Tópicos de Análise de Algoritmos

Parte destes slides são adaptações de slides

do Prof. Paulo Feofiloff e do Prof. José Coelho de Pina.

Aula 3

Multiplicação de inteiros grandes

Secs 5.5 do KT e 28.2 do CLRS.

Multiplicação de inteiros gigantescos

$n :=$ número de algarismos.

Problema: Dados dois números inteiros $X[1\,..\,n]$ e $Y[1\,..\,n]$, calcular o produto $X \cdot Y$.

Entra: Exemplo com $n = 12$

	12											1
X	9	2	3	4	5	5	4	5	6	2	9	8
Y	0	6	3	2	8	4	9	9	3	8	4	4

Multiplicação de inteiros gigantescos

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Entra: Exemplo com $n = 12$

	12											1
X	9	2	3	4	5	5	4	5	6	2	9	8
Y	0	6	3	2	8	4	9	9	3	8	4	4

Sai:

23											$X \cdot Y$											1
5	8	4	4	0	8	7	2	8	6	7	0	2	7	1	4	1	0	2	9	5	1	2

Algoritmo do ensino fundamental

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&& * & * & * & * & * & * & * & * \\
&&&&&&&& \times & * & * & * & * & * & * & * & * \\
\cline{9-17}
&&&&&&&& * & * & * & * & * & * & * & * & * \\
&&&&&&& * & * & * & * & * & * & * & * & * & \\
&&&&&& * & * & * & * & * & * & * & * & * && \\
&&&&& * & * & * & * & * & * & * & * & * &&& \\
&&&& * & * & * & * & * & * & * & * & * &&&& \\
&&& * & * & * & * & * & * & * & * & * &&&&& \\
&& * & * & * & * & * & * & * & * & * &&&&&& \\
& * & * & * & * & * & * & * & * & * &&&&&&& \\
\hline
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & *
\end{array}$$

O algoritmo do ensino fundamental é $\Theta(n^2)$.

Da Wiki

In 1960, Kolmogorov organized a seminar on mathematical problems in cybernetics at the Moscow State University, where, among other problems in the complexity of computation, he stated the $\Omega(n^2)$ conjecture, that stated that the traditional algorithm for multiplication of two n-digit numbers was asymptotically optimal, meaning that any algorithm for that task would require $\Omega(n^2)$ elementary operations.

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Kolmogorov was very excited about the discovery; he communicated it at the next meeting of the seminar, which was then terminated. Kolmogorov gave some lectures on the Karatsuba result at conferences all over the world (see, for example, *Proceedings of the International Congress of Mathematicians 1962*, pp. 351–356, and also *6 Lectures delivered at the International Congress of Mathematicians in Stockholm, 1962*) and published the method in 1962, in the *Proceedings of the USSR Academy of Sciences*.

Da Wiki

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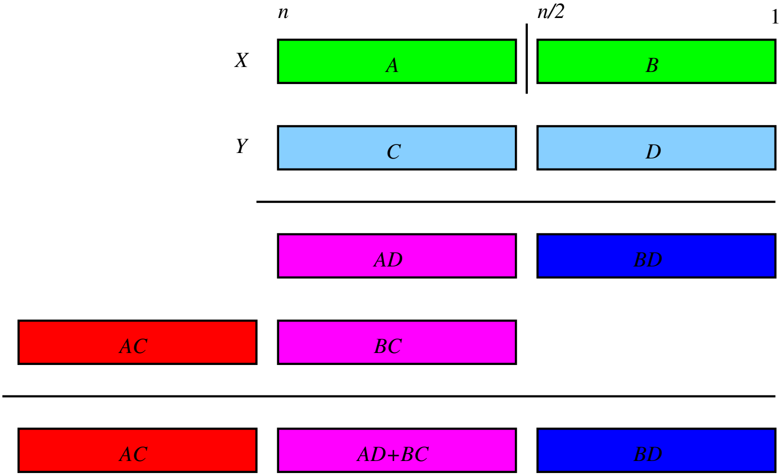
The article had been written by Kolmogorov and contained two results on multiplication, Karatsuba's algorithm and a separate result by Yuri Ofman; it listed "A. Karatsuba and Yu. Ofman" as the authors. Karatsuba only became aware of the paper when he received the reprints from the publisher.

