Wavelet-Smoothed Empirical Copula Estimators

Resumo

O objetivo deste artigo é introduzir um estimador de cópulas baseado na suavização de cópulas empíricas, para o caso de séries temporais. As propriedades desse estimador são avaliadas por meio de simulações e seu desempenho é comparado com outros estimadores. Também são feitas aplicações a dados reais.

Palavras chaves: cópula, cópula empírica, série temporal, ondaleta.

Código JEL: C14

Abstract

We introduce a copula estimator based on wavelet smoothing of empirical copulas for the case of time series data. We then study the properties of this estimator via simulations and compare its performance with other estimators. Applications to real data are also given.

Keywords: Copula, empirical copula, time series, wavelets, wavelet estimators

1 Introduction

Copulas provide a convenient tool for describing the dependence between variables. Copula techniques have been developed basically for the independent, identically distributed (i.i.d.) case, which would prevent, at least theoretically, their applications to dependent data, eg time series data, appearing in economics, finance and other areas. The presence of serial correlation and time-varying heteroscedasticity in financial time series, for example, calls for the development of new methodologies for analyzing this kind of data, especially in the field of copula estimation.

For i.i.d. samples of bivariate or multivariate distributions, parametric and nonparametric methods of analysis are well known, with several such approaches comonly used for copula estimation. If the copula is assumed to belong to some parametric family of copulas, consistent and asymptotically normal estimators of the parameters can be obtained by the method of maximum likelihood (ML); see Genest and Rivest (1993) and Shih and Louis (1995). A two-step procedure called *inference function for margins* can be used: first the parameters of the marginals are estimated and then the parameters of the (parametric) copula are estimated, both via ML. See for example Joe and Xu (1996). These estimators are consistent and asymptotically normal and also almost as efficient as the full MLE.

Another approach is to use the so-called empirical copulas, introduced by Deheuvels (1979, 1981a,b). These are highly discontinuous, so some form of smoothing is necessary to obtain better estimates. One approach, of Fermanian et al. (2004), uses kernel estimates based on empirical copulas.

Concerning the estimation of copulas for time series, to our knowledge the only works are those of Fermanian and Scaillet (2003), using nonparametric techniques with kernels, and of Morettin et al. (2008), using wavelets. A related paper by Chen and Fan (2004) focuses on stationary Markov processes of order one, while assuming a parametric form for the copula function. For the case of independent samples, see Genest et al. (2009) and Autin et al. (2008).

In the present paper we propose wavelet estimators based on the empirical copula.

The plan of the article is as follows. In Section 2 we set down the necessary background on copulas and wavelets. In Section 3 we describe the estimation of copulas for the time series setting, first discussing the fitting of volatility models before estimating copulas and then introducing our nonparametric estimators. In Section 4 we perform some simulation studies and in Section 5 we apply the proposed techniques for some sets of real data. In Section 6 we onclude with additional remarks

2 Background

In this section we present some basic notions on copulas and wavelets.

2.1 Copulas

For ease of notation we restrict our attention to the bivariate case; the extensions to the n-dimensional case are straightforward.

A *copula* can be viewed as a function C defined on $I^2 = [0,1]^2$ with values in I, satisfying, for $0 \leq x \leq 1$ and $x_1 \leq x_2,\ y_1 \leq y_2,\ (x_1, y_1), (x_2, y_2) \in I^2$, the conditions

$$C(x,1) = C(1,x) = x, \ \ C(x,0) = C(0,x) = 0, \tag{1}$$

$$C(x_2, y_2) - C(x_2, y_1) - C(x_1, y_2) + C(x_1, y_1) \geq 0. \tag{2}$$

Property (1) means uniformity of the margins, while (2), the *n-increasing property* (with $n = 2$) means that $P(x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2) \geq 0$ for (X, Y) with distribution function (d.f.) C.

See Nelsen (2006) for a general definition and further details on copulas. The following important theorem links the definition of copula with a d.f. and its marginal distributions; a proof can be found in Sklar (1959).

Theorem 1 (i) Let C be a copula and F_1, F_2 univariate d.f.'s. Then

$$F(x,y) = C(F_1(x), F_2(y)), \quad (x,y) \in \mathbb{R}^2 \tag{3}$$

defines a d.f. F with marginals F_1, F_2.
(ii) Conversely, for a two-dimensional d.f. F with marginals F_1, F_2, there exists a copula C satisfying (3); this is unique if F_1, F_2 are continuous and then, for every $(u, v) \in I^2$,

$$C(u,v) = F(F_1^{-1}(u), F_2^{-1}(v)), \tag{4}$$

where F_1^{-1}, F_2^{-1} denote the generalized left-continuous inverses of F_1, F_2.

Briefly, copulas are bivariate or, more generally, multivariate d.f.'s with uniform univariate marginals. See also Schweizer (1991), Kolev et al. (2006) and Charpentier et al. (2006) for good reviews on copulas. In what follows we assume that F_1 and F_2 are continuous.

We now introduce empirical copulas. Let (X_i, Y_i), $i = 1, \ldots, n$, be an i.i.d. sample from (X, Y) and let

$$F_n(x, y) = \frac{1}{n} \sum_{i=1}^{n} I\{X_i \le x, Y_i \le y\}, \quad -\infty < x, y < +\infty \tag{5}$$

be the empirical d.f. and let $F_{1n}(x), F_{2n}(y)$ be the corresponding marginal d.f.'s, namely

$$F_{1n}(x) = F_n(x, +\infty), \quad F_{2n}(y) = F_n(+\infty, y), \quad -\infty < x, y < +\infty.$$

Then the *empirical copula function* is defined by

$$C_n(u, v) = F_n(F_{1n}^{-1}(u), F_{2n}^{-1}(v)), \quad 0 \le u, v \le 1, \tag{6}$$

and the *empirical copula process* is defined by

$$Z_n(u, v) = \sqrt{n}(C_n - C)(u, v), \quad 0 \le u, v \le 1. \tag{7}$$

Deheuvels (1979) proved uniform consistency of the empirical copula, while Deheuvels (1981a, 1981b) obtained results concerning limits for Z_n in the case of independent marginals. In particular, he proposed a Kolmogorov-Smirnov-type statistic for testing the independence hypothesis that $C(u, v) = uv$ and obtained its asymptotic distribution under the null hypothesis. Fermanian et al. (2004) proved that the empirical copula process converges weakly to a Gaussian process in $L_\infty[0, 1]^2$ (the space of a.e. bounded functions on I^2 with sup-norm), under the assumption that C has continuous partial derivatives. See also Ibragimov (2005) for a similar result in the case of a stationary β-mixing process.

