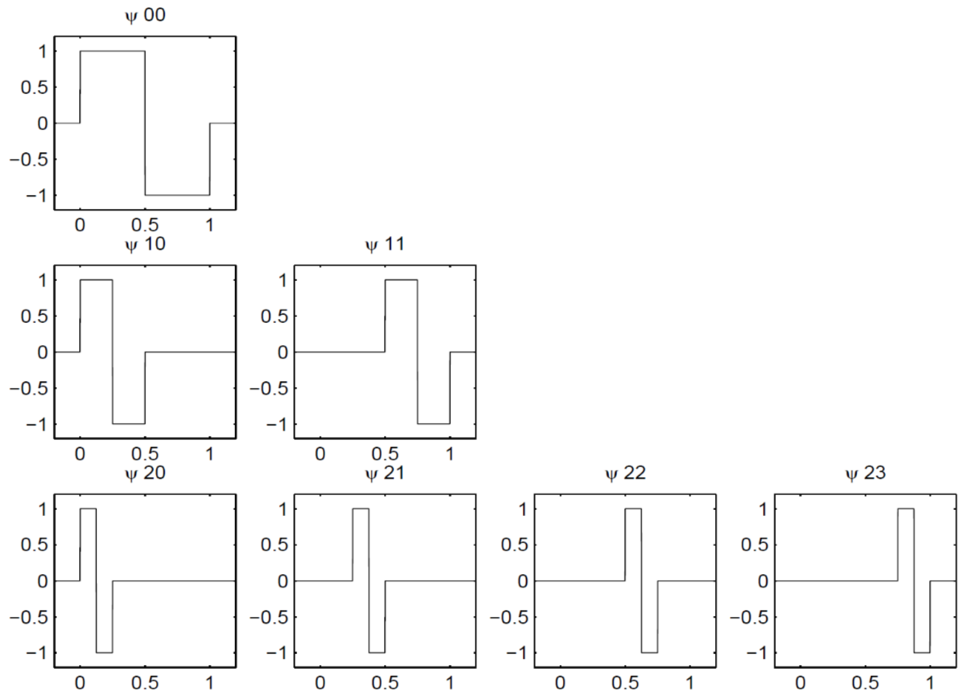
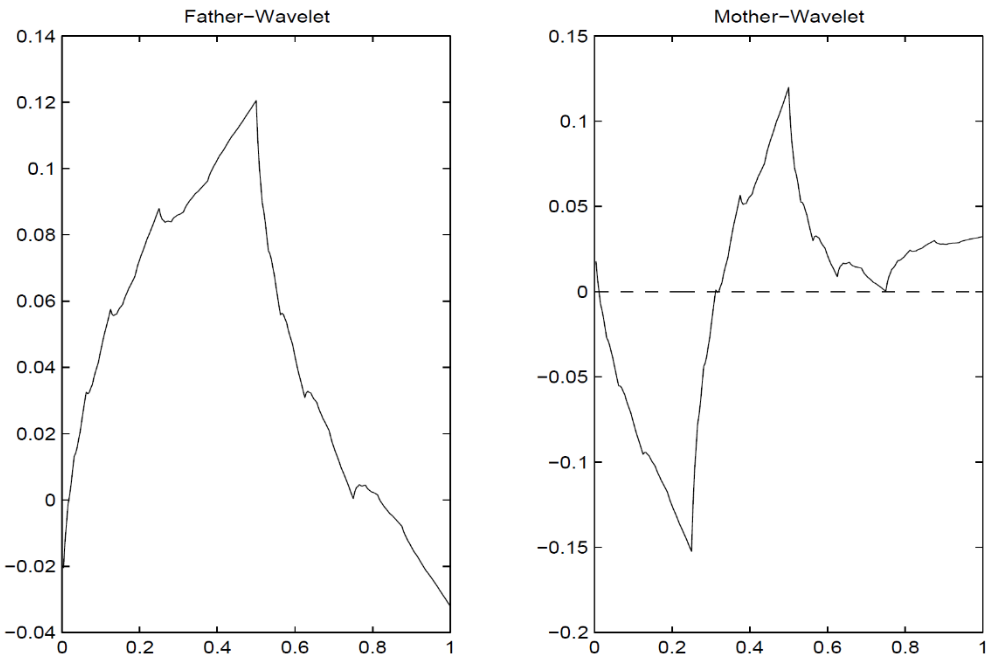


Figure 3: The Daubechies(4) Father and Mother Wavelet



A Transformada de Ondaletas

The wavelet transform consists of the vector of all coefficients $w = [\, c_0, c_{00}, c_{10}, c_{11}, c_{20}, c_{21}, \ldots \,]$.

