MAE 5871 – Aula 5

Análise Espectral de Processos Estacionários

Considere uma realização particular do processo, que chamaremos simplesmente de $X(t)$, para simplificar a notação. Vamos lembrar que $X(t)$ é, agora, uma função não aleatória de t.

Defina a função

$$Y(t) = \begin{cases} X(t), & \text{se } -T \leq t \leq T \\ 0, & \text{se } |t| > T. \end{cases} \tag{3.1}$$

Para essa função, podemos definir a transformada de Fourier

$$F_Y(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(t)e^{-i\lambda t}dt = \frac{1}{2\pi}\int_{-T}^{T} X(t)e^{-i\lambda t}dt, \tag{3.2}$$

com

$$Y(t) = \int_{-\infty}^{\infty} F_Y(\lambda)^{i\lambda t}d\lambda. \tag{3.3}$$

$$|F_Y(\lambda)|^2 d\lambda \tag{3.4}$$

representa a contribuição à energia total daquelas componentes de $Y(t)$, com frequências em $(\lambda, \lambda + d\lambda)$. Segue-se que

$$J^{(T)}(\lambda) = \frac{|F_Y(\lambda)|^2}{2T} \tag{3.5}$$

representa uma *função densidade de potência* de $Y(t)$, de modo que

$$\lim_{T\to\infty} \frac{|F_Y(\lambda)|^2}{2T} \tag{3.6}$$

descreveria as propriedades espectrais de $X(t)$.

$f(\lambda)$ de *função densidade espectral* de X(t), ou simplesmente *espectro* de X(t):

$$f(\lambda) = \lim_{T\to\infty} E\{J^{(T)}(\lambda)\}, \tag{3.7}$$

a média das contribuições, das componentes de X(t) com frequências em (λ, λ + dλ), à potência total

$$f(\lambda) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-2T}^{2T} [1 - \frac{|\tau|}{2T}]\gamma(\tau)e^{-i\lambda\tau} d\tau, \qquad (3.8)$$

onde $\gamma(\tau)$ é a função de autocovariância de $X(t)$. Uma condição suficiente para que o limite em (3.8) exista é

$$\int_{-\infty}^{\infty} |\gamma(\tau)| d\tau < \infty, \qquad (3.9)$$

ou seja, $\gamma(\tau) \to 0, |\tau| \to \infty$, indicando que valores do processo suficientemente afastados são fracamente correlacionados. Se (3.9) estiver satisfeita, segue-se de (3.8) que

$$f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma(\tau)e^{-i\lambda\tau} d\tau, \qquad (3.10)$$

obtendo-se o espectro como a transformada de Fourier da função de autocovariância. A transformada inversa resulta

$$\gamma(\tau) = \int_{-\infty}^{\infty} f(\lambda)e^{i\lambda\tau} d\lambda. \qquad (3.11)$$

Em particular, para $\tau = 0$ em (3.11), obtemos

$$\gamma(0) = Var\{X(t)\} = \int_{-\infty}^{\infty} f(\lambda) d\lambda, \qquad (3.12)$$

o que mostra que a variância do processo é uma "soma"de contribuições devidas às diversas componentes de frequências presentes em $X(t)$, sendo que as componentes com frequências em $(\lambda, \lambda + d\lambda)$ contribuem para a variância com $f(\lambda)d\lambda$.

Se tivermos um processo estacionário discreto $\{X_t, t \in \mathbb{Z}\}$, o argumento acima pode ser repetido, com somas substituindo integrais. Se

$$\sum_{k=-\infty}^{\infty} |\gamma_k| < \infty \qquad (3.13)$$

estiver satisfeita, obtemos o espectro de X_t como

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k e^{-i\lambda k} \ , \ -\pi < \lambda < \pi. \qquad (3.14)$$

Teorema 3.1. *O espectro $f(\lambda)$, definido em (3.14), é limitado, não negativo, uniformemente contínuo, par e periódico, de período 2π.*

$$\gamma_k = \int_{-\pi}^{\pi} e^{i\lambda k} f(\lambda) d\lambda \quad , k = 0, \pm 1, \; \ldots \; . \tag{3.15}$$

Suponha, agora, que $\{X(t), t \in \mathbb{R}\}$ seja *real*. Como $\gamma(\tau)$ é par, temos

$$f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\cos(\lambda\tau) - i\mathrm{sen}(\lambda\tau)] \gamma(\tau) d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(\lambda\tau)\gamma(\tau)d\tau - \frac{i}{2\pi} \int_{-\infty}^{\infty} \mathrm{sen}(\lambda\tau)\gamma(\tau)d\tau$$

e, portanto, como a última integral se anula (pois o integrando é uma função ímpar), obtemos

$$f(\lambda) = \frac{1}{\pi} \int_{0}^{\infty} \cos(\lambda\tau)\gamma(\tau)d\tau.$$