2.2 Wavelets

We will need two-dimensional wavelets in this paper, but start for motivation with the one-dimensional case. For additional background see Daubechies (1992) and Meyer (1993). From a *mother wavelet* ψ and a *father wavelet* ϕ (or scaling function), an orthonormal system for $L_2(\mathbb{R})$ is generated by setting $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$, $j \ge j_0$ and $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$, for some coarse scale j_0, which we take as zero. Hence, for any $f \in L_2(\mathbb{R})$ we may write, uniquely,

$$f(x) = \sum_k \alpha_{0,k}\phi_{0,k}(x) + \sum_{j \ge 0} \sum_k \beta_{j,k}\psi_{j,k}(x), \tag{8}$$

where the wavelet coefficients are given by

$$\alpha_{0,k} = \int f(x)\phi_{0,k}(x)dx, \tag{9}$$

$$\beta_{j,k} = \int f(x)\psi_{j,k}(x)dx. \tag{10}$$

In our case, copulas belongs to $L_2(I^2)$, so we may first consider periodized wavelets in the interval $[0,1]$, defined by

$$\tilde{\phi}_{j,k}(x) = \sum_n \phi_{j,k}(x-n), \quad \tilde{\psi}_{j,k}(x) = \sum_n \psi_{j,k}(x-n),$$

see Vidakovic (1999). For notational simplicity we will suppress the upper tilde from now on. For any function $f \in L_2(I^2)$ we have a similar expansion to (8), where the wavelets are obtained as products of one-dimensional wavelets. One can, for example, consider a basis with a single scale. For this we define the bivariate scaling function as $\Phi(x,y) = \phi(x)\phi(y)$ and the wavelets by $\Psi^h(x,y) = \phi(x)\psi(y)$, $\Psi^v(x,y) = \psi(x)\phi(y)$ and $\Psi^d(x,y) = \psi(x)\psi(y)$, where h, v and d indicate the horizontal, vertical and diagonal directions, respectively. Let $\mathbf{k} = (k_1, k_2)$. Then a wavelet expansion for $f(x,y)$ is

$$f(x,y) = c_{0,\mathbf{0}} + \sum_{j=0}^{\infty} \sum_{\mathbf{k}} \sum_{\mu=h,v,d} d^{\mu}_{j,\mathbf{k}} \Psi^{\mu}_{j,\mathbf{k}}(x,y), \tag{11}$$

with the wavelet coefficients given by

$$c_{0,\mathbf{0}} = \int f(x,y)dxdy, \;\; d^{\mu}_{j,\mathbf{k}} = \int f(x,y)\Psi^{\mu}_{j,\mathbf{k}}(x,y)dxdy. \tag{12}$$

Another possibility is to build a basis as the tensor product of two one-dimensional bases with different scales for each dimension; see Morettin et al. (2008).

We mention the comprehensive treatments of wavelets for economics and finance in the paper by Ramsey (2002) and the book by Gençay et al. (2002).

3 Estimation for time series data

As remarked in Section 1, most of the results available in the literature of copulas apply to i.i.d. samples $(X_i, Y_i), i = 1, \ldots, n$, from a distribution function F. As remarked by Mikosch (2006), "it is contradictory that in risk management, where one observes a lot of dependence through time, copulas are applied most frequently".

In this section we discuss copula estimation techniques in the presence of time series data. We repeat that one approach often used is to apply directly the methods available for i.i.d. data (mostly using parametric copula models), however may be misleading.

3.1 Fitting univariate and multivariate models

This method, used for example by Dias and Embrechts (2007a,b) and Patton (2006), consists in estimating the copula for the standardized residuals after fitting linear and/or non-linear univariate or multivariate models to the series. In the case of high-frequency data (intraday, for example) it is often necessary to deseasonalize the data first. The deseasonalized data in turn may reveal the presence of time-varying variance (heteroskedasticity) and heavy tails, so it may be appropriate to fit ARMA-GARCH models to each of the marginal series, with proper innovations (for example the use of t-distributions). The effect of asymmetric impacts of negative returns may be also incorporated in these models.

After the fitting of the models for both series, some parametric family of copulas may be used for the standardized residuals. Possible families are: t, Gaussian, Frank, Gumbel, Clayton etc. See Nelsen (2006) for details. Some criterion, like AIC or BIC, may be used to choose the best fit amongst the possible choices. This procedure does not produce, of course, i.i.d. samples, but at least the autocorrelation of each series is removed.

Another possibility is to fit a bivariate GARCH-type model to both series and then apply a copula family to the bivariate standardized residuals, with a time-dependent parameter vector θ_t. An issue here is the choice of a suitable dynamics for θ_t. See Dias and Embrechts (2007a,b), Patton (2006) and Rockinger and Jondeau (2001).

3.2 Nonparametric estimation

By (4), to estimate the copula C we need to estimate the marginal d.f.'s, F, followed by the quantiles $F_1^{-1}(u), F_2^{-1}(v)$.

Fermanian and Scaillet (2003) (written FS from here on) use kernel estimates for C: they estimate the marginal density and distribution functions, then the joint density and distribution functions, next estimate the quantiles and finally the copula. FS prove some asymptotic results for the various estimators, assuming that the process is strongly mixing plus further conditions on F_j and the bandwidths of the kernels.

Morettin et al. (2008) (referred to as MTCM from here on) follow the same route, but using wavelets instead of kernels. Both FS and MTCM present simulations and applications to real data. In Section 4 these estimators will be compared with the estimators proposed in the present paper.

We now propose a wavelet-smoothed empirical copula estimator. This approach is different from MTCM in the sense that the copula is estimated directly, without a need for estimating densities, distribution functions and quantiles. Assume that $\{(X_t, Y_t), t \in \mathbb{Z}\}$ is a strictly stationary process.

Since the copula $C(u,v)$ is in $L_2([0,1]^2)$, we can consider its wavelet expansion

$$C(u,v) = c_{0,\mathbf{0}} + \sum_{j=0}^{\infty} \sum_{\mathbf{k}} \sum_{\mu=h,v,d} d^{\mu}_{j,\mathbf{k}} \Psi^{\mu}_{j,\mathbf{k}}(u,v), \tag{13}$$

with the wavelet coefficients given by

$$c_{0,\mathbf{0}} = \int C(u,v)dudv, \quad d^{\mu}_{j,\mathbf{k}} = \int C(u,v)\Psi^{\mu}_{j,\mathbf{k}}(u,v)dudv. \tag{14}$$

For estimates of the wavelet coefficients we shall take the *empirical wavelet coefficients*,

$$\hat{d}^{\mu}_{j,\mathbf{k}} = \int C_T(u,v)\Psi^{\mu}_{j,\mathbf{k}}(u,v)dudv, \tag{15}$$

with a similar expression for $\hat{c}_{0,\mathbf{0}}$, where C_T is the empirical copula function based on observations $(X_t, Y_t), t = 1, \ldots, T$ and defined in (6).