Let y be a data vector with 2^n elements that can be represented by a piecewise constant function, $f(x)$ on $[0,1]$.[3] The wavelet transformation of f(x) is then given by

$$f(x) = c_0\phi(x) + \sum_{j=0}^{n-1}\sum_{k=0}^{2^j-1} c_{jk}\psi_{jk}(x) \; .$$

Here, $\phi(x)$ is the father wavelet, also referred to as the scaling function that represents the coarsest components or the smooth baseline trend of the function. For the simplest wavelet, the Haar

Decomposição e reconstrução

$$f_N = f_{N-1} + g_{N-1},$$

where $f_{N-1} \in V_{N-1}$ and $g_{N-1} \in W_{N-1}$. By repeating this process, we have

$$f_N = g_{N-1} + g_{N-2} + \cdots + g_{N-M} + f_{N-M} \qquad (1.6.1)$$

where $f_j \in V_j$ and $g_j \in W_j$ for any j, and M is so chosen that f_{N-M} is sufficiently "blurred". The "decomposition" in (1.6.1), which is unique, is

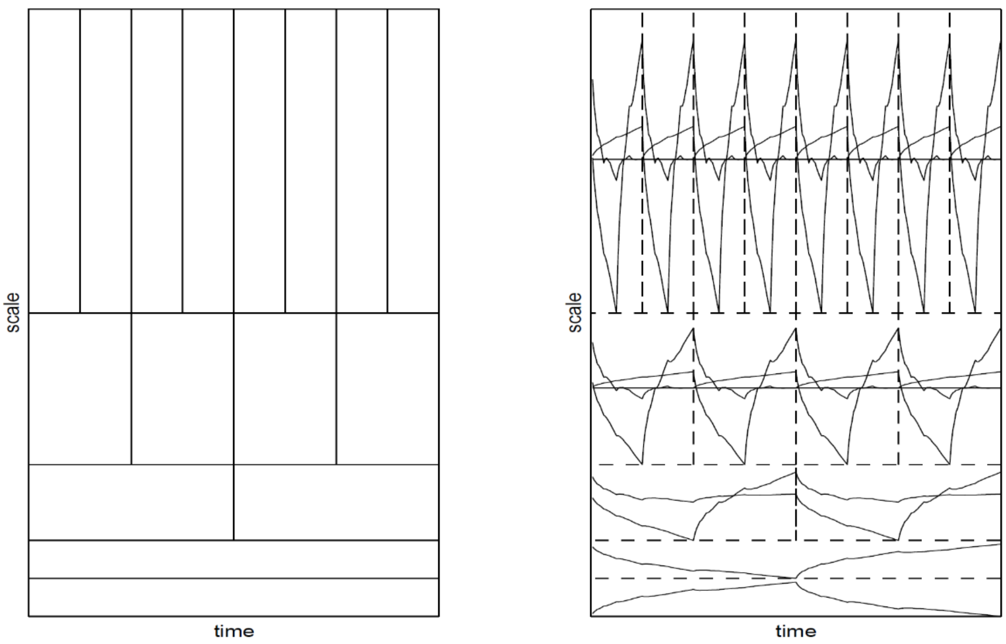
$$\begin{array}{ccccccccc}
 & & \mathbf{d}^{N-1} & & \mathbf{d}^{N-2} & & & & \mathbf{d}^{N-M} \\
 & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\
\mathbf{c}^{N} & \longrightarrow & \mathbf{c}^{N-1} & \longrightarrow & \mathbf{c}^{N-2} & \longrightarrow & \cdots & \longrightarrow & \mathbf{c}^{N-M}
\end{array}$$

Figure 1.6.1. Wavelet decomposition.

$$\begin{array}{ccccccccc}
\mathbf{d}^{N-M} & & \mathbf{d}^{N-M+1} & & & & \mathbf{d}^{N-1} & & \\
 & \searrow & & \searrow & & & & \searrow & \\
\mathbf{c}^{N-M} & \longrightarrow & \mathbf{c}^{N-M+1} & \longrightarrow & \cdots & & \mathbf{c}^{N-1} & \longrightarrow & \mathbf{c}^{N}
\end{array}$$

Figure 1.6.2. Wavelet reconstruction.

