

Tópicos de Análise de Algoritmos

Parte destes slides são adaptações de slides
do Prof. Paulo Feofiloff e do Prof. José Coelho de Pina.

Aula 4

Multiplicação de inteiros grandes

Secs 5.5 do KT e 28.2 do CLRS.

Multiplicação de inteiros gigantescos

n := número de algarismos.

Problema: Dados dois números inteiros $X[1..n]$ e $Y[1..n]$, calcular o **produto** $X \cdot Y$.

Entra: Exemplo com $n = 12$

		12										1	
X		9	2	3	4	5	5	4	5	6	2	9	8
Y		0	6	3	2	8	4	9	9	3	8	4	4

Algoritmo do ensino fundamental

The diagram illustrates the elementary school multiplication algorithm for two 8-digit numbers, represented by asterisks (*). The numbers are aligned to the right, with the multiplier on top and the multiplicand below it. A horizontal line separates the two numbers. The multiplication process is shown by a series of partial products, each shifted one position to the left relative to the previous one. The partial products are separated by horizontal lines. The final result is shown at the bottom, aligned to the right.

```
      * * * * * * * *
      * * * * * * * *
      ×
      -----
      * * * * * * * *
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      * * * * * * * *
```

O algoritmo do ensino fundamental é $\Theta(n^2)$.

Da Wiki

In 1960, Kolmogorov organized a seminar on mathematical problems in cybernetics at the Moscow State University, where, among other problems in the complexity of computation, he stated the $\Omega(n^2)$ conjecture, that stated that **the traditional algorithm for multiplication of two n -digit numbers was asymptotically optimal**, meaning that any algorithm for that task would require $\Omega(n^2)$ elementary operations.

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Kolmogorov was very excited about the discovery; he communicated it at the next meeting of the seminar, which was then terminated. Kolmogorov gave some lectures on the Karatsuba result at conferences all over the world (see, for example, *Proceedings of the International Congress of Mathematicians 1962*, pp. 351–356, and also *6 Lectures delivered at the International Congress of Mathematicians in Stockholm, 1962*) and published the method in 1962, in the *Proceedings of the USSR Academy of Sciences*.

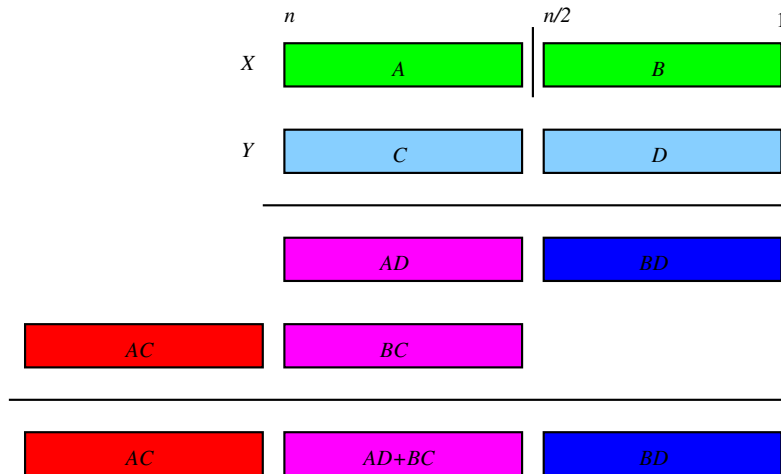
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The article had been written by Kolmogorov and contained two results on multiplication, Karatsuba’s algorithm and a separate result by Yuri Ofman; it listed “A. Karatsuba and Yu. Ofman” as the authors. Karatsuba only became aware of the paper when he received the reprints from the publisher.

Divisão e conquista



$$X \cdot Y = A \cdot C \times 10^{2\lceil n/2 \rceil} + (A \cdot D + B \cdot C) \times 10^{\lceil n/2 \rceil} + B \cdot D$$