The corresponding estimator for $C(u,v)$ is then:

$$\hat{C}(u,v) = \hat{c}_{0,\mathbf{0}} + \sum_{j,\mathbf{k}} \sum_{\mu} \delta(\hat{d}^{\mu}_{j,\mathbf{k}}, \lambda)\Psi^{\mu}_{j,\mathbf{k}}(u,v), \tag{16}$$

where $\delta(\cdot, \lambda)$ is a threshold. Both hard and soft thresholds are often used; see Donoho et al. (1995) for details on thresholds. In this paper we will take for threshold a high quantile. The sums in (16) will be computed for $0 \leq j \leq J$, where $J = J(T)$ is the maximum scale analyzed, chosen (for theoretical purposes) in such a way that $J \to \infty$, $J/T \to 0$, as $T \to \infty$. In turn, k_1, k_2 will vary from zero to $2^j - 1$.

There are several possible choices for the wavelets to be employed: Haar wavelets, compactly supported Daubechies wavelets, or Shannon, Meyer, Mexican hat or Morlet wavelet. The last of these is often used in physical sciences problems. Sometimes one encounters categorical type data, for which the Haar wavelet may be appropriate. In other situations compactly supported wavelets are more suitable, for example in theoretical considerations. In Morettin et al. (2008) we used Haar wavelets for the simulations and applications. Our choice here will be the d8 wavelet of the Daubechies family. We note that the choice of the wavelet family parallels the choice of the kernel in the case of kernel estimators. In this case the choice of the smoothing parameters (i.e. bandwidths) is crucial. In our situation, the choice of J is as important as the choice of the wavelet, as discussed further below.

We will not develop here theoretical properties of these smoothed wavelet estimators; this will be done elsewhere. Here, instead, we present some simulations to assess their performance, and we compare this with FS and MTCM results. We also provide applications with real data.

4 Simulations

In this section we present simulation examples of the wavelet estimators proposed in section 3, using the examples of Fermanian and Scaillet (2003) for ease of comparison. The choice of J is implemented by an heuristic approach. For $J = 2, 3, 4, 5$ we calculated biases, MSE, minimum and maximum values of these quantities and associated ranges. Then a value of J was chosen, looking at an overall performance of the estimator according to these measures. We mention that another possibility would be to use the following rule of thumb: truncate the series expansion at some level $J(T)$ such that $2^{J(T)}$ is approximately of order $T^{1/2}$.

(1) We consider the smoothed empirical copula estimator (16) in the case of a stationary bivariate autoregressive process of order one:

$$\mathbf{X}_t = \mathbf{A} + \mathbf{B}\mathbf{X}_{t-1} + \epsilon_t, \tag{17}$$

where $\mathbf{X}_t = (X_{1t}, X_{2t})$, with independent components and thus $C(u_1, u_2) = u_1 u_2$, $\epsilon_t \sim N(0, \Sigma)$, $\mathbf{A} = (1, 1)'$, $\text{vec}(\mathbf{B}) = (0.25, 0, 0, 0.75)'$ and $\text{vec}(\Sigma) = (0.75, 0, 0, 1.25)'$.

The number of Monte Carlo replications is 1,000, while the data length is $T = 2^{10} = 1,024$.

Table 1 shows the bias, $\mathbb{E}(\hat{C}) - C$, and mean squared error (MSE), $\mathbb{E}[(\hat{C} - C)^2]$, computed for the Daubechies d8 wavelet using $J = 5$, chosen as above. All values (the true value of the copula, the bias and MSE) are expressed as multiples of 10^{-4}. These results are satisfactory in terms of bias and MSE, comparable with those of MTCM, but are outperformed by those of FS. However, we remark that FS used series of the same length 1,024, but with 5,000 replications, while MTCM used a different wavelet, namely Haar, with $T = 1,024$ and 500 replications. Figure 1 shows the estimated copula and the contour plot.

Table A1 in the Appendix gives a larger grid of values for the wavelet estimator considered in this paper, showing an overall good performance both in terms of bias and of mean square error.

Table 1: Biases and MSE of estimators: independent case

(a) d8 wavelet estimator (1,000 replications)

x10^{-4}	C(.01,.01)	C(.05,.05)	C(.25,.25)	C(.50,.50)	C(.75,.75)	C(.95,.95)	C(.99,.99)
True	1.00	25.00	625.00	2500.00	5625.00	9025.00	9801.00
Bias	0.66	1.31	1.76	11.59	16.16	3.38	9.27
MSE	0.00	0.03	0.42	0.42	0.40	0.02	0.01

(b) MTCM Haar wavelet estimator (500 replications)

x10^{-4}	C(.01,.01)	C(.05,.05)	C(.25,.25)	C(.50,.50)	C(.75,.75)	C(.95,.95)	C(.99,.99)
True	1.00	25.00	625.00	2500.00	5625.00	9025.00	9801.00
Bias	0.05	0.72	9.06	25.32	28.57	13.56	5.32
MSE	0.00	0.00	0.02	0.11	0.15	0.03	0.01

(c) FS estimator with product of two Gaussian kernels (5,000 replications)

x10^{-4}	C(.01,.01)	C(.05,.05)	C(.25,.25)	C(.50,.50)	C(.75,.75)	C(.95,.95)	C(.99,.99)
True	1.00	25.00	625.00	2500.00	5625.00	9025.00	9801.00
Bias	-.09	-0.08	0.40	1.12	-0.90	-0.04	4.66
MSE	0.00	0.01	0.25	0.48	0.25	0.01	0.05

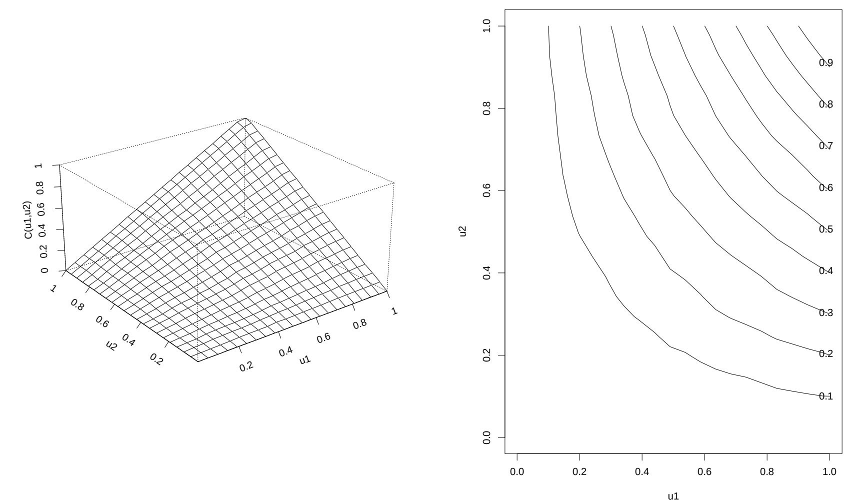


Figure 1: d8 wavelet estimated Copula based on empirical copulas and contour plot: independent case.