Divisão e conquista



$$X \cdot Y = A \cdot C \times 10^{2\lceil n/2 \rceil} + (A \cdot D + B \cdot C) \times 10^{\lceil n/2 \rceil} + B \cdot D$$

Exemplo

	4			1
X	3	1	4	1

	4			1
Y	5	9	3	6

Exemplo

	4			1
X	3	1	4	1

	4			1
Y	5	9	3	6

A	3	1

B	4	1

C	5	9

D	3	6

Exemplo

$$\begin{array}{c|c|c|c|c|c}
 & 4 & & & 1 \\
X & 3 & 1 & 4 & 1
\end{array}
\qquad
\begin{array}{c|c|c|c|c|c}
 & 4 & & & 1 \\
Y & 5 & 9 & 3 & 6
\end{array}$$

$$A\ \boxed{3\,|\,1} \qquad B\ \boxed{4\,|\,1} \qquad C\ \boxed{5\,|\,9} \qquad D\ \boxed{3\,|\,6}$$

$$X \cdot Y = A \cdot C \times 10^4 + (A \cdot D + B \cdot C) \times 10^2 + B \cdot D$$

$A \cdot C = 1829$ $\qquad (A \cdot D + B \cdot C) = 1116 + 2419 = 3535$

$B \cdot D = 1476$

$$\begin{array}{lcccccccc}
A \cdot C & 1 & 8 & 2 & 9 & 0 & 0 & 0 & 0 \\
(A \cdot D + B \cdot C) & & & 3 & 5 & 3 & 5 & 0 & 0 \\
B \cdot D & & & & & 1 & 4 & 7 & 6 \\
\hline
X \cdot Y = & 1 & 8 & 6 & 4 & 4 & 9 & 7 & 6
\end{array}$$

Algoritmo de Multi-DC

Algoritmo recebe inteiros $X[1\,..\,n]$ e $Y[1\,..\,n]$ e devolve $X \cdot Y$.

```
MULT $(X, Y, n)$
    se $n = 1$ devolva $X \cdot Y$
    $q \leftarrow \lceil n/2 \rceil$
    $A \leftarrow X[q+1\,..\,n]$        $B \leftarrow X[1\,..\,q]$
    $C \leftarrow Y[q+1\,..\,n]$        $D \leftarrow Y[1\,..\,q]$
    $E \leftarrow$ MULT$(A, C, \lfloor n/2 \rfloor)$
    $F \leftarrow$ MULT$(B, D, \lceil n/2 \rceil)$
    $G \leftarrow$ MULT$(A, D, \lceil n/2 \rceil)$
    $H \leftarrow$ MULT$(B, C, \lceil n/2 \rceil)$
    $R \leftarrow E \times 10^{2\lceil n/2 \rceil} + (G + H) \times 10^{\lceil n/2 \rceil} + F$
    devolva $R$
```

$T(n)$ = consumo de tempo do algoritmo para multiplicar dois inteiros com n algarismos.

Consumo de tempo

linha	todas as execuções da linha	
1	$=$	$\Theta(1)$
2	$=$	$\Theta(1)$
3	$=$	$\Theta(n)$
4	$=$	$\Theta(n)$
5	$=$	$T(\lfloor n/2 \rfloor)$
6	$=$	$T(\lceil n/2 \rceil)$
7	$=$	$T(\lceil n/2 \rceil)$
8	$=$	$T(\lceil n/2 \rceil)$
9	$=$	$\Theta(n)$
10	$=$	$\Theta(n)$
total	$=$	$T(\lfloor n/2 \rfloor) + 3\,T(\lceil n/2 \rceil) + \Theta(n)$

Consumo de tempo

Sabemos que

$$\begin{array}{rcl} T(1) & = & \Theta(1) \\ T(n) & = & T(\lfloor n/2 \rfloor) + 3\,T(\lceil n/2 \rceil) + \Theta(n) \quad \text{para } n = 2, 3, 4, \ldots \end{array}$$

está na mesma classe Θ que a solução de

$$T'(n) \;=\; 4\,T'(n/2) + n$$

n	1	2	4	8	16	32	64	128	256	512
$T'(n)$	1	6	28	120	496	2016	8128	32640	130816	523776

Conclusões

$T'(n)$ é $\Theta(n^2)$.

$T(n)$ é $\Theta(n^2)$.

O consumo de tempo do algoritmo MULT é $\Theta(n^2)$.

Tanto trabalho por nada . . .

Será?!?

Pensar pequeno

Olhar para números com 2 algarismos (n=2).

Suponha $X = a\,b$ e $Y = c\,d$.
Se cada multiplicação custa R$ 1,00 e
cada soma custa R$ 0,01, quanto custa $X \cdot Y$?