Exemplo

X

	4		1	
3	1	4	1	

Y

	4		1	
5	9	3	6	

Exemplo

X

	4		1	
3	1	4	1	

Y

	4		1	
5	9	3	6	

A

3	1
---	---

B

4	1
---	---

C

5	9
---	---

D

3	6
---	---

Exemplo

$$X \begin{array}{c} \textcolor{red}{4} \qquad \qquad \textcolor{blue}{1} \\ \boxed{3} \boxed{1} \boxed{4} \boxed{1} \end{array}$$

$$Y \begin{array}{c} \textcolor{red}{4} \qquad \qquad \textcolor{blue}{1} \\ \boxed{5} \boxed{9} \boxed{3} \boxed{6} \end{array}$$

$$A \boxed{3} \boxed{1}$$

$$B \boxed{4} \boxed{1}$$

$$C \boxed{5} \boxed{9}$$

$$D \boxed{3} \boxed{6}$$

$$X \cdot Y = \textcolor{red}{A} \cdot \textcolor{red}{C} \times 10^4 + (\textcolor{red}{A} \cdot \textcolor{blue}{D} + \textcolor{blue}{B} \cdot \textcolor{red}{C}) \times 10^2 + \textcolor{blue}{B} \cdot \textcolor{blue}{D}$$

$$\textcolor{red}{A} \cdot \textcolor{red}{C} = 1829 \qquad (\textcolor{red}{A} \cdot \textcolor{blue}{D} + \textcolor{blue}{B} \cdot \textcolor{red}{C}) = 1116 + 2419 = 3535$$

$$\textcolor{blue}{B} \cdot \textcolor{blue}{D} = 1476$$

$$\begin{array}{r} \textcolor{red}{A} \cdot \textcolor{red}{C} \qquad \qquad \qquad \textcolor{red}{1} \ \textcolor{red}{8} \ \textcolor{red}{2} \ \textcolor{red}{9} \ 0 \ 0 \ 0 \ 0 \\ (\textcolor{red}{A} \cdot \textcolor{blue}{D} + \textcolor{blue}{B} \cdot \textcolor{red}{C}) \qquad \qquad \textcolor{red}{3} \ \textcolor{red}{5} \ \textcolor{red}{3} \ \textcolor{red}{5} \ 0 \ 0 \\ \textcolor{blue}{B} \cdot \textcolor{blue}{D} \qquad \qquad \qquad \qquad \qquad \textcolor{blue}{1} \ \textcolor{blue}{4} \ \textcolor{blue}{7} \ \textcolor{blue}{6} \\ \hline X \cdot Y = \qquad \qquad \qquad 1 \ 8 \ 6 \ 4 \ 4 \ 9 \ 7 \ 6 \end{array}$$

Algoritmo de Multi-DC

Algoritmo recebe inteiros $X[1..n]$ e $Y[1..n]$ e devolve $X \cdot Y$.

MULT (X, Y, n)

```
1  se  $n = 1$  devolva  $X \cdot Y$ 
2   $q \leftarrow \lceil n/2 \rceil$ 
3   $A \leftarrow X[q + 1..n]$      $B \leftarrow X[1..q]$ 
4   $C \leftarrow Y[q + 1..n]$      $D \leftarrow Y[1..q]$ 
5   $E \leftarrow \text{MULT}(A, C, \lfloor n/2 \rfloor)$ 
6   $F \leftarrow \text{MULT}(B, D, \lceil n/2 \rceil)$ 
7   $G \leftarrow \text{MULT}(A, D, \lceil n/2 \rceil)$ 
8   $H \leftarrow \text{MULT}(B, C, \lceil n/2 \rceil)$ 
9   $R \leftarrow E \times 10^{2\lceil n/2 \rceil} + (G + H) \times 10^{\lceil n/2 \rceil} + F$ 
10 devolva  $R$ 
```

$T(n)$ = consumo de tempo do algoritmo para multiplicar dois inteiros com n algarismos.

Consumo de tempo

linha	todas as execuções da linha
1	$= \Theta(1)$
2	$= \Theta(1)$
3	$= \Theta(n)$
4	$= \Theta(n)$
5	$= T(\lfloor n/2 \rfloor)$
6	$= T(\lceil n/2 \rceil)$
7	$= T(\lceil n/2 \rceil)$
8	$= T(\lceil n/2 \rceil)$
9	$= \Theta(n)$
10	$= \Theta(n)$
total	$= T(\lfloor n/2 \rfloor) + 3 T(\lceil n/2 \rceil) + \Theta(n)$

Consumo de tempo

Sabemos que

$$T(1) = \Theta(1)$$

$$T(n) = T(\lfloor n/2 \rfloor) + 3 T(\lceil n/2 \rceil) + \Theta(n) \quad \text{para } n = 2, 3, 4, \dots$$

está na **mesma classe Θ** que a solução de

$$T'(n) = 4T'(n/2) + n$$

n	1	2	4	8	16	32	64	128	256	512
$T'(n)$	1	6	28	120	496	2016	8128	32640	130816	523776

Conclusões

$$T'(n) \text{ é } \Theta(n^2).$$

$$T(n) \text{ é } \Theta(n^2).$$

O consumo de tempo do algoritmo **MULT** é $\Theta(n^2)$.