(2) We now turn to the case where the components of $\mathbf{X}_t$ are dependent processes, with $\mathbf{A} = (1,1)^{'}$, $\text{vec}(\mathbf{B}) = (0.25, 0.2, 0.2, 0.75)^{'}$ and $\text{vec}(\Sigma) = (0.75, 0.5, 0.5, 1.25)^{'}$.

Since X_{1t} and X_{2t} are positively dependent, we have $C(u_1, u_2) > u_1 u_2$. For 1,000 Monte Carlo replications with the data length $T = 1,024$, our results are reported in Table 2. These results outperform those of Fermanian and Scaillet and are comparable with those of Morettin et al. (2008). Our previous remarks concerning the number of observations and replications again hold here. Figure 2 shows the estimated copula and contour plot. Again, we have used the d8 wavelet with $J = 5$.

Table A2 in the Appendix shows biases and MSE for the wavelet estimator of this paper, for a larger grid of quantiles.

Table 2: Biases and MSE of estimators: dependent case

(a) d8 wavelet estimator (1,000 replications)

x10^{-4}	C(.01,.01)	C(.05,.05)	C(.25,.25)	C(.50,.50)	C(.75,.75)	C(.95,.95)	C(.99,.99)
True	27.08	197.95	1511.74	3747.68	6511.74	9197.95	9827.08
Bias	-0.30	0.96	3.89	7.37	9.56	0.05	1.99
MSE	0.02	0.11	0.50	0.66	0.52	0.11	0.02

(b) MTCM Haar wavelet estimator (500 replications)

x10^{-4}	C(.01,.01)	C(.05,.05)	C(.25,.25)	C(.50,.50)	C(.75,.75)	C(.95,.95)	C(.99,.99)
True	27.08	197.95	1511.74	3747.68	6511.74	9197.95	9827.08
Bias	0.60	-1.69	-21.32	-32.93	-13.27	9.22	16.75
MSE	0.02	0.11	0.46	0.73	0.50	0.11	0.05

(c) FS estimator with product of two Gaussian kernels (5,000 replications)

x10^{-4}	C(.01,.01)	C(.05,.05)	C(.25,.25)	C(.50,.50)	C(.75,.75)	C(.95,.95)	C(.99,.99)
True	27.08	197.95	1511.74	3747.68	6511.74	9197.95	9827.08
Bias	-7.47	-34.88	-130.32	-172.28	-130.53	-35.25	-7.65
MSE	0.01	0.18	1.98	3.36	1.99	0.18	0.01

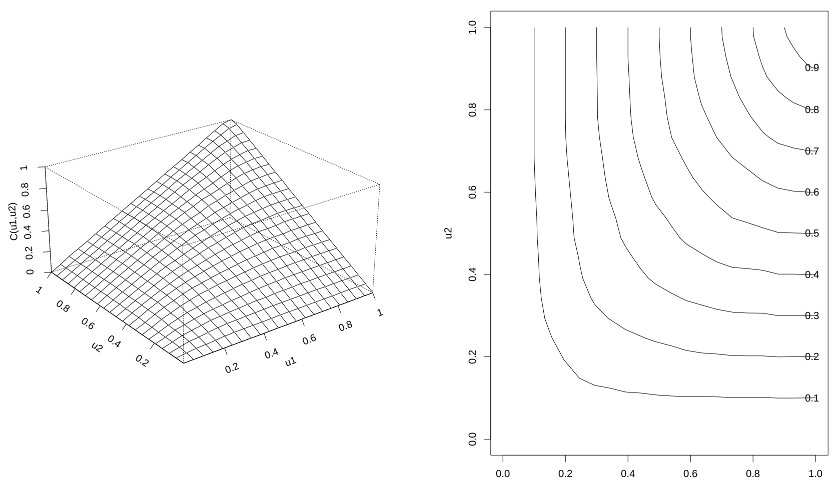


Figure 2: d8 wavelet estimated copula, based on empirical copulas, and contour plot: dependent case.

5 Empirical Applications

In this section we illustrate the estimation of copulas using the smoothed wavelet estimator given by (16), considering several pairs of series. In the first example we consider the daily returns of the following stock market indices: Ibovespa (Brazil) and IPC (Mexico), from September 4, 1995 to June 5, 2000 (with $T = 1,024$ observations). In the second example we consider daily returns of SP500 and DJIA, as in FS, recorded from 03/01/1994 to 07/07/2000 with 1,700 observations, but take only $T = 1,024$ so as to use a fast wavelet transform. Finally, in the third example we use again an example of FS, considering the pair of stock indices CAC40-DAX35, for the same period as the pair SP-DJ and the for same number of observations

(1) Figure 3 shows the scatter plot of Brazilian (Ibovespa) and Mexican (IPC) indices. The contemporaneous correlation coefficient is a moderate 0.552. In Figure 4 we have the plot of the estimated copula using (16) with the d8 wavelet, $J = 5$ (we have used this value based on the considerations made in the simulation results) and the corresponding contour plot, respectively. We have used the 0.90 percentile as the λ parameter of the thresholding procedure: all empirical wavelet coefficients smaller than λ were discarded. We see the same kind of behavior as in the simulated dependent case above, where the correlation coefficient was also moderate.

