4.8 Linear Filters

Some of the examples of the previous sections have hinted at the possibility the distribution of power or variance in a time series can be modified by making a linear transformation. In this section, we explore that notion further by defining a linear filter and showing how it can be used to extract signals from a time series. The linear filter modifies the spectral characteristics of a time series in a predictable way, and the systematic development of methods for taking advantage of the special properties of linear filters is an important topic in time series analysis.

A linear filter uses a set of specified coefficients a_j, for $j = 0, \pm1, \pm2, \ldots$, to transform an input series, x_t, producing an output series, y_t, of the form

$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |a_j| < \infty. \tag{4.99}$$

The form (4.99) is also called a convolution in some statistical contexts. The coefficients, collectively called the *impulse response function*, are required to satisfy absolute summability so y_t in (4.99) exists as a limit in mean square and the infinite Fourier transform

$$A_{yx}(\omega) = \sum_{j=-\infty}^{\infty} a_j\, e^{-2\pi i\omega j}, \tag{4.100}$$

called the *frequency response function*, is well defined. We have already encountered several linear filters, for example, the simple three-point moving average in Example 4.16, which can be put into the form of (4.99) by letting $a_{-1} = a_0 = a_1 = 1/3$ and taking $a_t = 0$ for $|j| \geq 2$.

$$\begin{aligned}
\gamma_{yy}(h) &= \operatorname{cov}(y_{t+h}, y_t) \\
&= \operatorname{cov}\left(\sum_r a_r x_{t+h-r}, \sum_s a_s x_{t-s}\right) \\
&= \sum_r \sum_s a_r \gamma_{xx}(h - r + s) a_s \\
&= \sum_r \sum_s a_r \left[\int_{-1/2}^{1/2} e^{2\pi i\omega(h-r+s)} f_{xx}(\omega) d\omega\right] a_s \\
&= \int_{-1/2}^{1/2} \left(\sum_r a_r e^{-2\pi i\omega r}\right)\left(\sum_s a_s e^{2\pi i\omega s}\right) e^{2\pi i\omega h} f_{xx}(\omega)\, d\omega \\
&= \int_{-1/2}^{1/2} e^{2\pi i\omega h} |A_{yx}(\omega)|^2 f_{xx}(\omega)\, d\omega,
\end{aligned}$$

Property 4.7 Output Spectrum of a Filtered Stationary Series

The spectrum of the filtered output y_t in (4.99) is related to the spectrum of the input x_t by

$$f_{yy}(\omega) = |A_{yx}(\omega)|^2\ f_{xx}(\omega), \tag{4.101}$$

where the frequency response function $A_{yx}(\omega)$ is defined in (4.100).

Example 4.19 First Difference and Moving Average Filters

We illustrate the effect of filtering with two common examples, the first difference filter

$$y_t = \nabla x_t = x_t - x_{t-1}$$

and the symmetric moving average filter

$$y_t = \tfrac{1}{24}(x_{t-6} + x_{t+6}) + \tfrac{1}{12}\sum_{r=-5}^{5} x_{t-r},$$

which is a modified Daniell kernel with $m = 6$.