Tanto trabalho por nada ...

Será?!?

Pensar pequeno

Olhar para números com 2 algarismos ($n=2$).

Suponha $X = a b$ e $Y = c d$.

Se cada multiplicação custa R\$ 1,00 e
cada soma custa R\$ 0,01, quanto custa $X \cdot Y$?

$X \cdot Y$ por apenas R\$ 3,06

X		a	b
Y		c	d
		ad	bd
		ac	bc
$X \cdot Y$		$ad + bc$	

$X \cdot Y$ por apenas R\$ 3,06

X	a	b
Y	c	d
	ad	bd
	ac	bc
$X \cdot Y$	ac	$ad + bc$
		bd

$$(a + b)(c + d) = ac + ad + bc + bd \Rightarrow$$

$$ad + bc = (a + b)(c + d) - ac - bd$$

$$g = (a + b)(c + d) \quad e = ac \quad f = bd \quad h = g - e - f$$

$$X \cdot Y \text{ (por R\$ 3,06)} = e \times 10^2 + h \times 10^1 + f$$

Exemplo

$$\begin{array}{lll} X = 2133 & Y = 2312 & X \cdot Y = ? \\ \textcolor{red}{a}c = ? & \textcolor{blue}{b}d = ? & (a + b)(c + d) = ? \end{array}$$

Exemplo

$$\begin{array}{lll} X = 2133 & Y = 2312 & X \cdot Y = ? \\ \textcolor{red}{a}c = ? & \textcolor{blue}{b}d = ? & (a + b)(c + d) = ? \end{array}$$

$$\begin{array}{lll} X = 21 & Y = 23 & X \cdot Y = ? \\ \textcolor{red}{a}c = ? & \textcolor{blue}{b}d = ? & (a + b)(c + d) = ? \end{array}$$

Exemplo

$$\begin{array}{llll} X = & 2133 & Y = & 2312 \\ \textcolor{red}{a}c = & ? & \textcolor{blue}{b}d = & ? \end{array} \quad X \cdot Y = ? \quad (a+b)(c+d) = ?$$

$$\begin{array}{llll} X = & 21 & Y = & 23 \\ \textcolor{red}{a}c = & ? & \textcolor{blue}{b}d = & ? \end{array} \quad X \cdot Y = ? \quad (a+b)(c+d) = ?$$

$$X = 2 \quad Y = 2 \quad X \cdot Y = 4$$

Exemplo

$$\begin{array}{lll} X = 2133 & Y = 2312 & X \cdot Y = ? \\ \textcolor{red}{a}c = ? & \textcolor{blue}{b}d = ? & (a + b)(c + d) = ? \end{array}$$

$$\begin{array}{lll} X = 21 & Y = 23 & X \cdot Y = ? \\ \textcolor{red}{a}c = 4 & \textcolor{blue}{b}d = ? & (a + b)(c + d) = ? \end{array}$$

Exemplo

$$\begin{array}{lll} X = 2133 & Y = 2312 & X \cdot Y = ? \\ \text{ac} = ? & \text{bd} = ? & (a+b)(c+d) = ? \end{array}$$

$$\begin{array}{lll} X = 21 & Y = 23 & X \cdot Y = ? \\ \text{ac} = 4 & \text{bd} = ? & (a+b)(c+d) = ? \end{array}$$

$$X = 1 \quad Y = 3 \quad X \cdot Y = 3$$

Exemplo

$$\begin{array}{lll} X = 2133 & Y = 2312 & X \cdot Y = ? \\ \textcolor{red}{a}c = ? & \textcolor{blue}{b}d = ? & (a + b)(c + d) = ? \end{array}$$

$$\begin{array}{lll} X = 21 & Y = 23 & X \cdot Y = ? \\ \textcolor{red}{a}c = 4 & \textcolor{blue}{b}d = 3 & (a + b)(c + d) = ? \end{array}$$

