Fibonacci

Fonte: `http://www.geek.com/geek-cetera/`

PF 2.3 S 5.2

http://www.ime.usp.br/~pf/algoritmos/aulas/recu.html

Números de Fibonacci

$$F_0 = 0 \qquad F_1 = 1 \qquad F_n = F_{n-1} + F_{n-2}$$

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34

Algoritmo recursivo para F_n:

```
def fibonacciR(n)
    if n == 0:  return 0
    if n == 1:  return 1
    return fibonacciR(n-1) +
           fibonacciR(n-2)
```

fibonacciR(4)

```
fibonacciR(4)
  fibonacciR(3)
    fibonacciR(2)
      fibonacciR(1)
      fibonacciR(0)
    fibonacciR(1)
  fibonacciR(2)
    fibonacciR(1)
    fibonacciR(0)
fibonacci(4) = 3.
```

Fibonacci iterativo

```
def fibonacciI(n)
    if n == 0:  return 0
    if n == 1:  return 1
    anterior = 0
    atual = 1
    for i in range(1,n,1):
          proximo = atual + anterior
          anterior = atual
          atual = proximo
    return atual
```

Qual é mais eficiente?

```
meu_prompt> time ./fibonacciI.py 10
fibonacci(10)=55
real                        0m0.028s
user                        0m0.024s
sys                         0m0.000s

meu_prompt> time ./fibonacciR.py 10
fibonacci(10)=55
real                        0m0.028s
user                        0m0.024s
sys                         0m0.000s
```

Qual é mais eficiente?

```
meu_prompt> time ./fibonacciI.py 20
fibonacci(20) = 6765
real                       0m0.028s
user                       0m0.024s
sys                        0m0.000s

meu_prompt> time ./fibonacciR.py 20
fibonacci(20) = 6765
real                       0m0.030s
user                       0m0.024s
sys                        0m0.000s
```

Qual é mais eficiente?

```
meu_prompt> time ./fibonacciI.py 30
fibonacci(30) = 832040
real                    0m0.028s
user                    0m0.024s
sys                     0m0.000s

meu_prompt> time ./fibonacciR.py 30
fibonacci(30) = 832040
real                    0m0.584s
user                    0m0.580s
sys                     0m0.000s
```

Qual é mais eficiente?

```
meu_prompt> time ./fibonacciI.py 40
fibonacci(40) = 102334155
real                    0m0.026s
user                    0m0.024s
sys                     0m0.000s

meu_prompt> time ./fibonacciR.py 40
fibonacci(40) = 102334155
real                    1m8.530s
user                    1m8.508s
sys                     0m0.004s
```

Qual é mais eficiente?

```
meu_prompt> time ./fibonacciI.py 45
fibonacci(45) = 1134903170
real                        0m0.032s
user                        0m0.028s
sys                         0m0.000s

meu_prompt> time ./fibonacciR.py 45
fibonacci(45) = 1134903170
real                      12m47.577s
user                      12m47.248s
sys                         0m0.080s
```

fibonacciR(5)

fibonacciR resolve subproblemas muitas vezes.

```
fibonacciR(5)
  fibonacciR(4)
    fibonacciR(3)
      fibonacciR(2)
        fibonacciR(1)
        fibonacciR(0)
      fibonacciR(1)
    fibonacciR(2)
      fibonacciR(1)
      fibonacciR(0)
  fibonacciR(3)
    fibonacciR(2)
      fibonacciR(1)
      fibonacciR(0)
    fibonacciR(1)
fibonacci(5) = 5.
```

fibonacciR(8)

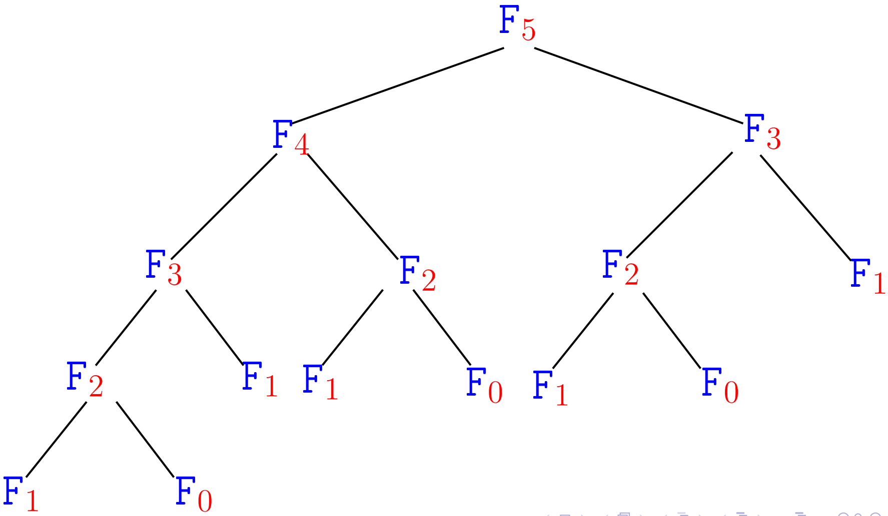
fibonacciR resolve subproblemas muitas vezes.

```
fibonacciR(8)
  fibonacciR(7)
    fibonacciR(6)
      fibonacciR(5)
        fibonacciR(4)
          fibonacciR(3)
            fibonacciR(2)
              fibonacciR(1)
              fibonacciR(0)
            fibonacciR(1)
          fibonacciR(2)
            fibonacciR(1)
            fibonacciR(0)
        fibonacciR(3)
          fibonacciR(2)
            fibonacciR(1)
            fibonacciR(0)
          fibonacciR(1)
      fibonacciR(4)
        fibonacciR(3)
          fibonacciR(2)
            fibonacciR(1)
            fibonacciR(0)
            fibonacciR(1)
          fibonacciR(2)
            fibonacciR(1)
            fibonacciR(0)
        fibonacciR(5)
          fibonacciR(4)
            fibonacciR(3)
              fibonacciR(2)
                fibonacciR(1)
                fibonacciR(0)
              fibonacciR(1)
            fibonacciR(2)
              fibonacciR(1)
              fibonacciR(0)
          fibonacciR(3)
            fibonacciR(2)
              fibonacciR(1)
              fibonacciR(0)
            fibonacciR(1)
      fibonacciR(6)
        fibonacciR(5)
          fibonacciR(4)
            fibonacciR(3)
                fibonacciR(2)
                  fibonacciR(1)
                  fibonacciR(0)
                fibonacciR(1)
              fibonacciR(2)
                fibonacciR(1)
                fibonacciR(0)
            fibonacciR(3)
              fibonacciR(2)
                fibonacciR(1)
                fibonacciR(0)
              fibonacciR(1)
          fibonacciR(4)
            fibonacciR(3)
              fibonacciR(2)
                fibonacciR(1)
                fibonacciR(0)
              fibonacciR(1)
            fibonacciR(2)
              fibonacciR(1)
              fibonacciR(0)
fibonacci(8) = 21.
```