Pensar pequeno

Olhar para números com 2 algarismos (n=2).

Suponha $X = a\,b$ e $Y = c\,d$.
Se cada multiplicação custa R$ 1,00 e
cada soma custa R$ 0,01, quanto custa $X \cdot Y$?

Eis $X \cdot Y$ por R$ 4,03:

$$\begin{array}{cccc} X & & a & b \\ Y & & c & d \\ \hline & & ad & bd \\ & ac & bc & \\ \hline X \cdot Y & ac & ad + bc & bd \end{array}$$

$$X \cdot Y = ac \times 10^2 + (ad + bc) \times 10^1 + bd$$

Pensar pequeno

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$$X \cdot Y = ac \times 10^2 + (ad + bc) \times 10^1 + bd$$

Solução mais barata?

Pensar pequeno

Olhar para números com 2 algarismos (n=2).

Suponha $X = a\,b$ e $Y = c\,d$.
Se cada multiplicação custa R$ 1,00 e
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Eis $X \cdot Y$ por R$ 4,03:

$$\begin{array}{lccc}
X & & a & b \\
Y & & c & d \\
\hline
 & & ad & bd \\
 & ac & bc & \\
\hline
X \cdot Y & ac & ad + bc & bd
\end{array}$$

$$X \cdot Y = ac \times 10^2 + (ad + bc) \times 10^1 + bd$$

Solução mais barata?
Gauss faz por R$ 3,06!

$X \cdot Y$ por apenas R$ 3,06

$$
\begin{array}{lccc}
X & & a & b \\
Y & & c & d \\
\hline
 & & ad & bd \\
 & ac & bc & \\
\hline
X \cdot Y & ac & ad + bc & bd
\end{array}
$$

$X \cdot Y$ por apenas R$ 3,06

$$\begin{array}{cccc}
X & & a & b \\
Y & & c & d \\
\hline
 & & ad & bd \\
 & ac & bc & \\
\hline
X \cdot Y & ac & ad + bc & bd
\end{array}$$

$$(a+b)(c+d) = ac + ad + bc + bd \Rightarrow$$
$$ad + bc = (a+b)(c+d) - ac - bd$$

$$g = (a+b)(c+d) \qquad e = ac \qquad f = bd \qquad h = g - e - f$$

$$\boxed{X \cdot Y \ \ \text{(por R\$ 3,06)} = e \times 10^2 + h \times 10^1 + f}$$

Exemplo

$$\begin{array}{rlrlrl} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & ? & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

Exemplo

$$\begin{array}{rlrlrl} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & ? & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{rlrlrl} X = & 21 & Y = & 23 & X \cdot Y = & ? \\ ac = & ? & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

Exemplo

$$\begin{aligned} X &= 2133 & Y &= 2312 & X \cdot Y &= ? \\ ac &= ? & bd &= ? & (a+b)(c+d) &= ? \end{aligned}$$

$$\begin{aligned} X &= 21 & Y &= 23 & X \cdot Y &= ? \\ ac &= ? & bd &= ? & (a+b)(c+d) &= ? \end{aligned}$$

$$\begin{aligned} X &= 2 & Y &= 2 & X \cdot Y &= 4 \end{aligned}$$

Exemplo

$$\begin{array}{llllrl} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & ? & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{llllrl} X = & 21 & Y = & 23 & X \cdot Y = & ? \\ ac = & 4 & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

Exemplo

$$\begin{array}{rlrlrl} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & ? & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{rlrlrl} X = & 21 & Y = & 23 & X \cdot Y = & ? \\ ac = & 4 & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{rlrlrl} X = & 1 & Y = & 3 & X \cdot Y = & 3 \end{array}$$

Exemplo

$$\begin{aligned} X &= 2133 & Y &= 2312 & X \cdot Y &= ? \\ ac &= ? & bd &= ? & (a+b)(c+d) &= ? \end{aligned}$$

$$\begin{aligned} X &= 21 & Y &= 23 & X \cdot Y &= ? \\ ac &= 4 & bd &= 3 & (a+b)(c+d) &= ? \end{aligned}$$

Exemplo

$$\begin{array}{rlrlrl} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & ? & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{rlrlrl} X = & 21 & Y = & 23 & X \cdot Y = & ? \\ ac = & 4 & bd = & 3 & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{rlrlrl} X = & 3 & Y = & 5 & X \cdot Y = & 15 \end{array}$$