Exemplo

$$\begin{array}{lll} X = 2133 & Y = 2312 & X \cdot Y = ? \\ \textcolor{red}{a}c = ? & \textcolor{blue}{b}d = ? & (a + b)(c + d) = ? \end{array}$$

$$\begin{array}{lll} X = 21 & Y = 23 & X \cdot Y = ? \\ \textcolor{red}{a}c = 4 & \textcolor{blue}{b}d = 3 & (a + b)(c + d) = ? \end{array}$$

$$X = 3 \quad Y = 5 \quad X \cdot Y = 15$$

Exemplo

$$\begin{array}{lll} X = 2133 & Y = 2312 & X \cdot Y = ? \\ ac = ? & bd = ? & (a+b)(c+d) = ? \end{array}$$

$$\begin{array}{lll} X = 21 & Y = 23 & X \cdot Y = 483 \\ ac = 4 & bd = 3 & (a+b)(c+d) = 15 \end{array}$$

Exemplo

$$\begin{array}{llll} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ \textcolor{red}{a}c = & 483 & \textcolor{blue}{b}d = & ? & (a + b)(c + d) = & ? \end{array}$$

Exemplo

$$\begin{array}{llll} X = & 2133 & Y = & 2312 \\ \textcolor{red}{ac} = & 483 & \textcolor{blue}{bd} = & ? \end{array} \quad \begin{array}{ll} X \cdot Y = & ? \\ (a + b)(c + d) = & ? \end{array}$$

$$\begin{array}{llll} X = & 33 & Y = & 12 \\ \textcolor{red}{ac} = & ? & \textcolor{blue}{bd} = & ? \end{array} \quad \begin{array}{ll} X \cdot Y = & ? \\ (a + b)(c + d) = & ? \end{array}$$

Exemplo

$$\begin{array}{llll} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ \textcolor{red}{ac} = & 483 & \textcolor{blue}{bd} = & ? & (a + b)(c + d) = & ? \end{array}$$

$$\begin{array}{llll} X = & 33 & Y = & 12 & X \cdot Y = & 396 \\ \textcolor{red}{ac} = & 3 & \textcolor{blue}{bd} = & 6 & (a + b)(c + d) = & 18 \end{array}$$

Exemplo

$$\begin{array}{llll} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ \textcolor{red}{ac} = & 483 & \textcolor{blue}{bd} = & 396 & (a + b)(c + d) = & ? \end{array}$$

Exemplo

$$\begin{array}{llll} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ \textcolor{red}{ac} = & 483 & \textcolor{blue}{bd} = & 396 & (a + b)(c + d) = & ? \end{array}$$

$$\begin{array}{llll} X = & 54 & Y = & 35 & X \cdot Y = & ? \\ \textcolor{red}{ac} = & ? & \textcolor{blue}{bd} = & ? & (a + b)(c + d) = & ? \end{array}$$

Exemplo

$$\begin{array}{llll} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ \textcolor{red}{ac} = & 483 & \textcolor{blue}{bd} = & 396 & (a + b)(c + d) = & ? \end{array}$$

$$\begin{array}{llll} X = & 54 & Y = & 35 & X \cdot Y = & 1890 \\ \textcolor{red}{ac} = & 15 & \textcolor{blue}{bd} = & 20 & (a + b)(c + d) = & 72 \end{array}$$

Exemplo

$$\begin{array}{llll} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ \textcolor{red}{a}c = & 483 & \textcolor{blue}{b}d = & 396 & (a + b)(c + d) = & 1890 \end{array}$$

Exemplo

$$\begin{array}{llll} X = & 2133 & Y = & 2312 \\ \textcolor{red}{a}c = & 483 & \textcolor{blue}{b}d = & 396 \end{array} \quad X \cdot Y = 4931496 \quad (a+b)(c+d) = 1890$$

Algoritmo de Karatsuba e Ofman

Algoritmo recebe inteiros $X[1..n]$ e $Y[1..n]$ e devolve $X \cdot Y$.

KARATSUBA (X, Y, n)

```
1  se  $n \leq 3$  devolva  $X \cdot Y$ 
2   $q \leftarrow \lceil n/2 \rceil$ 
3   $A \leftarrow X[q+1..n]$      $B \leftarrow X[1..q]$ 
4   $C \leftarrow Y[q+1..n]$      $D \leftarrow Y[1..q]$ 
5   $E \leftarrow \text{KARATSUBA}(A, C, \lfloor n/2 \rfloor)$ 
6   $F \leftarrow \text{KARATSUBA}(B, D, \lceil n/2 \rceil)$ 
7   $G \leftarrow \text{KARATSUBA}(A+B, C+D, \lceil n/2 \rceil + 1)$ 
8   $H \leftarrow G - F - E$ 
9   $R \leftarrow E \times 10^{2\lceil n/2 \rceil} + H \times 10^{\lceil n/2 \rceil} + F$ 
10 devolva  $R$ 
```

$T(n)$ = consumo de tempo do algoritmo para multiplicar dois inteiros com n algarismos.