Árvore da recursão

Consumo de tempo é **exponencial.**

`fibonacciR` resolve subproblemas muitas vezes.



Consumo de tempo

$T(n)$:= número de somas feitas por `fibonacciR(n)`

```
def fibonacciR(n)
    if n == 0:  return 0
    if n == 1:  return 1
    return fibonacciR(n-1) +
           fibonacciR(n-2)
```

Consumo de tempo

linha	número de somas
1	$= 0$
2	$= 0$
3	$= \mathtt{T}(\mathtt{n}-1)$
4	$= \mathtt{T}(\mathtt{n}-2) + 1$

$$\mathtt{T}(\mathtt{n}) \quad = \mathtt{T}(\mathtt{n}-1) + \mathtt{T}(\mathtt{n}-2) + 1$$

Recorrência

$$\mathtt{T}(0) = 0$$
$$\mathtt{T}(1) = 0$$
$$\mathtt{T}(\mathtt{n}) = \mathtt{T}(\mathtt{n}-1) + \mathtt{T}(\mathtt{n}-2) + 1 \quad \text{para } \mathtt{n} = 2, 3, \ldots$$

Uma estimativa para $\mathtt{T}(\mathtt{n})$?

Recorrência

$$\mathtt{T}(0) = 0$$
$$\mathtt{T}(1) = 0$$
$$\mathtt{T}(\mathtt{n}) = \mathtt{T}(\mathtt{n}-1) + \mathtt{T}(\mathtt{n}-2) + 1 \quad \text{para } \mathtt{n} = 2, 3, \ldots$$

Uma estimativa para $\mathtt{T}(\mathtt{n})$?

Solução: $\mathtt{T}(\mathtt{n}) > (3/2)^{\mathtt{n}}$ para $\mathtt{n} \geq 6$.

$\mathtt{n}$	0	1	2	3	4	5	6	7	8	9
$\mathtt{T}_{\mathtt{n}}$	0	0	1	2	4	7	12	20	33	54
$(3/2)^{\mathtt{n}}$	1	1.5	2.25	3.38	5.06	7.59	11.39	17.09	25.63	38.44

Recorrência

Prova: $\mathtt{T}(6) = 12 > 11.40 > (3/2)^6$ e
$\mathtt{T}(7) = 20 > 18 > (3/2)^7$.
Se $\mathtt{n} \geq 8$, então

$$
\begin{aligned}
\mathtt{T}(\mathtt{n}) &= \mathtt{T}(\mathtt{n}-1) + \mathtt{T}(\mathtt{n}-2) + 1 \\
&\overset{\text{hi}}{>} (3/2)^{\mathtt{n}-1} + (3/2)^{\mathtt{n}-2} + 1 \\
&= (3/2+1)\,(3/2)^{\mathtt{n}-2} + 1 \\
&> (5/2)\,(3/2)^{\mathtt{n}-2} \\
&> (9/4)\,(3/2)^{\mathtt{n}-2} \\
&= (3/2)^2 (3/2)^{\mathtt{n}-2} \\
&= (3/2)^{\mathtt{n}}\,.
\end{aligned}
$$

Logo, $\mathtt{T}(\mathtt{n}) \geq (3/2)^{\mathtt{n}}$. Consumo de tempo é **exponencial**.

Conclusão

O consumo de tempo é da função `fibonacciI(n)` é proporcional a `n`.

O consumo de tempo da função `fibonacciR` é **exponencial.**

Exercícios

Prove que

$$\mathtt{T}(\mathtt{n}) = \frac{\phi^{\mathtt{n}+1} - \hat{\phi}^{\mathtt{n}+1}}{\sqrt{5}} - 1 \quad \text{para } \mathtt{n} = 0, 1, 2, \ldots$$

onde

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1{,}61803 \quad \text{e} \quad \hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -0{,}61803.$$

Prove que $1 + \phi = \phi^2$.

Prove que $1 + \hat{\phi} = \hat{\phi}^2$.