Figure 4: Daubechies(4) Wavelet Basis



Usualmente, não se consideram todos os níveis de resolução, J, mas um valor J_0, que corresponde à escala mais fina, 2^{J_0}. Nesse caso, podemos escrever

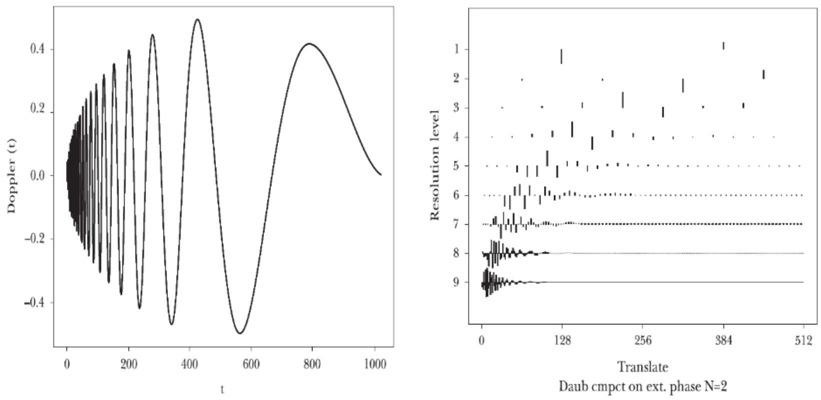
$$f(x) = c_{0,0}\phi_{0,0}(x) + \sum_{j=0}^{J_0-1} \sum_{k=0}^{2^j-1} d_{j,k}\psi_{j,k}(x). \tag{4.38}$$

Exemplo 4.7. Na Figura 4.7 (a), temos representada a função

$$f(t) = \sqrt{t(1-t)}\text{sen}(2,1\pi/(t+0,05)), \quad 0 \leq t \leq 1, \tag{4.39}$$

conhecida como *Doppler* e calculada em n=1024 pontos igualmente espaçados. Na Figura 4.7 (b) temos os coeficientes de ondaletas, computados pelo pacote WaveThresh, com a ondaleta de Daubechies $d2$.

Observamos que as frequências mais altas, presentes no começo da função, resultam em coeficientes de ondaletas maiores nas escalas mais finas (7,8 e 9), enquanto as frequências mais baixas aparecem nos coeficientes de escalas mais grossas.



(a) (b)

Figura 4.7: (a) A função Doppler; (b)Coeficientes de ondaletas da função Doppler.

Consider, for example, the four-dimensional vector $y = [\,2, 5, 2, 7\,]'$. Its wavelet representation is given by $y = W\,c$, where W contains the Haar basis vectors, $W = \begin{bmatrix}\phi_{00} & \psi_{00} & \psi_{10} & \psi_{11}\end{bmatrix}$:

$$\begin{bmatrix}2\\5\\2\\7\end{bmatrix} = \begin{bmatrix}1 & 1 & \sqrt{2} & 0\\ 1 & 1 & -\sqrt{2} & 0\\ 1 & -1 & 0 & \sqrt{2}\\ 1 & -1 & 0 & -\sqrt{2}\end{bmatrix} \cdot \begin{bmatrix}c_0\\c_{00}\\c_{10}\\c_{11}\end{bmatrix}.$$

The matrix of basis vectors W can be inverted easily, since the inverse of any orthogonal matrix is equal to its transpose divided by 4.[4] The solution for the wavelet coefficients is then given by

4. In general, the inverse of a (real valued) orthogonal matrix A with dimension n is $\alpha A'$, with $\alpha = |A|^{-\frac{2}{n}}$.

$$\begin{bmatrix}c_0\\c_{00}\\c_{10}\\c_{11}\end{bmatrix} = \frac{1}{4}\cdot\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ \sqrt{2} & -\sqrt{2} & 0 & 0\\ 0 & 0 & \sqrt{2} & -\sqrt{2}\end{bmatrix}\cdot\begin{bmatrix}2\\5\\2\\7\end{bmatrix} = \begin{bmatrix}4\\ -1/2\\ -\dfrac{3}{2\sqrt{2}}\\ -\dfrac{5}{2\sqrt{2}}\end{bmatrix}.$$