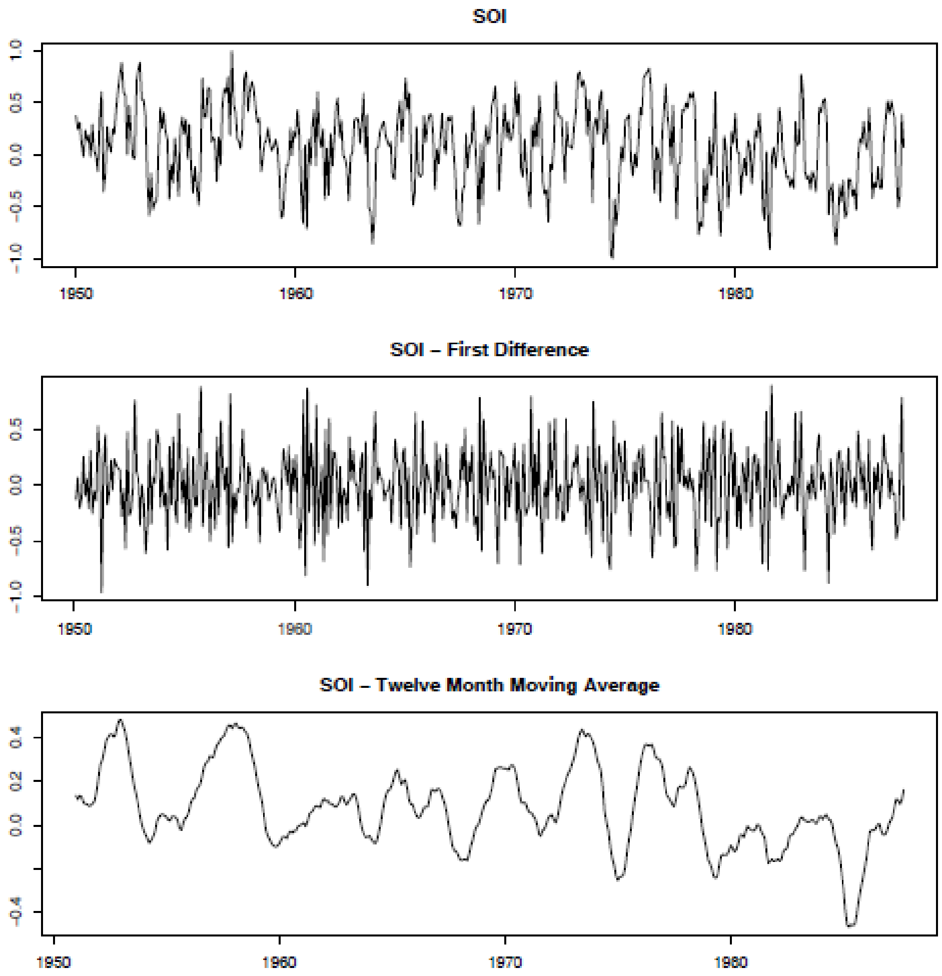

Fig. 4.14. SOI series (top) compared with the differenced SOI (middle) and a centered 12-month moving average (bottom).

Notice that the effect of differencing is to roughen the series because it tends to retain the higher or faster frequencies. The centered moving average smoothes the series because it retains the lower frequencies and tends to attenuate the higher frequencies. In general, differencing is na example of a high-pass filter because it retains or passes the higher frequencies, whereas the moving average is a low-pass filter because it passes the lower or slower frequencies.

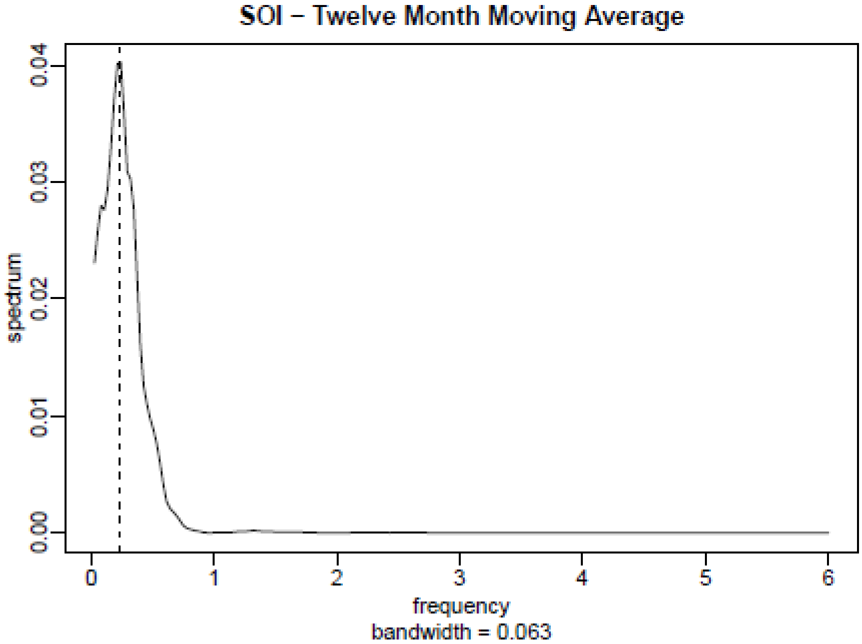

Fig. 4.15. Spectral analysis of SOI after applying a 12-month moving average filter. The vertical line corresponds to the 52-month cycle.