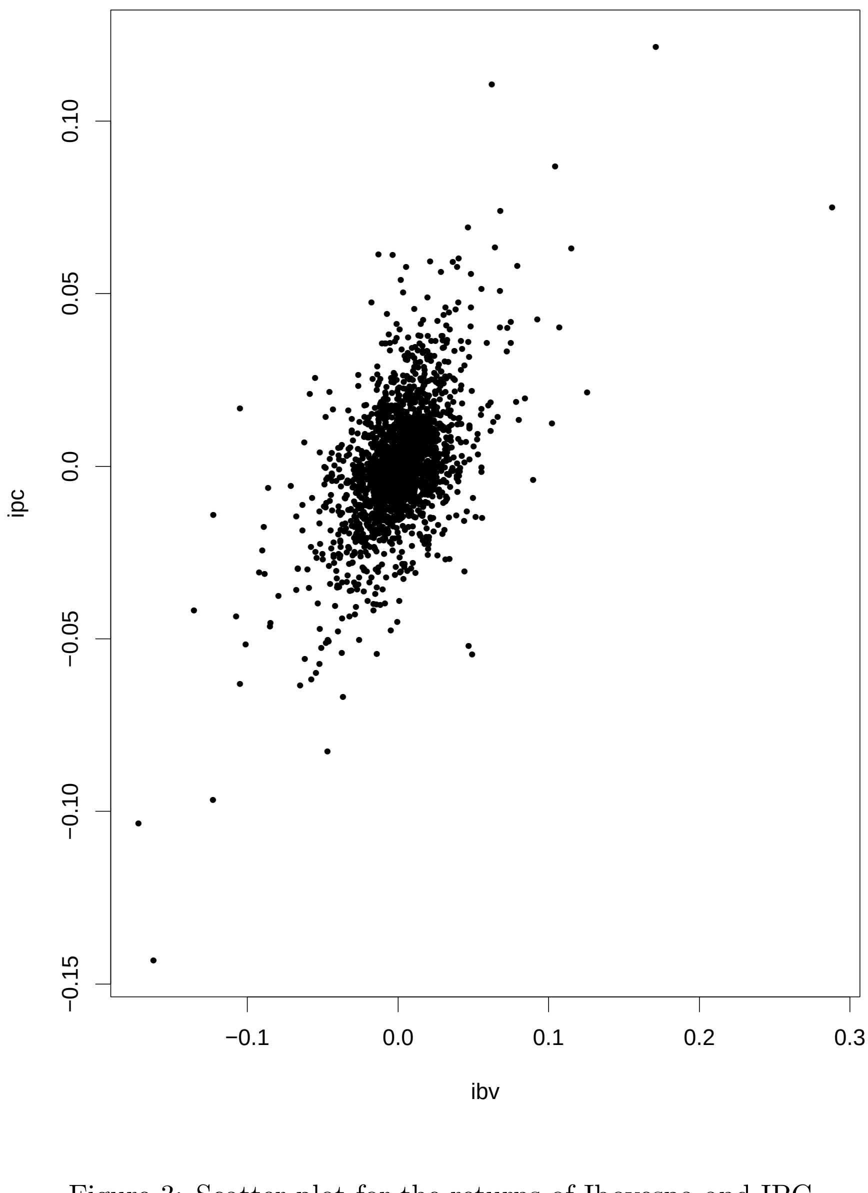


Figure 3: Scatter plot for the returns of Ibovespa and IPC.

(2) Figure 5 shows the scatter plot of the returns of SP500 and DJIA. There is a high correlation between both series, specifically the contemporaneous correlation coefficient is 0.933. The wavelet estimator, using again the d8 wavelet and $J = 5$, is presented in Figure 6, left panel. We see the expected comonotonic behaviour, due to the large dependence. The Kendall and Spearman coefficients are $\tau = 0.7341$ and $\rho_S = 0.9009$, respectively.

After fitting ARMA-GARCH models to the series, the (standardized) residual series have a correlation coefficient of 0.926. Specifically, an AR(3)-GARCH(1,1) model with t-errors was fitted to the SP500 series, and an AR(10)-GARCH(1,1) model also with t-errors was fitted to the DJIA series. These models passed the usual diagnostic checks; details are available upon request from the authors.

The wavelet estimator of the copula between residuals is shown in the right panel of Figure 6, and this plot is similar to the plot obtained for the original series. In turn these two plots are quite similar to the kernel copula estimator of FS. Lastly in Figure 7 we have the contour plots of the estimated Gaussian copula for the residuals, at left the distribution function and at right the density function; note the similarity of Figures 6 and 7.

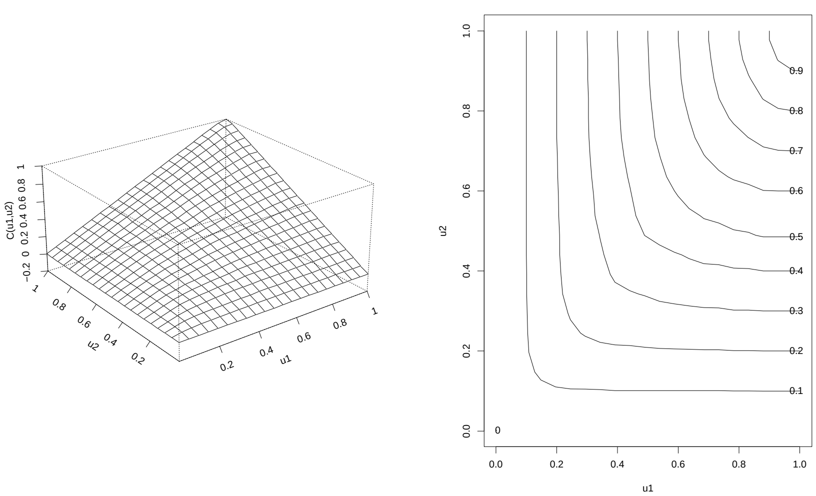


Figure 4: d8 wavelet estimated copula for Ibovespa and IPC and contour plot.