Exemplo

$$\begin{array}{rlrlrl} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & ? & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{rlrlrl} X = & 21 & Y = & 23 & X \cdot Y = & 483 \\ ac = & 4 & bd = & 3 & (a+b)(c+d) = & 15 \end{array}$$

Exemplo

$$\begin{array}{rlrlrl} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & 483 & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

Exemplo

$$\begin{array}{rlrlrl} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & 483 & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{rlrlrl} X = & 33 & Y = & 12 & X \cdot Y = & ? \\ ac = & ? & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

Exemplo

$$\begin{aligned} X &= 2133 & Y &= 2312 & X \cdot Y &= ? \\ ac &= 483 & bd &= ? & (a+b)(c+d) &= ? \end{aligned}$$

$$\begin{aligned} X &= 33 & Y &= 12 & X \cdot Y &= 396 \\ ac &= 3 & bd &= 6 & (a+b)(c+d) &= 18 \end{aligned}$$

Exemplo

$$\begin{array}{llllrl} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ {\color{red}ac} = & 483 & {\color{blue}bd} = & 396 & (a+b)(c+d) = & ? \end{array}$$

Exemplo

$$\begin{array}{rlrlrl} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & 483 & bd = & 396 & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{rlrlrl} X = & 54 & Y = & 35 & X \cdot Y = & ? \\ ac = & ? & bd = & ? & (a+b)(c+d) = & ? \end{array}$$

Exemplo

$$\begin{array}{llllrl} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ \color{red}{ac} = & 483 & \color{blue}{bd} = & 396 & (a+b)(c+d) = & ? \end{array}$$

$$\begin{array}{llllrl} X = & \color{olive}{5}4 & Y = & \color{olive}{3}5 & X \cdot Y = & 1890 \\ \color{red}{ac} = & 15 & \color{blue}{bd} = & 20 & (a+b)(c+d) = & 72 \end{array}$$

Exemplo

$$\boxed{\begin{array}{rlrlrl} X = & 2133 & Y = & 2312 & X \cdot Y = & ? \\ ac = & 483 & bd = & 396 & (a+b)(c+d) = & 1890 \end{array}}$$

Exemplo

$$
\begin{array}{rlrlrl}
X = & 2133 & Y = & 2312 & X \cdot Y = & 4931496 \\
ac = & 483 & bd = & 396 & (a+b)(c+d) = & 1890
\end{array}
$$

Algoritmo de Karatsuba e Ofman

Algoritmo recebe inteiros $X[1\,.\,.\,n]$ e $Y[1\,.\,.\,n]$ e devolve $X \cdot Y$.

```
KARATSUBA $(X, Y, n)$
    se $n \leq 3$ devolva $X \cdot Y$
    $q \leftarrow \lceil n/2 \rceil$
    $A \leftarrow X[q+1\,.\,.\,n]$        $B \leftarrow X[1\,.\,.\,q]$
    $C \leftarrow Y[q+1\,.\,.\,n]$        $D \leftarrow Y[1\,.\,.\,q]$
    $E \leftarrow$ KARATSUBA$(A, C, \lfloor n/2 \rfloor)$
    $F \leftarrow$ KARATSUBA$(B, D, \lceil n/2 \rceil)$
    $G \leftarrow$ KARATSUBA$(A+B, C+D, \lceil n/2 \rceil + 1)$
    $H \leftarrow G - F - E$
    $R \leftarrow E \times 10^{2\lceil n/2 \rceil} + H \times 10^{\lceil n/2 \rceil} + F$
    devolva $R$
```

$T(n)$ = consumo de tempo do algoritmo para multiplicar dois inteiros com n algarismos.

Consumo de tempo

linha		todas as execuções da linha
1	$=$	$\Theta(1)$
2	$=$	$\Theta(1)$
3	$=$	$\Theta(n)$
4	$=$	$\Theta(n)$
5	$=$	$T(\lfloor n/2 \rfloor)$
6	$=$	$T(\lceil n/2 \rceil)$
7	$=$	$T(\lceil n/2 \rceil + 1) + \Theta(n)$
8	$=$	$\Theta(n)$
9	$=$	$\Theta(n)$
10	$=$	$\Theta(n)$
total	$=$	$T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(\lceil n/2 \rceil + 1) + \Theta(n)$