De modo análogo, temos

$$\gamma(\tau) = 2 \int_{0}^{\infty} \cos(\lambda\tau) f(\lambda) d\lambda.$$

No caso de um processo discreto real, teremos, respectivamente,

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k \cos(\lambda k) = \frac{\sigma_X^2}{2\pi} \{1 + 2 \sum_{k=1}^{\infty} \rho_k \cos(\lambda k)\}, \tag{3.16}$$

$$\gamma_k = 2 \int_{0}^{\pi} \cos(\lambda k) f(\lambda) d\lambda. \tag{3.17}$$

Example 4.3 A Periodic Stationary Process

Consider a periodic stationary random process given by (4.2), with a fixed frequency ω_0, say,

$$x_t = U_1 \cos(2\pi\omega_0 t) + U_2 \sin(2\pi\omega_0 t),$$

where U_1 and U_2 are independent zero-mean random variables with equal variance σ^2. The number of time periods needed for the above series to complete one cycle is exactly $1/\omega_0$, and the process makes exactly ω_0 cycles per point for $t = 0, \pm1, \pm2, \ldots$. It is easily shown that[3]

$$\begin{aligned}\gamma(h) &= \sigma^2 \cos(2\pi\omega_0 h) = \frac{\sigma^2}{2} e^{-2\pi i\omega_0 h} + \frac{\sigma^2}{2} e^{2\pi i\omega_0 h} \\ &= \int_{-1/2}^{1/2} e^{2\pi i\omega h} dF(\omega)\end{aligned}$$

[3] Some identities may be helpful here: $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$ and consequently, $\cos(\alpha) = (e^{i\alpha} + e^{-i\alpha})/2$ and $\sin(\alpha) = (e^{i\alpha} - e^{-i\alpha})/2i$.

using a Riemann–Stieltjes integration, where $F(\omega)$ is the function defined by

$$F(\omega) = \begin{cases} 0 & \omega < -\omega_0, \\ \sigma^2/2 & -\omega_0 \le \omega < \omega_0, \\ \sigma^2 & \omega \ge \omega_0. \end{cases}$$

The function $F(\omega)$ behaves like a cumulative distribution function for a discrete random variable, except that $F(\infty) = \sigma^2 = \text{var}(x_t)$ instead of one. In fact, $F(\omega)$ is a cumulative distribution function, not of probabilities, but rather of variances associated with the frequency ω_0 in an analysis of variance, with $F(\infty)$ being the total variance of the process x_t. Hence, we term $F(\omega)$ the *spectral distribution function*.

Property 4.2 The Spectral Density

If the autocovariance function, $\gamma(h)$, of a stationary process satisfies

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty, \tag{4.10}$$

then it has the representation

$$\gamma(h) = \int_{-1/2}^{1/2} \mathrm{e}^{2\pi i\omega h}\ f(\omega)\ d\omega \quad h = 0, \pm1, \pm2, \ldots \tag{4.11}$$

as the inverse transform of the spectral density, which has the representation

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h)\mathrm{e}^{-2\pi i\omega h} \quad -1/2 \leq \omega \leq 1/2. \tag{4.12}$$

This spectral density is the analogue of the probability density function; the fact that $\gamma(h)$ is non-negative definite ensures

$$f(\omega) \geq 0$$

for all ω (see Appendix C, Theorem C.3 for details). It follows immediately from (4.12) that

$$f(\omega) = f(-\omega) \quad \text{and} \quad f(\omega) = f(1-\omega),$$

verifying the spectral density is an even function of period one. Because of the evenness, we will typically only plot $f(\omega)$ for $\omega \geq 0$. In addition, putting $h = 0$ in (4.11) yields

$$\gamma(0) = \mathrm{var}(x_t) = \int_{-1/2}^{1/2} f(\omega)\ d\omega,$$

which expresses the total variance as the integrated spectral density over all of the frequencies. We show later on, that a linear filter can isolate the variance in certain frequency intervals or bands.