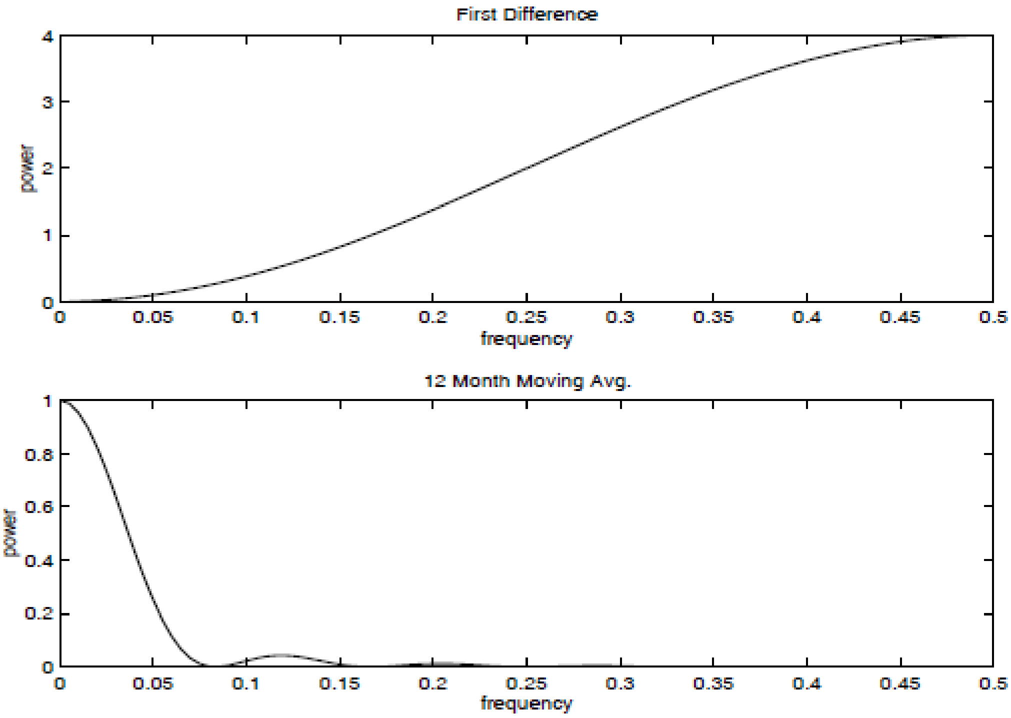

Fig. 4.16. Squared frequency response functions of the first difference and 12-month moving average filters.

Now, having done the filtering, it is essential to determine the exact way in which the filters change the input spectrum. We shall use (4.100) and (4.101) for this purpose. The first difference filter can be written in the form (4.99) by letting $a_0 = 1, a_1 = -1$, and $a_r = 0$ otherwise. This implies that

$$A_{yx}(\omega) = 1 - e^{-2\pi i\omega},$$

and the squared frequency response becomes

$$|A_{yx}(\omega)|^2 = (1 - e^{-2\pi i\omega})(1 - e^{2\pi i\omega}) = 2[1 - \cos(2\pi\omega)]. \tag{4.102}$$

