico:

f(t) = 2,5.cos(2t); A = 2,5; ν = 2



Cyclical Behavior and Periodicity

As in (1.5), we consider the periodic process

$$x_t = A\cos(2\pi\omega t + \phi) \tag{4.1}$$

for $t = 0, \pm 1, \pm 2, \ldots$, where ω is a frequency index, defined in cycles per unit time with A determining the height or *amplitude* of the function and ϕ, called the *phase*, determining the start point of the cosine function. We can introduce random variation in this time series by allowing the amplitude and phase to vary randomly.

$$x_t = U_1 \cos(2\pi\omega t) + U_2 \sin(2\pi\omega t), \tag{4.2}$$

where $U_1 = A\cos\phi$ and $U_2 = -A\sin\phi$ are often taken to be normally distributed random variables. In this case, the amplitude is $A = \sqrt{U_1^2 + U_2^2}$ and the phase is $\phi = \tan^{-1}(-U_2/U_1)$. From these facts we can show that if, and only if, in (4.1), A and ϕ are independent random variables, where A^2 is chi-squared with 2 degrees of freedom, and ϕ is uniformly distributed on $(-\pi, \pi)$, then U_1 and U_2 are independent, standard normal random variables

Consider a generalization of (4.2) that allows mixtures of periodic series with multiple frequencies and amplitudes,

$$x_t = \sum_{k=1}^{q} \left[U_{k1} \cos(2\pi\omega_k t) + U_{k2} \sin(2\pi\omega_k t)\right], \tag{4.3}$$

where U_{k1}, U_{k2}, for $k = 1, 2, \ldots, q$, are independent zero-mean random variables with variances σ_k^2, and the ω_k are distinct frequencies. Notice that (4.3) exhibits the process as a sum of independent components, with variance σ_k^2 for frequency ω_k. Using the independence of the Us and the trig identity in footnote 1, it is easy to show[2] (Problem 4.3) that the autocovariance function of the process is

$$\gamma(h) = \sum_{k=1}^{q} \sigma_k^2 \cos(2\pi\omega_k h), \tag{4.4}$$

and we note the autocovariance function is the sum of periodic components with weights proportional to the variances σ_k^2. Hence, x_t is a mean-zero stationary processes with variance

[2] For example, for x_t in (4.2) we have $\text{cov}(x_{t+h}, x_t) = \sigma^2\{\cos(2\pi\omega[t+h])\cos(2\pi\omega t) + \sin(2\pi\omega[t+h])\sin(2\pi\omega t)\} = \sigma^2\cos(2\pi\omega h)$, noting that $\text{cov}(U_1, U_2) = 0$.

$$\gamma(0) = E(x_t^2) = \sum_{k=1}^{q} \sigma_k^2, \tag{4.5}$$

which exhibits the overall variance as a sum of variances of each of the component parts.

Example 4.1 A Periodic Series

Figure 4.1 shows an example of the mixture (4.3) with $q = 3$ constructed in the following way. First, for $t = 1, \ldots, 100$, we generated three series
$x_{t1} = 2\cos(2\pi t\, 6/100) + 3\sin(2\pi t\, 6/100)$
$x_{t2} = 4\cos(2\pi t\, 10/100) + 5\sin(2\pi t\, 10/100)$
$x_{t3} = 6\cos(2\pi t\, 40/100) + 7\sin(2\pi t\, 40/100)$
These three series are displayed in Figure 4.1 along with the corresponding frequencies and squared amplitudes. For example, the squared amplitude of x_{t1} is $A^2 = 2^2 + 3^2 = 13$. Hence, the maximum and minimum values that x_{t1} will attain are $\pm\sqrt{13} = \pm 3.61$.

Finally, we constructed

$$x_t = x_{t1} + x_{t2} + x_{t3}$$

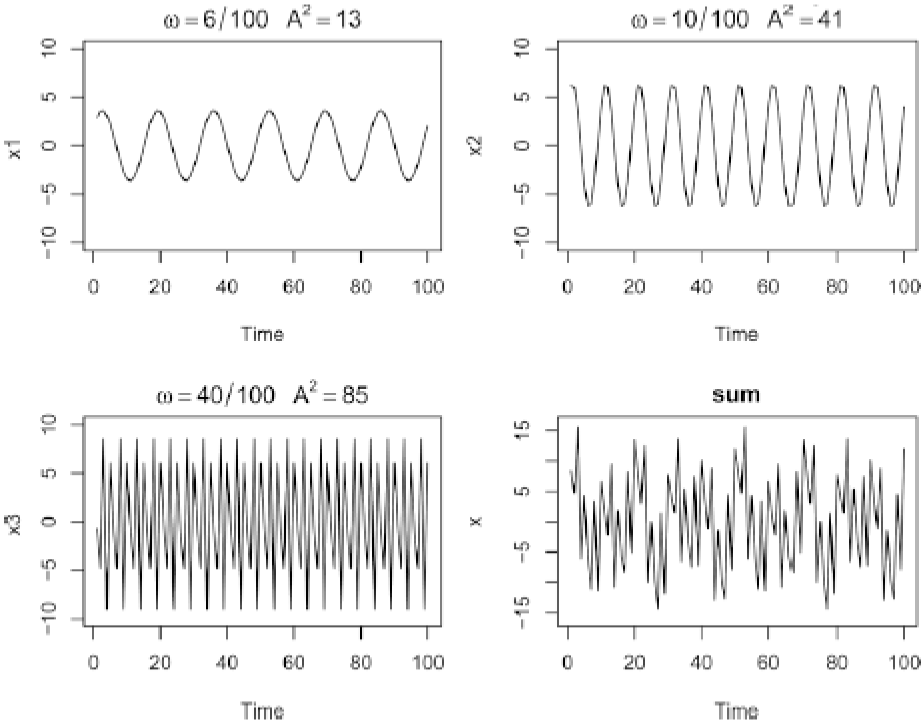


Fig. 4.1. Periodic components and their sum as described in Example 4.1.