Consumo de tempo

linha	todas as execuções da linha
1	$= \Theta(1)$
2	$= \Theta(1)$
3	$= \Theta(n)$
4	$= \Theta(n)$
5	$= T(\lfloor n/2 \rfloor)$
6	$= T(\lceil n/2 \rceil)$
7	$= T(\lceil n/2 \rceil + 1) + \Theta(n)$
8	$= \Theta(n)$
9	$= \Theta(n)$
10	$= \Theta(n)$
total	$= T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(\lceil n/2 \rceil + 1) + \Theta(n)$

Consumo de tempo

Sabemos que

$$T(n) = \Theta(1) \text{ para } n = 1, 2, 3$$

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(\lceil n/2 \rceil + 1) + \Theta(n) \quad n \geq 4$$

está na mesma classe Θ que a solução de

$$T'(n) = 3T'(n/2) + n$$

n	1	2	4	8	16	32	64	128	256	512
$T'(n)$	1	5	19	65	211	665	2059	6305	19171	58025

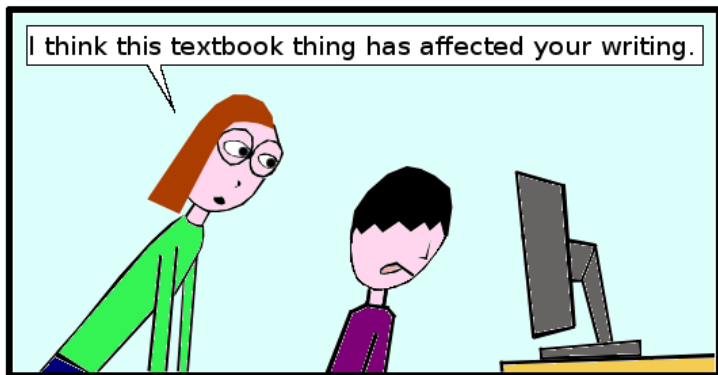
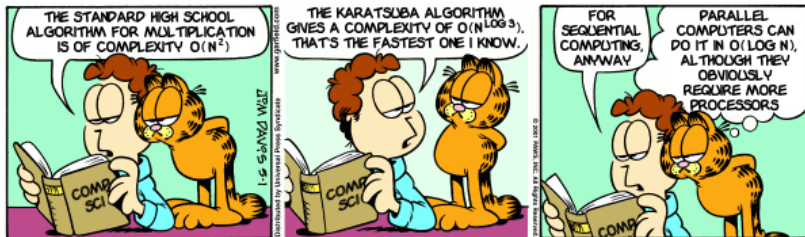
Conclusões

$$T'(n) \text{ é } \Theta(n^{\lg 3}).$$

$$\text{Logo } T(n) \text{ é } \Theta(n^{\lg 3}).$$

O consumo de tempo do algoritmo **KARATSUBA** é $\Theta(n^{\lg 3})$
($1,584 < \lg 3 < 1,585$).

Karatsuba em tirinhas



Mais conclusões

Consumo de tempo de
algoritmos para multiplicação de inteiros:

Jardim de infância

$$\Theta(n 10^n)$$

Ensino fundamental

$$\Theta(n^2)$$

Karatsuba e Ofman'60

$$O(n^{1.585})$$

Toom e Cook'63

$$O(n^{1.465})$$

(divisão e conquista; generaliza o acima)

Schönhage e Strassen'71

$$O(n \lg n \lg \lg n)$$

(FFT em anéis de tamanho específico)

Fürer'07

$$O(n \lg n 2^{O(\log^* n)})$$

Harvey e van der Hoeven'19

$$O(n \lg n)$$

Ambiente experimental

A **plataforma utilizada** nos experimentos é um PC rodando Linux Debian ?? com um processador Pentium II de 233 MHz e 128MB de memória RAM .

Os **códigos estão compilados** com o gcc versão 2.7.2.1 e opção de compilação -O2.