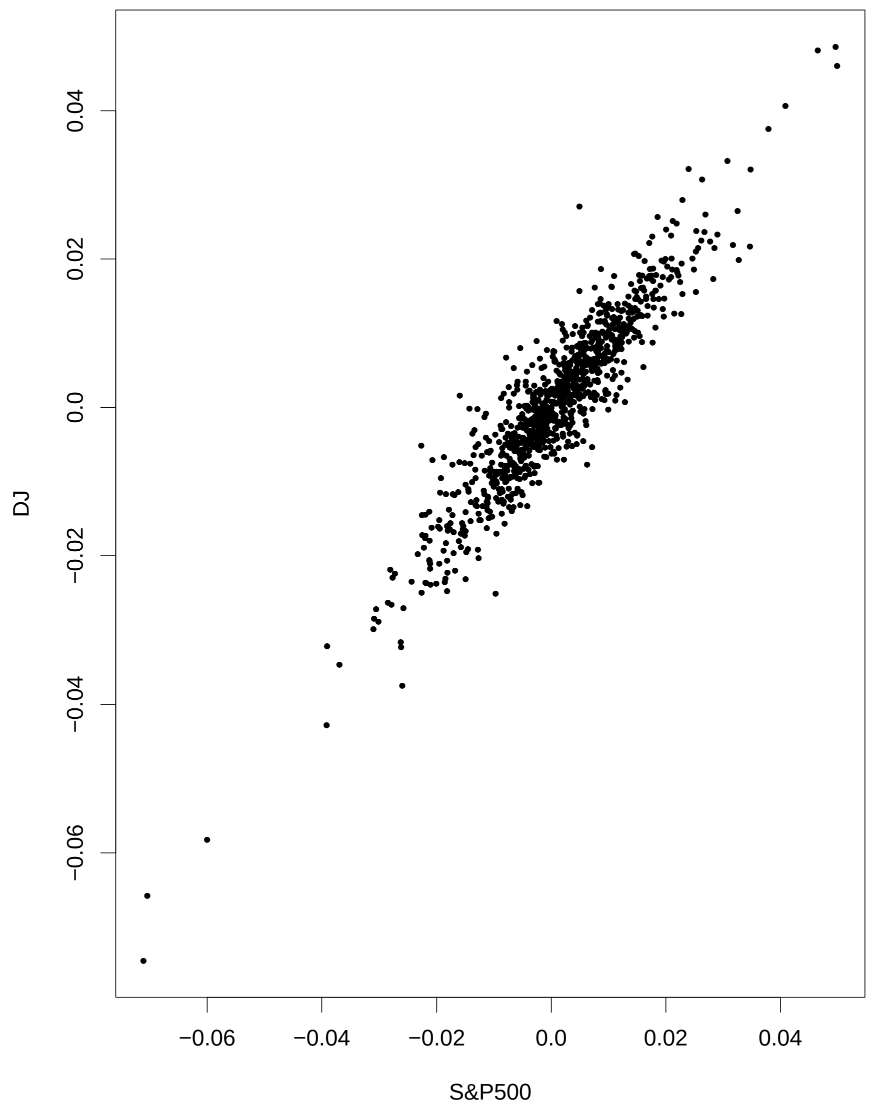


Figure 5: Scatter plot for the returns of SP500 and DJIA.

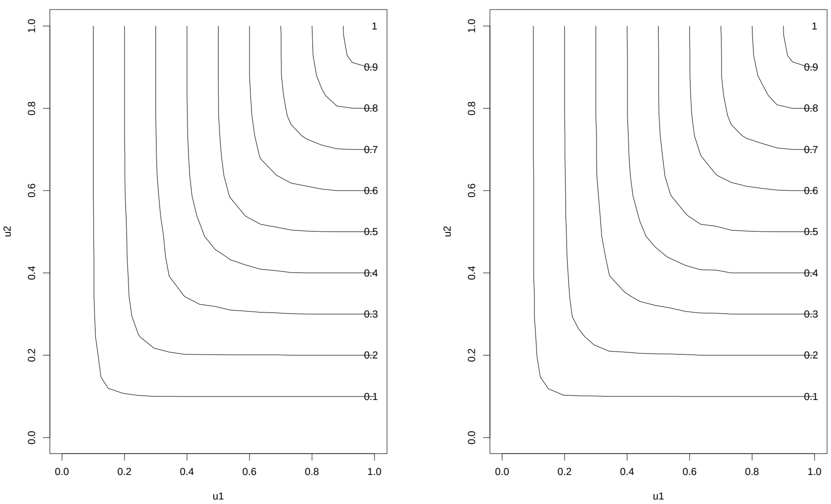


Figure 6: Contour plots of d8 wavelet estimated copulas: before and after fitting ARMA-GARCH models.

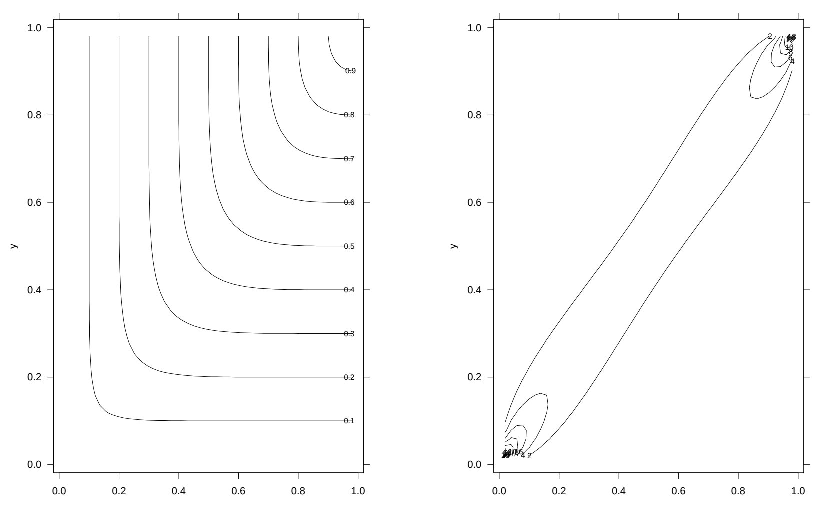
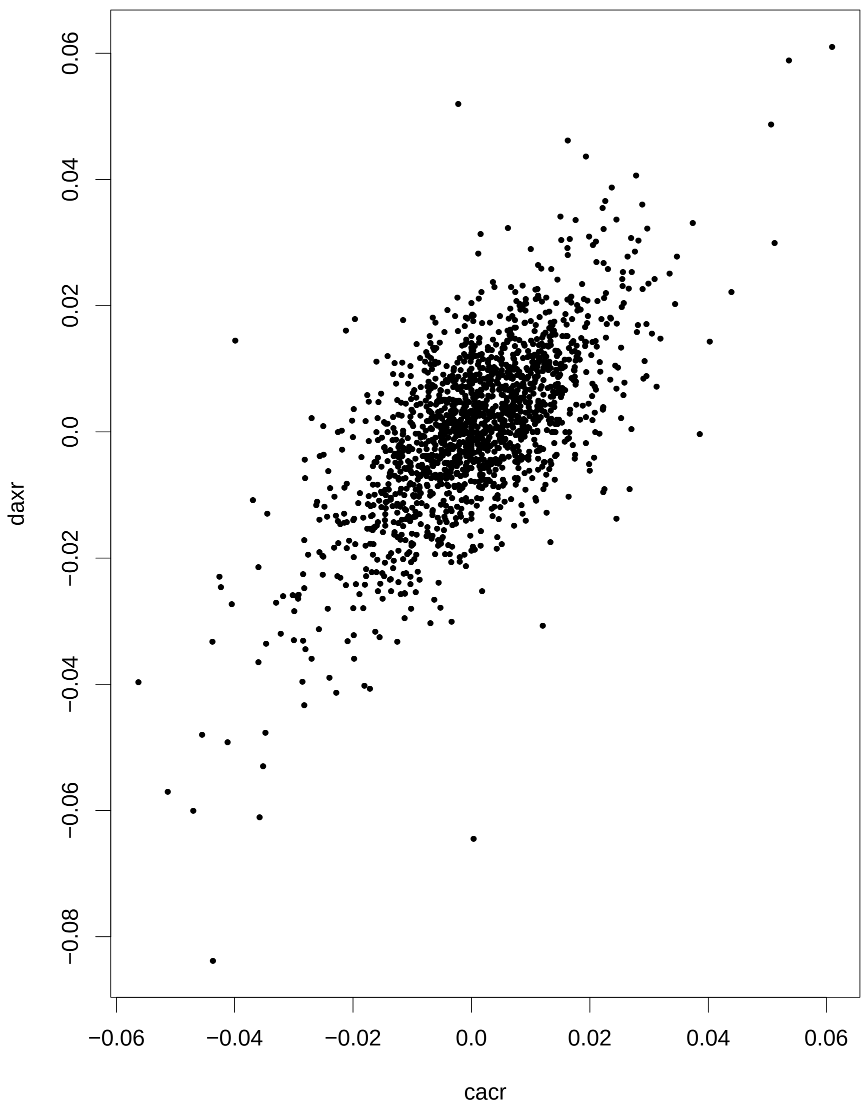