Quatro situações:

1. Tempo continuo e frequência discreta;
2. Tempo continuo e frequência contínua;
3. Tempo discreto e frequência contínua;
4. Tempo discreto e frequência discreta.

Tempo Contínuo e frequência Discreta

Let $L^2(0, 2\pi)$ denote the collection of all measurable functions f defined on the interval $(0, 2\pi)$ with

$$\int_0^{2\pi} |f(x)|^2 dx < \infty.$$

For the reader who is not familiar with the basic Lebesgue theory, the sacrifice is very minimal by assuming that f is a piecewise continuous function. It will always be assumed that functions in $L^2(0, 2\pi)$ are extended periodically to the real line

$$\mathbb{R} := (-\infty, \infty),$$

namely: $f(x) = f(x - 2\pi)$ for all x. Hence, the collection $L^2(0, 2\pi)$ is often called the space of 2π-periodic square-integrable functions. That $L^2(0, 2\pi)$ is

a vector space can be verified very easily. Any f in $L^2(0, 2\pi)$ has a Fourier series representation:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad (1.1.1)$$

where the constants c_n, called the Fourier coefficients of f, are defined by

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx. \qquad (1.1.2)$$

The convergence of the series in (1.1.1) is in $L^2(0, 2\pi)$, meaning that

$$\lim_{M,N\to\infty} \int_0^{2\pi} \left| f(x) - \sum_{n=-M}^{N} c_n e^{inx} \right|^2 dx = 0.$$

There are two distinct features in the Fourier series representation (1.1.1). First, we mention that f is decomposed into a sum of infinitely many mutually orthogonal components $g_n(x) := c_n e^{inx}$, where orthogonality means that

$$\langle g_m, g_n \rangle^* = 0, \quad \text{for all} \quad m \neq n, \tag{1.1.3}$$

with the "*inner product*" in (1.1.3) being defined by

$$\langle g_m, g_n \rangle^* := \frac{1}{2\pi} \int_0^{2\pi} g_m(x) \overline{g_n(x)} dx. \tag{1.1.4}$$

That (1.1.3) holds is a consequence of the important, yet simple fact that

$$w_n(x) := e^{inx}, \qquad n = \cdots, -1, 0, 1, \ldots, \tag{1.1.5}$$

is an orthonormal (o.n.) basis of $L^2(0, 2\pi)$. The second distinct feature of the Fourier series representation (1.1.1) is that the o.n. basis $\{w_n\}$ is generated by "*dilation*" of a single function

$$w(x) := e^{ix}; \tag{1.1.6}$$

that is, $w_n(x) = w(nx)$ for all integers n. This will be called *integral dilation*.

Let us summarize this remarkable fact by saying that *every 2π-periodic square-integrable function is generated by a "superposition" of integral dilations of the basic function $w(x) = e^{ix}$.*

We also remark that from the o.n. property of $\{w_n\}$, the Fourier series representation (1.1.1) also satisfies the so-called *Parseval Identity*:

$$\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{1.1.7}$$

Uma função periódica de período T e de quadrado integrável:

$$\int_{-T/2}^{T/2} |f(t)|^2 dt < \propto$$

Então

$$f(t) = \sum_{n=-\propto}^{\propto} c_n \, e^{i\lambda_n t}, \qquad (1)$$

$\lambda_n = 2\pi n/T$: frequências de Fourier.

(1): representação em série de Fourier de f(t).

Os coeficientes c_n (coeficientes de Fourier) são dados por

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-i\lambda_n t}dt \quad (2)$$

Teorema de Parseval:

$$1/T \int_{-T/2}^{T/2} |f(t)|^2 dt = \sum_{n=-\propto}^{n} |c_n|^2$$

O espectro discreto de f(t) é a sequência $\{|c_n|^2, n = 0, \pm 1, \ldots\}$

Forma alternativa de (1):

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\propto}(a_n \cos(\lambda_n t) + b_n sen(\lambda_n t))$$

Em que:

$C_n = (a_n - i\, b_n)/2$, n >=1;

$= a_0/2$, n = 0;

$= (a_{|n|} + i\, b_{|n|})/2$, n <= -1.

Os coeficientes de a_n e b_n:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\lambda_n t)\, dt, n \geq 0;$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\text{sen}(\lambda_n t)\, dt, n \geq 1.$$

n = 1: harmônico fundamental e corresponde a uma onda cosseno (seno) de período igual a T (igual ao de f(t));

n =2: o primeiro harmônico de período igual à T/2 (metade do período de f(t));

e assim por diante.

Como c_n é um número complexo, ele pode ser escrito na forma:

$$c_n = R_n e^{i\phi_n}$$

$$R_n = |c_n| = \frac{1}{2}(a_n^2 + b_n^2)^{1/2}$$

$$\Phi_n = \text{arctg}(-b_n/a_n)\ ,\ n = 0, 1, 2, \ldots$$

Referência complementar:

Chui, C.K. (1992). An Introduction to Wavelets. Academic Press.