As implementações comparadas neste experimento são as do algoritmo do ensino fundamental e do algoritmo **KARATSUBA**.

O programa foi escrito por Carl Burch:

<http://www-2.cs.cmu.edu/~cburch/251/karat/>.

Resultados experimentais

n	Ensino Fund.	KARATSUBA
4	0.005662	0.005815
8	0.010141	0.010600
16	0.020406	0.023643
32	0.051744	0.060335
64	0.155788	0.165563
128	0.532198	0.470810
256	1.941748	1.369863
512	7.352941	4.032258

Tempos em 10^3 segundos.

Multiplicação de matrizes

Problema: Dadas duas matrizes $X[1..n, 1..n]$ e $Y[1..n, 1..n]$ calcular o **produto** $X \cdot Y$.

O algoritmo tradicional de multiplicação de matrizes consome tempo $\Theta(n^3)$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

$$r = ae + bg$$

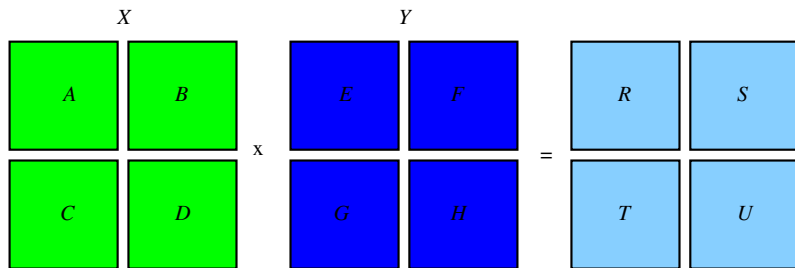
$$s = af + bh$$

$$t = ce + dg$$

$$u = cf + dh \tag{1}$$

Solução custa R\$ 8,04

Divisão e conquista



$$R = AE + BG$$

$$S = AF + BH$$

$$T = CE + DG$$

$$U = CF + DH$$

(2)

Algoritmo de multiplicação de matrizes

Algoritmo recebe inteiros $X[1..n]$ e $Y[1..n]$ e devolve $X \cdot Y$.

MULTI-M (X, Y, n)

- 1 **se** $n = 1$ **devolva** $X \cdot Y$
- 2 $(A, B, C, D) \leftarrow \text{PARTICIONE}(X, n)$
- 3 $(E, F, G, H) \leftarrow \text{PARTICIONE}(Y, n)$
- 4 $R \leftarrow \text{MULTI-M}(A, E, n/2) + \text{MULTI-M}(B, G, n/2)$
- 5 $S \leftarrow \text{MULTI-M}(A, F, n/2) + \text{MULTI-M}(B, H, n/2)$
- 6 $T \leftarrow \text{MULTI-M}(C, E, n/2) + \text{MULTI-M}(D, G, n/2)$
- 7 $U \leftarrow \text{MULTI-M}(C, F, n/2) + \text{MULTI-M}(D, H, n/2)$
- 8 $P \leftarrow \text{CONSTRÓI-MAT}(R, S, T, U)$
- 9 **devolva** P

$T(n)$ = consumo de tempo do algoritmo para
multiplicar duas matrizes de n linhas e n colunas.

Consumo de tempo

linha	todas as execuções da linha
1	$= \Theta(1)$
2	$= \Theta(n^2)$
3	$= \Theta(n^2)$
4	$= T(n/2) + T(n/2)$
5	$= T(n/2) + T(n/2)$
6	$= T(n/2) + T(n/2)$
7	$= T(n/2) + T(n/2)$
8	$= \Theta(n^2)$
9	$= \Theta(n^2)$
total	$= 8 T(n/2) + \Theta(n^2)$

Consumo de tempo

As dicas no nosso estudo de recorrências sugere que a solução da recorrência

$$\begin{aligned}T(1) &= \Theta(1) \\T(n) &= 8 T(n/2) + \Theta(n^2) \quad \text{para } n = 2, 3, 4, \dots\end{aligned}$$

está na **mesma classe Θ** que a solução de

$$T'(n) = 8T'(n/2) + n^2$$

n	1	2	4	8	16	32	64	128	256
$T'(n)$	1	12	112	960	7936	64512	520192	4177920	33488896

Conclusões

$T'(n)$ é $\Theta(n^3)$.

Logo $T(n)$ é $\Theta(n^3)$.

O consumo de tempo do algoritmo **MULTI-M** é $\Theta(n^3)$.

