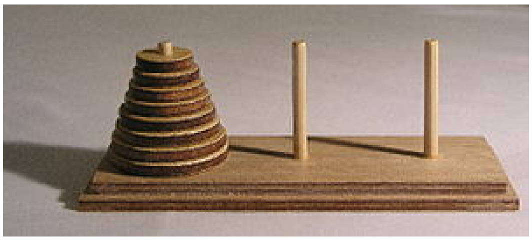
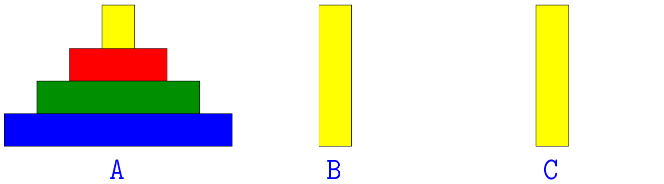
Torres de Hanoi: epílogo



Fonte: http://en.wikipedia.org/

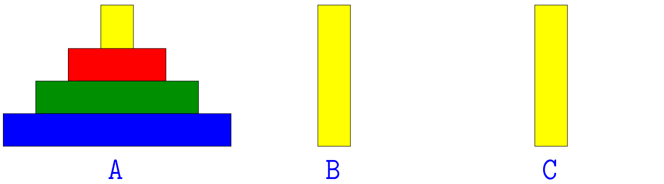
Torres de Hanoi



Desejamos transferir n discos do pino A para o pino C usando o pino B como auxiliar e repeitando as regras:

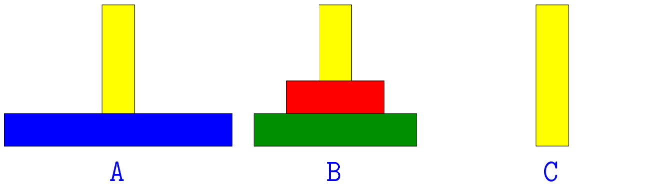
- podemos mover apenas um disco por vez;
- nunca um disco de diâmetro maior poderá ser colocado sobre um disco de diâmetro menor.

Algoritmo recursivo



Para resolver Hanoi(n,A,B,C) basta:

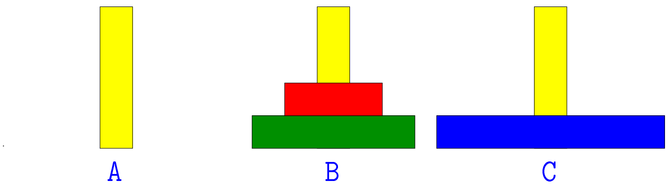
Algoritmo recursivo



Para resolver Hanoi(n,A,B,C) basta:

1. resolver Hanoi(n-1,A,C,B)

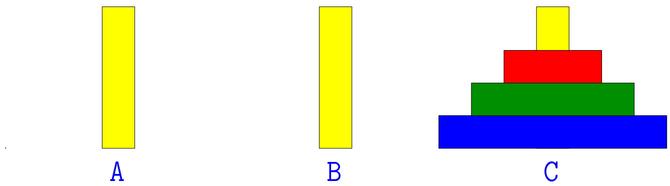
Algoritmo recursivo



Para resolver Hanoi(n,A,B,C) basta:

1. resolver Hanoi(n-1,A,C,B)
2. mover o disco n de A para C

Algoritmo recursivo



Para resolver Hanoi(n,A,B,C) basta:

1. resolver Hanoi(n-1,A,C,B)
2. mover o disco n de A para C
3. resolver Hanoi(n-1,B,A,C)

Base: sabemos revolver Hanoi(**0**,...,...,...)

The Tower of Hanoi Story

Taken From W.W. Rouse Ball & H.S.M. Coxeter, Mathematical Recreations and Essays, 12th edition. Univ. of Toronto Press, 1974. The De Parville account of the origen from La Nature, Paris, 1884, part I, pp. 285-286.

> *In the great temple at Benares beneath the dome that marks the centre of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty-four discs of pure gold, the largest disk resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the tower of Bramah. Day and night unceasingly the priest transfer the discs from one diamond needle to another according to the fixed and immutable laws of Bramah, which require that the priest on duty must not move more than one disc at a time and that he must place this disc on a needle so that there is no smaller disc below it. When the sixty-four discs shall have been thus transferred from the needle which at creation God placed them, to one of the other needles, tower, temple, and Brahmins alike will crumble into dust and with a thunderclap the world will vanish. The number of separate transfers of single discs which the Brahmins must make to effect the transfer of the tower is two raised to the sixty-fourth power minus 1 or 18,446,744,073,709,551,615 moves. Even if the priests move one disk every second, it would take more than 500 billion years to relocate the initial tower of 64 disks.*

http://www.rci.rutgers.edu/~cfs/472_html/AI_SEARCH/Story_TOH.html

Número de movimentos

Seja $\texttt{T}(\texttt{n})$ o número de movimentos feitos pelo algoritmo para resolver o problema das torres de Hanoi com $\texttt{n}$ disco.

Temos que

$$\texttt{T}(0) = 0$$
$$\texttt{T}(\texttt{n}) = 2\,\texttt{T}(\texttt{n}-1) + 1 \quad \text{para } \texttt{n} = 1, 2, 3 \ldots$$

Quanto vale $\texttt{T}(\texttt{n})$?

Recorrência

Temos que

$$\begin{aligned}
\texttt{T}(\texttt{n}) &= 2\,\texttt{T}(\texttt{n}-1) + 1 \\
&= 2\,(2\,\texttt{T}(\texttt{n}-2) + 1) + 1 \\
&= 2\,(2\,(2\,\texttt{T}(\texttt{n}-3) + 1) + 1) + 1 \\
&= 2\,(2\,(2\,(2\,\texttt{T}(\texttt{n}-4) + 1) + 1) + 1) + 1 \\
&= \cdots \\
&= 2\,(2\,(2\,(2\,(\cdots(2\,\texttt{T}(0) + 1)) + 1) + 1) + 1
\end{aligned}$$

Recorrência

Logo,

$$\begin{aligned}
\texttt{T}(\texttt{n}) &= 2^{\texttt{n}-1} + \cdots + 2^3 + 2^2 + 2 + 1 \\
&= 2^{\texttt{n}} - 1\,.
\end{aligned}$$

n	0	1	2	3	4	5	6	7	8	9
T(n)	0	1	3	7	15	31	63	127	255	511

Conclusões

> O número de movimentos feitos pela chamada `hanoi(n, ..., ..., ...)` é
>
> $$2^{\texttt{n}} - 1\,.$$

Notemos que a função `hanoi` faz o **número mínimo** de movimentos: **não é possível** resolver o quebra-cabeça com menos movimentos.

Enquanto isto ... os monges ...

$$\texttt{T}(64) = 18.446.744.073.709.551.615 \approx 1{,}84 \times 10^{19}$$

Suponha que os monges façam o movimento de 1 disco por segundo(!).

$$\begin{aligned}
1{,}84 \times 10^{19}\ \text{seg} &\approx 3{,}07 \times 10^{17}\ \text{min} \\
&\approx 5{,}11 \times 10^{15}\ \text{horas} \\
&\approx 2{,}13 \times 10^{14}\ \text{dias} \\
&\approx 5{,}83 \times 10^{11}\ \text{anos.} \\
&= \textbf{583}\ \text{bilhões de anos.}
\end{aligned}$$

A idade da Terra é **4,54 bilhões de anos**.