We note that the autocovariance function, $\gamma(h)$, in (4.11) and the spectral density, $f(\omega)$, in (4.12) are Fourier transform pairs. In particular, this means that if $f(\omega)$ and $g(\omega)$ are two spectral densities for which

$$\gamma_f(h) = \int_{-1/2}^{1/2} f(\omega)\mathrm{e}^{2\pi i\omega h}\ d\omega = \int_{-1/2}^{1/2} g(\omega)\mathrm{e}^{2\pi i\omega h}\ d\omega = \gamma_g(h) \tag{4.13}$$

for all $h = 0, \pm1, \pm2, \ldots$, then

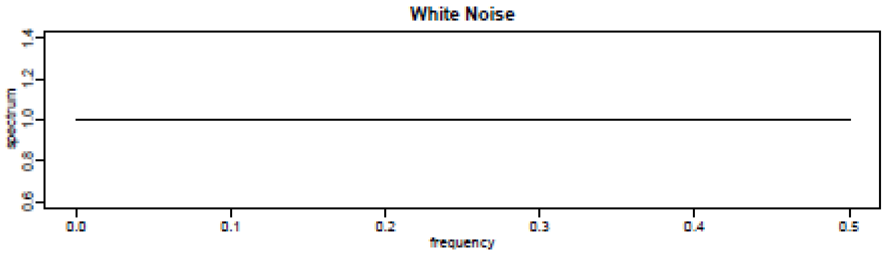
$$f(\omega) = g(\omega). \tag{4.14}$$

Example 4.4 White Noise Series

As a simple example, consider the theoretical power spectrum of a sequence of uncorrelated random variables, w_t, with variance σ_w^2. A simulated set of data is displayed in the top of Figure 1.8. Because the autocovariance function was computed in Example 1.16 as $\gamma_w(h) = \sigma_w^2$ for $h = 0$, and zero, otherwise, it follows from (4.12), that

$$f_w(\omega) = \sigma_w^2$$

for $-1/2 \le \omega \le 1/2$. Hence the process contains equal power at all frequencies. This property is seen in the realization, which seems to contain all different frequencies in a roughly equal mix. In fact, the name white noise comes from the analogy to white light, which contains all frequencies in the color spectrum at the same level of intensity. Figure 4.3 shows a plot of the white noise spectrum for $\sigma_w^2 = 1$.



If x_t is ARMA, its spectral density can be obtained explicitly using the fact that it is a linear process, i.e., $x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$, where $\sum_{j=0}^{\infty} |\psi_j| < \infty$. In the following property, we exhibit the form of the spectral density of an ARMA model. The proof of the property follows directly from the proof of a more general result, Property 4.7 given on page 222, by using the additional fact that $\psi(z) = \theta(z)/\phi(z)$; recall Property 3.1.

Property 4.3 The Spectral Density of ARMA

If x_t is ARMA(p, q), $\phi(B)x_t = \theta(B)w_t$, its spectral density is given by

$$f_x(\omega) = \sigma_w^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2} \tag{4.15}$$

where $\phi(z) = 1 - \sum_{k=1}^{p} \phi_k z^k$ and $\theta(z) = 1 + \sum_{k=1}^{q} \theta_k z^k$.

Example 4.5 Moving Average

As an example of a series that does not have an equal mix of frequencies, we consider a moving average model. Specifically, consider the MA(1) model given by

$$x_t = w_t + .5w_{t-1}.$$

A sample realization is shown in the top of Figure 3.2 and we note that the series has less of the higher or faster frequencies. The spectral density will verify this observation.

The autocovariance function is displayed in Example 3.4 on page 90, and for this particular example, we have

$$\gamma(0) = (1 + .5^2)\sigma_w^2 = 1.25\sigma_w^2; \quad \gamma(\pm 1) = .5\sigma_w^2; \quad \gamma(\pm h) = 0 \text{ for } h > 1.$$

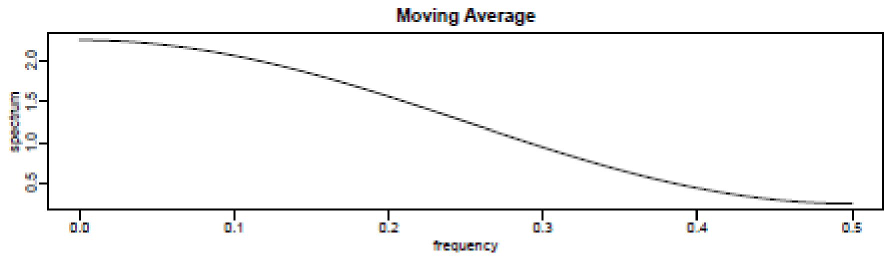
Substituting this directly into the definition given in (4.12), we have

$$\begin{aligned} f(\omega) &= \sum_{h=-\infty}^{\infty} \gamma(h)\, \mathrm{e}^{-2\pi\iota\omega h} = \sigma_w^2 \left[1.25 + .5\left(\mathrm{e}^{-2\pi\iota\omega} + \mathrm{e}^{2\pi\omega}\right)\right] \\ &= \sigma_w^2 \left[1.25 + \cos(2\pi\omega)\right]. \end{aligned} \tag{4.16}$$