Consumo de tempo

Sabemos que

$$\begin{array}{rcl} T(n) & = & \Theta(1) \quad \text{para } n = 1, 2, 3 \\ T(n) & = & T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(\lceil n/2 \rceil + 1) + \Theta(n) \quad n \geq 4 \end{array}$$

está na mesma classe Θ que a solução de

$$T'(n) = 3T'(n/2) + n$$

n	1	2	4	8	16	32	64	128	256	512
$T'(n)$	1	5	19	65	211	665	2059	6305	19171	58025

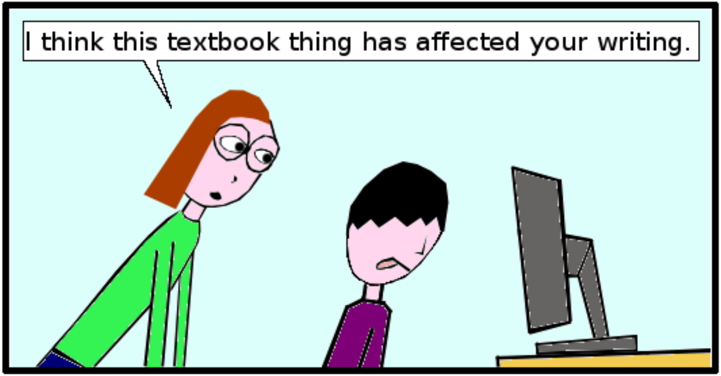
Conclusões

$T'(n)$ é $\Theta(n^{\lg 3})$.

Logo $T(n)$ é $\Theta(n^{\lg 3})$.

O consumo de tempo do algoritmo KARATSUBA é $\Theta(n^{\lg 3})$ $(1{,}584 < \lg 3 < 1{,}585)$.

Karatsuba em tirinhas



Mais conclusões

Consumo de tempo de
algoritmos para multiplicação de inteiros:

Jardim de infância	$\Theta(n\,10^n)$
Ensino fundamental	$\Theta(n^2)$
Karatsuba e Ofman'60	$\mathrm{O}(n^{1.585})$
Toom e Cook'63	$\mathrm{O}(n^{1.465})$
(divisão e conquista; generaliza o acima)	
Schönhage e Strassen'71	$\mathrm{O}(n \lg n \lg \lg n)$
(FFT em aneis de tamanho específico)	
Fürer'07	$\mathrm{O}(n \lg n\, 2^{\mathrm{O}(\log^* n)})$
Harvey e van der Hoeven'19	$\mathrm{O}(n \lg n)$

Ambiente experimental

A plataforma utilizada nos experimentos é um notebook rodando macOS Big Sur com um processador 1.3 GHz Dual-Core Intel Core i5 e 4GB de memória RAM 1600 MHz DDR3.

Os códigos estão compilados com o Apple clang version 11.0.0.

As implementações comparadas neste experimento são as do algoritmo do ensino fundamental e do algoritmo KARATSUBA.

O programa foi escrito por Carl Burch:

`http://www-2.cs.cmu.edu/~cburch/251/karat/.`

Resultados experimentais

n	Ensino Fund.	KARATSUBA
8	0.001156	0.001296
16	0.002211	0.002547
32	0.005947	0.006269
64	0.019024	0.017304
128	0.066732	0.051762
256	0.264311	0.157111
512	0.987317	0.455365
1024	4.162332	1.376201

Tempos em 10^{-3} segundos.

Multiplicação de matrizes

Problema: Dadas duas matrizes $X[1\,..\,n, 1\,..\,n]$ e $Y[1\,..\,n, 1\,..\,n]$ calcular o produto $X \cdot Y$.

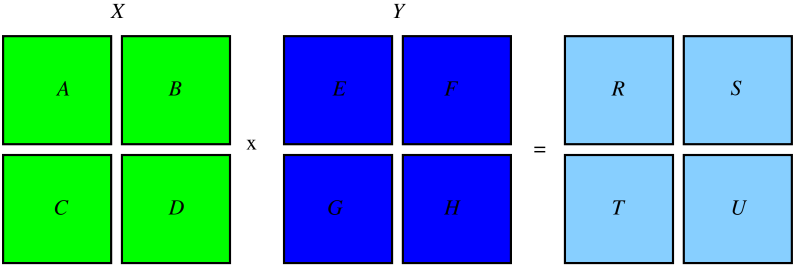
O algoritmo tradicional de multiplicação de matrizes consome tempo $\Theta(n^3)$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

$$\begin{aligned} r &= ae + bg \\ s &= af + bh \\ t &= ce + dg \\ u &= cf + dh \end{aligned} \tag{1}$$