For the centered 12-month moving average, we can take $a_{-6} = a_6 = 1/24$, $a_k = 1/12$ for $-5 \le k \le 5$ and $a_k = 0$ elsewhere. Substituting and recognizing the cosine terms gives

$$A_{yx}(\omega) = \tfrac{1}{12}\Big[1 + \cos(12\pi\omega) + 2\sum_{k=1}^{5}\cos(2\pi\omega k)\Big]. \tag{4.103}$$

the cross-spectrum satisfies

$$f_{yx}(\omega) = A_{yx}(\omega) f_{xx}(\omega),$$

so the frequency response is of the form

$$A_{yx}(\omega) = \frac{f_{yx}(\omega)}{f_{xx}(\omega)} \tag{4.104}$$

$$= \frac{c_{yx}(\omega)}{f_{xx}(\omega)} - i\frac{q_{yx}(\omega)}{f_{xx}(\omega)}, \tag{4.105}$$

where we have used (4.81) to get the last form. Then, we may write (4.105) in polar coordinates as

$$A_{yx}(\omega) = |A_{yx}(\omega)| \ \exp\{-i\ \phi_{yx}(\omega)\}, \tag{4.106}$$

where the amplitude and phase of the filter are defined by

$$|A_{yx}(\omega)| = \frac{\sqrt{c_{yx}^2(\omega) + q_{yx}^2(\omega)}}{f_{xx}(\omega)} \tag{4.107}$$

and

$$\phi_{yx}(\omega) = \tan^{-1}\left(-\frac{q_{yx}(\omega)}{c_{yx}(\omega)}\right). \tag{4.108}$$

A simple interpretation of the phase of a linear filter is that it exhibits time delays as a function of frequency in the same way as the spectrum represents the variance as a function of frequency. Additional insight can be gained by considering the simple delaying filter

$$y_t = Ax_{t-D},$$

where the series gets replaced by a version, amplified by multiplying by A and delayed by D points. For this case,

$$f_{yx}(\omega) = A\mathrm{e}^{-2\pi i\omega D} f_{xx}(\omega),$$

and the amplitude is $|A|$, and the phase is

$$\phi_{yx}(\omega) = -2\pi\omega D,$$

or just a linear function of frequency ω. For this case, applying a simple time delay causes phase delays that depend on the frequency of the periodic component being delayed. Interpretation is further enhanced by setting

$$x_t = \cos(2\pi\omega t),$$

in which case

$$y_t = A\cos(2\pi\omega t - 2\pi\omega D).$$

Thus, the output series, y_t, has the same period as the input series, x_t, but the amplitude of the output has increased by a factor of $|A|$ and the phase has been changed by a factor of $-2\pi\omega D$.

Example 4.20 Difference and Moving Average Filters

We consider calculating the amplitude and phase of the two filters discussed in Example 4.19. The case for the moving average is easy because $A_{yx}(\omega)$ given in (4.103) is purely real. So, the amplitude is just $|A_{yx}(\omega)|$ and the phase is $\phi_{yx}(\omega) = 0$. In general, symmetric $(a_j = a_{-j})$ filters have zero phase. The first difference, however, changes this, as we might expect from the example above involving the time delay filter. In this case, the squared amplitude is given in (4.102). To compute the phase, we write

$$\begin{aligned} A_{yx}(\omega) &= 1 - e^{-2\pi i\omega} = e^{-i\pi\omega}(e^{i\pi\omega} - e^{-i\pi\omega}) \\ &= 2ie^{-i\pi\omega}\sin(\pi\omega) = 2\sin^2(\pi\omega) + 2i\cos(\pi\omega)\sin(\pi\omega) \\ &= \frac{c_{yx}(\omega)}{f_{xx}(\omega)} - i\frac{q_{yx}(\omega)}{f_{xx}(\omega)}, \end{aligned}$$

so

$$\phi_{yx}(\omega) = \tan^{-1}\left(-\frac{q_{yx}(\omega)}{c_{yx}(\omega)}\right) = \tan^{-1}\left(\frac{\cos(\pi\omega)}{\sin(\pi\omega)}\right).$$

Noting that

$$\cos(\pi\omega) = \sin(-\pi\omega + \pi/2)$$

and that

$$\sin(\pi\omega) = \cos(-\pi\omega + \pi/2),$$

we get

$$\phi_{yx}(\omega) = -\pi\omega + \pi/2,$$

and the phase is again a linear function of frequency.