We can also compute the spectral density using Property 4.3, which states that for an MA, $f(\omega) = \sigma_w^2 |\theta(\mathrm{e}^{-2\pi\iota\omega})|^2$. Because $\theta(z) = 1 + .5z$, we have

$$\begin{aligned} |\theta(\mathrm{e}^{-2\pi\iota\omega})|^2 &= |1 + .5\mathrm{e}^{-2\pi\iota\omega}|^2 = (1 + .5\mathrm{e}^{-2\pi\iota\omega})(1 + .5\mathrm{e}^{2\pi\iota\omega}) \\ &= 1.25 + .5\left(\mathrm{e}^{-2\pi\iota\omega} + \mathrm{e}^{2\pi\omega}\right) \end{aligned}$$

which leads to agreement with (4.16).



Example 4.6 A Second-Order Autoregressive Series

We now consider the spectrum of an AR(2) series of the form

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} = w_t,$$

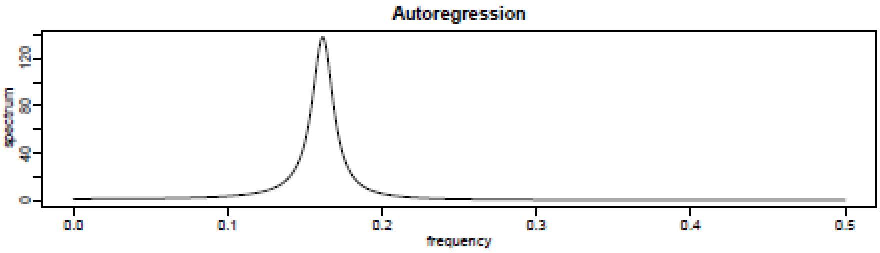
for the special case $\phi_1 = 1$ and $\phi_2 = -.9$. Figure 1.9 on page 14 shows a sample realization of such a process for $\sigma_w = 1$. We note the data exhibit a strong periodic component that makes a cycle about every six points.

To use Property 4.3, note that $\theta(z) = 1$, $\phi(z) = 1 - z + .9z^2$ and

$$\begin{aligned} |\phi(e^{-2\pi\iota\omega})|^2 &= (1 - \mathrm{e}^{-2\pi\iota\omega} + .9\mathrm{e}^{-4\pi\iota\omega})(1 - \mathrm{e}^{2\pi\iota\omega} + .9\mathrm{e}^{4\pi\iota\omega}) \\ &= 2.81 - 1.9(\mathrm{e}^{2\pi\iota\omega} + \mathrm{e}^{-2\pi\iota\omega}) + .9(\mathrm{e}^{4\pi\iota\omega} + \mathrm{e}^{-4\pi\iota\omega}) \\ &= 2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega). \end{aligned}$$

Using this result in (4.15), we have that the spectral density of x_t is

$$f_x(\omega) = \frac{\sigma_w^2}{2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)}.$$



The spectral density can also be obtained from first principles, without having to use Property 4.3. Because $w_t = x_t - x_{t-1} + .9x_{t-2}$ in this example, we have

$$\begin{aligned}\gamma_w(h) &= \text{cov}(w_{t+h}, w_t)\\ &= \text{cov}(x_{t+h} - x_{t+h-1} + .9x_{t+h-2},\, x_t - x_{t-1} + .9x_{t-2})\\ &= 2.81\gamma_x(h) - 1.9[\gamma_x(h+1) + \gamma_x(h-1)] + .9[\gamma_x(h+2) + \gamma_x(h-2)]\end{aligned}$$

Now, substituting the spectral representation (4.11) for $\gamma_x(h)$ in the above equation yields

$$\begin{aligned}\gamma_w(h) &= \int_{-1/2}^{1/2} [2.81 - 1.9(e^{2\pi i\omega} + e^{-2\pi i\omega}) + .9(e^{4\pi i\omega} + e^{-4\pi i\omega})] e^{2\pi i\omega h} f_x(\omega) d\omega\\ &= \int_{-1/2}^{1/2} [2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)] e^{2\pi i\omega h} f_x(\omega) d\omega.\end{aligned}$$

If the spectrum of the white noise process, w_t, is $g_w(\omega)$, the uniqueness of the Fourier transform allows us to identify

$$g_w(\omega) = [2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)]\, f_x(\omega).$$

But, as we have already seen, $g_w(\omega) = \sigma_w^2$, from which we deduce that

$$f_x(\omega) = \frac{\sigma_w^2}{2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)}$$

is the spectrum of the autoregressive series.