Figure 7: Contour plots of distribution (left) and density (right) of a Gaussian copula fitted to residuals of SP500 and DJIA.

(3) In Figure 8 we have the scatter plot of the returns of the stock indices CAC40-DAX35, as described above. The contemporaneous correlation coefficient is moderate, 0.67. Figure 9 shows the d8 wavelet estimator of the copula for the original series and Figure 10 the corresponding normal copula, after model fitting. The models fitted to the returns were AR(7)-GARCH(1,1) for CA40 and AR(6)-GARCH(1,1) for DAX35, respectively, both with errors following a t-distribution. These plots suggest a dependence, but not as strong as in the case of SP500-DJ. The Kendall τ is 0.4805 and the Spearman ρ_S is 0.6557.



0.06
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0.02
0.0
−0.02
−0.04
−0.06
−0.08
daxr
−0.06
−0.04
−0.02
0.0
0.02
0.04
0.06
cacr

Figure 8: Scatter plot of returns of CAC40 and DAX35 series.

6 Further remarks

In this work we have developed wavelet estimators of copulas based on empirical copulas. Although the idea here was to use these estimators with time series data, they can also be applied to i.i.d. samples. An advantage of the wavelet (and kernel) approach is that it can be used directly with the original series, as no model fitting is necessary. We have compared our proposal with two others: one using kernel methods and the other using wavelets, but with a different approach. The results seem satisfactory. Further studies are necessary to find the sample properties of the proposed estimator; this will be carried out elesewhere.