Strassen: $X \cdot Y$ por apenas R\$ 7,18

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

Strassen: $X \cdot Y$ por apenas R\$ 7,18

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

$$p_1 = a(f - h) = af - ah$$

$$p_2 = (a + b)h = ah + bh$$

$$p_3 = (c + d)e = ce + de$$

$$p_4 = d(g - e) = dg - de$$

$$p_5 = (a + d)(e + h) = ae + ah + de + dh$$

$$p_6 = (b - d)(g + h) = bg + bh - dg - dh$$

$$p_7 = (a - c)(e + f) = ae + af - ce - cf$$

Strassen: $X \cdot Y$ por apenas R\$ 7,18

$$p_1 = a(f - h) = af - ah$$

$$p_2 = (a + b)h = ah + bh$$

$$p_3 = (c + d)e = ce + de$$

$$p_4 = d(g - e) = dg - de$$

$$p_5 = (a + d)(e + h) = ae + ah + de + dh$$

$$p_6 = (b - d)(g + h) = bg + bh - dg - dh$$

$$p_7 = (a - c)(e + f) = ae + af - ce - cf$$

$$r = p_5 + p_4 - p_2 + p_6 = ae + bg$$

$$s = p_1 + p_2 = af + bh$$

$$t = p_3 + p_4 = ce + dg$$

$$u = p_5 + p_1 - p_3 - p_7 = cf + dh$$

Algoritmo de Strassen

STRASSEN (X, Y, n)

- 1 se $n = 1$ devolva $X \cdot Y$
- 2 $(A, B, C, D) \leftarrow \text{PARTICIONE}(X, n)$
- 3 $(E, F, G, H) \leftarrow \text{PARTICIONE}(Y, n)$
- 4 $P_1 \leftarrow \text{STRASSEN}(A, F - H, n/2)$
- 5 $P_2 \leftarrow \text{STRASSEN}(A + B, H, n/2)$
- 6 $P_3 \leftarrow \text{STRASSEN}(C + D, E, n/2)$
- 7 $P_4 \leftarrow \text{STRASSEN}(D, G - E, n/2)$
- 8 $P_5 \leftarrow \text{STRASSEN}(A + D, E + H, n/2)$
- 9 $P_6 \leftarrow \text{STRASSEN}(B - D, G + H, n/2)$
- 10 $P_7 \leftarrow \text{STRASSEN}(A - C, E + F, n/2)$
- 11 $R \leftarrow P_5 + P_4 - P_2 + P_6$
- 12 $S \leftarrow P_1 + P_2$
- 13 $T \leftarrow P_3 + P_4$
- 14 $U \leftarrow P_5 + P_1 - P_3 - P_7$
- 15 devolva $P \leftarrow \text{CONSTRÓI-MAT}(R, S, T, U)$

Consumo de tempo

linha	todas as execuções da linha
1	$= \Theta(1)$
2-3	$= \Theta(n^2)$
4-10	$= 7 T(n/2) + \Theta(n^2)$
11-14	$= \Theta(n^2)$
15	$= \Theta(n^2)$
total	$= 7 T(n/2) + \Theta(n^2)$

Consumo de tempo

As dicas no nosso estudo de recorrências sugerem que a solução da recorrência

$$\begin{aligned}T(1) &= \Theta(1) \\T(n) &= 7T(n/2) + \Theta(n^2) \quad \text{para } n = 2, 3, 4, \dots\end{aligned}$$

está na **mesma classe Θ** que a solução de

$$T'(n) = 7T'(n/2) + n^2$$

n	1	2	4	8	16	32	64	128	256
$T'(n)$	1	11	93	715	5261	37851	269053	1899755	13363821

Conclusões

$$T'(n) \text{ é } \Theta(n^{\lg 7}).$$

$$\text{Logo } T(n) \text{ é } \Theta(n^{\lg 7}).$$

O consumo de tempo do algoritmo **STRASSEN** é $\Theta(n^{\lg 7})$
($2,80 < \lg 7 < 2,81$).

Mais conclusões

Consumo de tempo de algoritmos para multiplicação de matrizes:

Ensino fundamental	$\Theta(n^3)$
Strassen (1969)	$O(n^{2.81})$
$\vdots$	$\vdots$
Coppersmith e Winograd (1987)	$O(n^{2.3755})$
Stothers (2010)	$O(n^{2.3736})$
Williams (2011)	$O(n^{2.3727})$