A similar representation could have been achieved by any orthogonal basis, such as the identity matrix I_4. However, the wavelet transform has the advantage of decomposing the data into different scales; that is, different levels of fineness. The vector of wavelet coefficients from our example consists of three levels (or scales): c_0, c_{00}, and $c_1 = (c_{10} c_{11})$. Setting the last level equal to zero and premultiplying with *W* sets the input vector *y* equal to [7/2 7/2 9/2 9/2]'; that is, the first two and the last two elements are averaged. In signal processing this is equivalent to applying a *low-pass filter*. Setting the last two levels equal to zero results in the transformed input vector [4 4 4 4]', the mean of *y*. Conversely, we could set all coefficients, except c_{00}, equal to zero and invert the transform by multiplying by *W*. The result would be the vector [-1/2 -1/2 1/2 1/2]', the difference between the mean and the second level of smoothness [7/2 7/2 9/2 9/2]'. Finally, setting all coefficients except c_{10} c_{11} equal to zero and reversing the transform gives the vector [-3/2 3/2 -5/2 5/2], the difference between the second level of smoothness and the original data. We can therefore use the wavelet decomposition to represent the vector *y* as the sum of its smooth component, S_2, and detail components D_2 and D_1:

... We can therefore use the wavelet decomposition to represent the vector *y* as the sum of it smooth component, S_2, and detail components D_2 and D_1:

$$y = S_2 + D_2 + D_1 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} + \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -3/2 \\ 3/2 \\ -5/2 \\ 5/2 \end{bmatrix} .$$

Figure 5 shows the wavelet transformation for a more complicated function, the *Doppler* function (taken from Vidakovic 1999).[5] Each additional level doubles the resolution and adds more detail to the function. It is also clear that the Haar wavelet is not the optimal choice for continuous and smooth functions, because a Daubechies wavelet achieves a much better approximation at four levels of depth.

Figure 5: Approximation of the Doppler Function using Different Levels of Fineness[6]

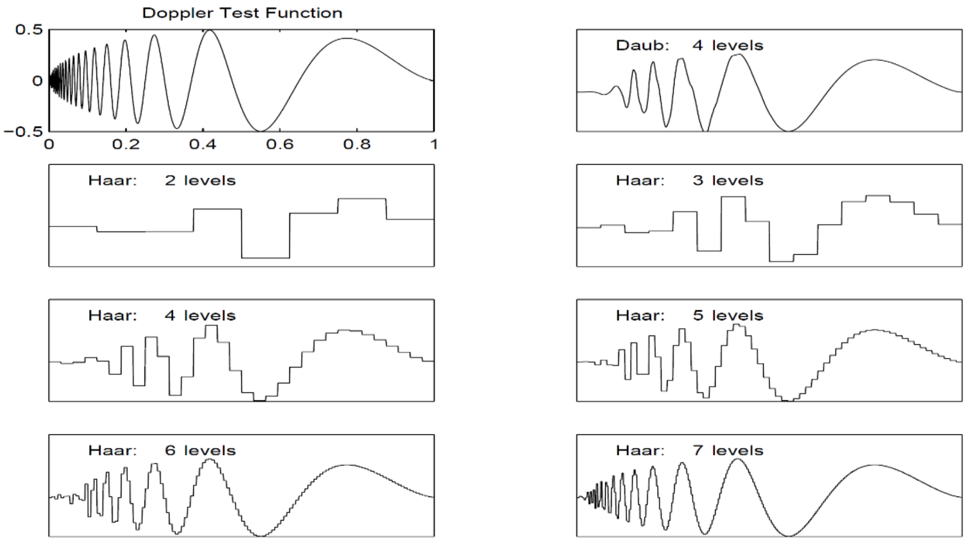


Figure 7: Multiresolution Analysis of the Vector $y = [\,4\ \ {-1}\ \ 3\ \ 2\ \ 1\ \ 4\ \ {-2}\ \ 2\,]$, using the Haar Wavelet Filter Coefficients

pair of filters: $h = \left[\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\right],\ g = \left[\frac{1}{\sqrt{2}}\ -\frac{1}{\sqrt{2}}\right].$