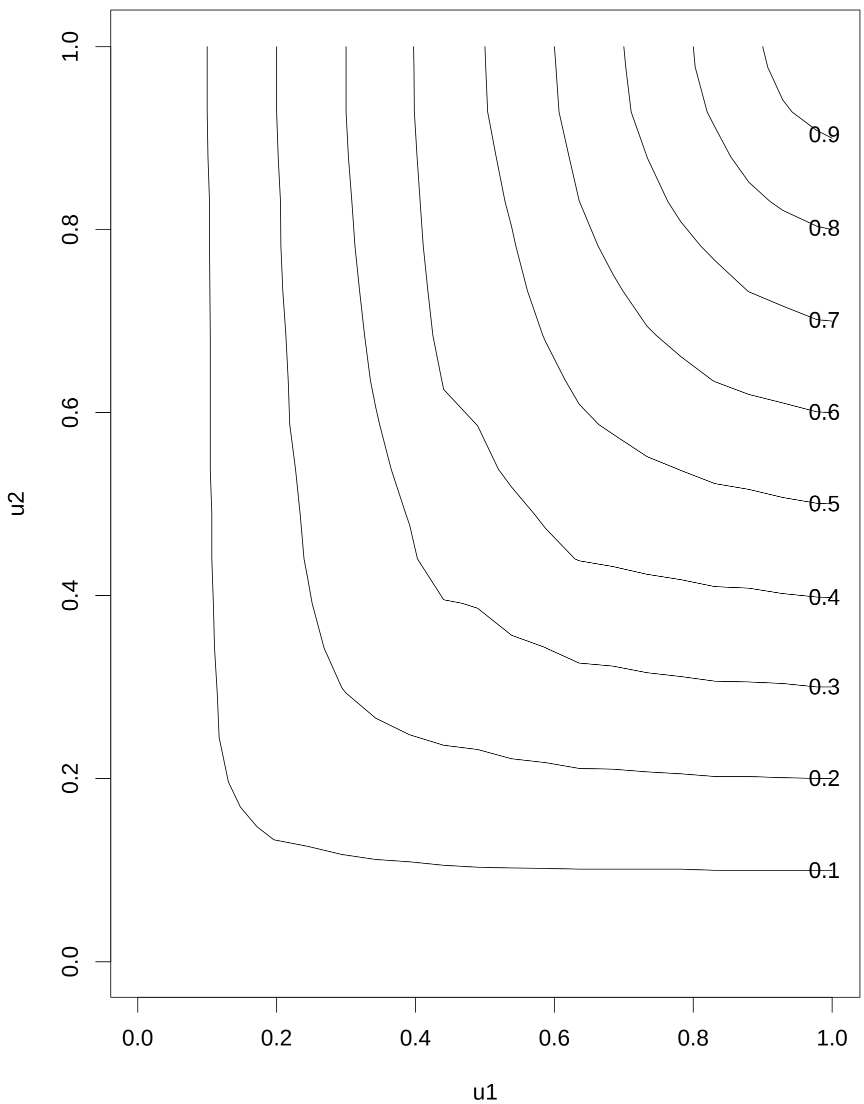

Figure 9: d8 wavelet estimator for the original stock indices returns of

CAC40 and DAX35

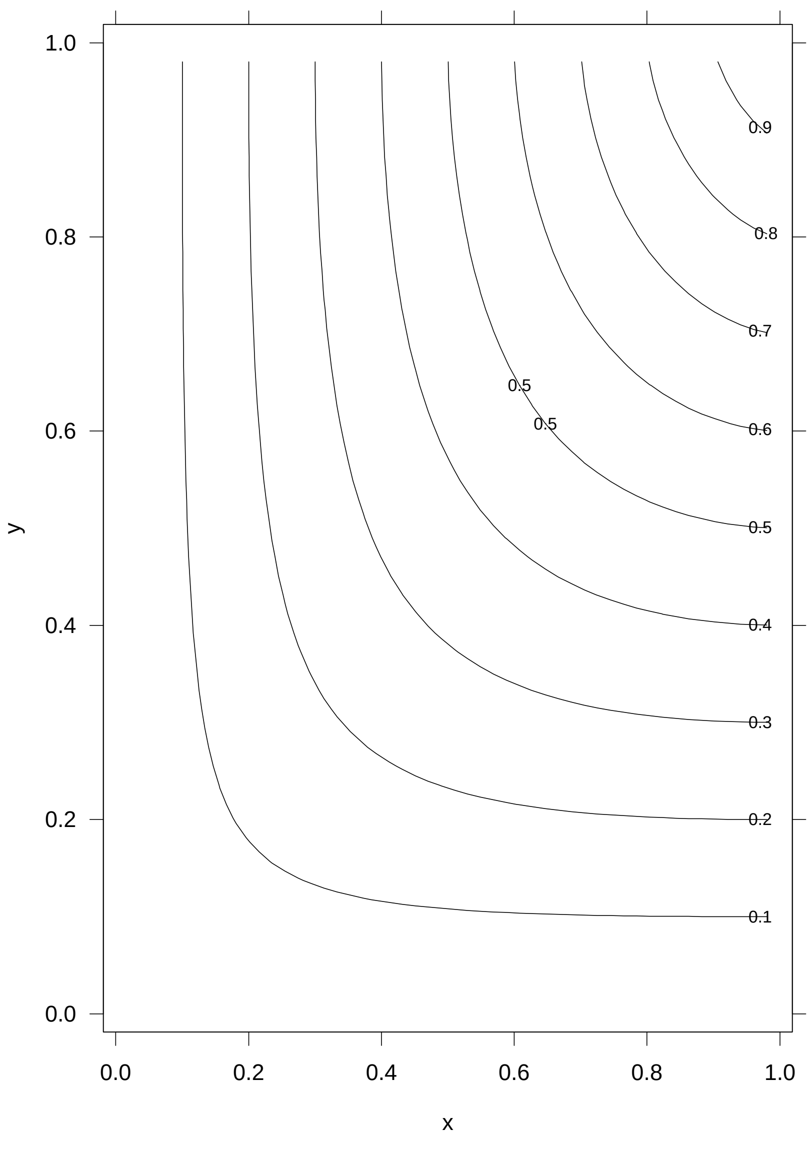

Figure 10. Normal copula after model fitting for the series CAC40-DAX35.

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Appendix

Table A1: Biases and MSE of wavelet estimates: independent case(10^{-4})

u/v		0.01	0.05	0.25	0.50	0.75	0.95	0.99
0.01	**True**	1.00	5.00	25.00	50.00	75.00	95.00	99.00
	Bias	0.66	0.39	1.73	3.57	5.79	7.24	7.78
	MSE	0.00	0.00	0.02	0.03	0.02	0.01	0.01
0.05	**True**	5.00	25.00	125.00	250.00	375.00	475.00	495.00
	Bias	0.24	1.31	1.63	4.21	7.42	8.30	9.56
	MSE	0.00	0.03	0.10	0.13	0.10	0.03	0.01
0.25	**True**	25.00	125.00	625.00	1250.00	1875.00	2375.00	2475.00
	Bias	1.15	2.09	1.76	6.87	9.20	10.39	12.66
	MSE	0.02	0.10	0.42	0.56	0.40	0.12	0.04
0.50	**True**	50.00	250.00	1250.00	2500.00	3750.00	4750.00	4950.00
	Bias	3.08	4.24	6.66	11.59	14.58	10.99	13.70
	MSE	0.03	0.14	0.55	0.75	0.57	0.15	0.04
0.75	**True**	75.00	375.00	1875.00	3750.00	5625.00	7125.00	7425.00
	Bias	4.77	6.01	6.85	11.47	16.16	10.96	14.04
	MSE	0.02	0.11	0.39	0.55	0.40	0.10	0.04
0.95	**True**	95.00	475.00	2375.00	4750.00	7125.00	9025.00	9405.00
	Bias	6.55	7.75	8.77	10.24	10.66	3.38	6.67
	MSE	0.01	0.03	0.10	0.13	0.10	0.02	0.01
0.99	**True**	99.00	495.00	2475.00	4950.00	7425.00	9405.00	9801.00
	Bias	0.01	0.01	0.03	0.04	0.04	0.01	0.01
	MSE	0.01	0.01	0.03	0.04	0.04	0.01	0.01

Table A2: Biases and MSE of wavelet estimates: dependent case(10^{-4})

u/v		0.01	0.05	0.25	0.50	0.75	0.95	0.99
0.01	**True**	26.92	61.82	94.74	99.45	99.97	100.00	100.00
	Bias	-0.30	1.43	6.18	7.17	7.41	7.42	7.33
	MSE	0.02	0.03	0.01	0.01	0.01	0.01	0.01
0.05	**True**	61.82	197.17	423.68	486.81	498.93	499.99	500.00
	Bias	1.90	0.96	3.84	6.48	7.56	7.82	7.87
	MSE	0.03	0.11	0.08	0.02	0.01	0.01	0.01
0.25	**True**	94.74	423.68	1508.83	2181.06	2447.04	2498.93	2499.97
	Bias	5.95	4.27	3.89	4.89	8.43	9.75	9.76
	MSE	0.01	0.08	0.50	0.33	0.07	0.01	0.01
0.50	**True**	99.45	486.81	2181.06	3739.88	4681.06	4986.81	4999.45
	Bias	7.14	7.25	7.20	7.37	6.30	8.92	9.60
	MSE	0.01	0.02	0.34	0.66	0.35	0.02	0.01
0.75	**True**	99.97	498.93	2447.04	4681.06	6508.83	7423.68	7494.74
	Bias	7.39	7.71	8.22	9.51	9.56	7.97	9.08
	MSE	0.01	0.01	0.07	0.33	0.51	0.09	0.01
0.95	**True**	100.00	499.99	2498.93	4986.81	7423.68	9197.17	9461.82
	Bias	7.41	7.81	9.65	9.16	8.22	0.05	0.91
	MSE	0.01	0.01	0.01	0.02	0.09	0.11	0.02
0.99	**True**	100.00	500.00	2499.97	4999.45	7494.74	9461.82	9826.92
	Bias	7.33	7.86	9.74	9.59	9.13	1.14	1.99
	MSE	0.01	0.01	0.01	0.01	0.01	0.02	0.02