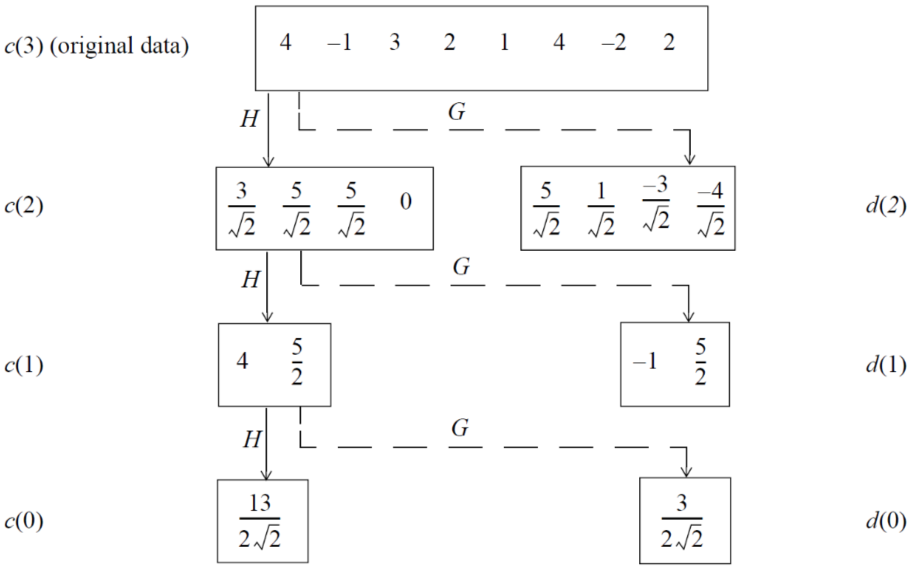


Figure 8: Reverse Wavelet Transform

$$c(0) \quad \boxed{\frac{13}{2\sqrt{2}}} \qquad \hat{H} \Rightarrow \boxed{\frac{13}{4} \quad \frac{13}{4}}$$

$$d(0) \quad \boxed{\frac{3}{2\sqrt{2}}} \qquad \hat{G} \Rightarrow \boxed{\frac{3}{4} \quad \frac{-3}{4}}$$

$$\Sigma: \boxed{4 \quad \frac{5}{2}}$$

$$c(1) \quad \boxed{4 \quad \frac{5}{2}} \qquad \hat{H} \Rightarrow \boxed{\frac{4}{\sqrt{2}} \quad \frac{4}{\sqrt{2}} \quad \frac{5}{2\sqrt{2}} \quad \frac{5}{2\sqrt{2}}}$$

$$d(1) \quad \boxed{-1 \quad \frac{5}{2}} \qquad \hat{G} \Rightarrow \boxed{\frac{-1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad \frac{5}{2\sqrt{2}} \quad \frac{-5}{2\sqrt{2}}}$$

$$\Sigma: \boxed{\frac{3}{\sqrt{2}} \quad \frac{5}{\sqrt{2}} \quad \frac{5}{\sqrt{2}} \quad 0}$$

$$c(2) \quad \boxed{\frac{3}{\sqrt{2}} \quad \frac{5}{\sqrt{2}} \quad \frac{5}{\sqrt{2}} \quad 0} \qquad \hat{H} \Rightarrow \boxed{\frac{3}{2} \quad \frac{3}{2} \quad \frac{5}{2} \quad \frac{5}{2} \quad \frac{5}{2} \quad \frac{5}{2} \quad 0 \quad 0}$$

$$d(2) \quad \boxed{\frac{5}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad \frac{-3}{\sqrt{2}} \quad \frac{-4}{\sqrt{2}}} \qquad \hat{G} \Rightarrow \boxed{\frac{5}{2} \quad \frac{-5}{2} \quad \frac{1}{2} \quad \frac{-1}{2} \quad \frac{-3}{2} \quad \frac{3}{2} \quad -2 \quad 2}$$

$$\Sigma: \boxed{4 \quad -1 \quad 3 \quad 2 \quad 1 \quad 4 \quad -2 \quad 2}$$