Solução custa R$ 8,04

Divisão e conquista



$$\begin{aligned} R &= AE + BG \\ S &= AF + BH \\ T &= CE + DG \\ U &= CF + DH \end{aligned} \tag{2}$$

Algoritmo de multiplicação de matrizes

Algoritmo recebe inteiros $X[1\,..\,n]$ e $Y[1\,..\,n]$ e devolve $X \cdot Y$.

```
MULTI-M ($X$, $Y$, $n$)
    se $n = 1$ devolva $X \cdot Y$
    $(A, B, C, D) \leftarrow$ PARTICIONE($X$, $n$)
    $(E, F, G, H) \leftarrow$ PARTICIONE($Y$, $n$)
    $R \leftarrow$ MULTI-M($A, E, n/2$) $+$ MULTI-M($B, G, n/2$)
    $S \leftarrow$ MULTI-M($A, F, n/2$) $+$ MULTI-M($B, H, n/2$)
    $T \leftarrow$ MULTI-M($C, E, n/2$) $+$ MULTI-M($D, G, n/2$)
    $U \leftarrow$ MULTI-M($C, F, n/2$) $+$ MULTI-M($D, H, n/2$)
    $P \leftarrow$ CONSTRÓI-MAT($R, S, T, U$)
    devolva $P$
```

$T(n) =$ consumo de tempo do algoritmo para
multiplicar duas matrizes de n linhas e n colunas.

Consumo de tempo

linha	todas as execuções da linha	
1	$=$	$\Theta(1)$
2	$=$	$\Theta(n^2)$
3	$=$	$\Theta(n^2)$
4	$=$	$T(n/2) + T(n/2)$
5	$=$	$T(n/2) + T(n/2)$
6	$=$	$T(n/2) + T(n/2)$
7	$=$	$T(n/2) + T(n/2)$
8	$=$	$\Theta(n^2)$
9	$=$	$\Theta(n^2)$
total	$=$	$8\,T(n/2) + \Theta(n^2)$

Consumo de tempo

As dicas no nosso estudo de recorrências sugere que a solução da recorrência

$$\begin{array}{rcl} T(1) & = & \Theta(1) \\ T(n) & = & 8\,T(n/2) + \Theta(n^2) \quad \text{para } n = 2, 3, 4, \ldots \end{array}$$

está na mesma classe Θ que a solução de

$$T'(n) \; = \; 8\,T'(n/2) + n^2$$

n	1	2	4	8	16	32	64	128	256
$T'(n)$	1	12	112	960	7936	64512	520192	4177920	33488896

Conclusões

$T'(n)$ é $\Theta(n^3)$.

Logo $T(n)$ é $\Theta(n^3)$.

O consumo de tempo do algoritmo MULTI-M é $\Theta(n^3)$.

Strassen: $X \cdot Y$ por apenas R$ 7,18

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

Strassen: $X \cdot Y$ por apenas R$ 7,18

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

$$\begin{aligned}
p_1 &= a(f-h) = af - ah \\
p_2 &= (a+b)h = ah + bh \\
p_3 &= (c+d)e = ce + de \\
p_4 &= d(g-e) = dg - de \\
p_5 &= (a+d)(e+h) = ae + ah + de + dh \\
p_6 &= (b-d)(g+h) = bg + bh - dg - dh \\
p_7 &= (a-c)(e+f) = ae + af - ce - cf
\end{aligned}$$

Strassen: $X \cdot Y$ por apenas R$ 7,18

$$\begin{aligned}
p_1 &= a(f-h) = af - ah\\
p_2 &= (a+b)h = ah + bh\\
p_3 &= (c+d)e = ce + de\\
p_4 &= d(g-e) = dg - de\\
p_5 &= (a+d)(e+h) = ae + ah + de + dh\\
p_6 &= (b-d)(g+h) = bg + bh - dg - dh\\
p_7 &= (a-c)(e+f) = ae + af - ce - cfd
\end{aligned}$$

$$\begin{aligned}
r &= p_5 + p_4 - p_2 + p_6 = ae + bg\\
s &= p_1 + p_2 = af + bh\\
t &= p_3 + p_4 = ce + dg\\
u &= p_5 + p_1 - p_3 - p_7 = cf + dh
\end{aligned}$$

Algoritmo de Strassen

```
STRASSEN $(X, Y, n)$
    se $n = 1$ devolva $X \cdot Y$
    $(A, B, C, D) \leftarrow$ PARTICIONE$(X, n)$
    $(E, F, G, H) \leftarrow$ PARTICIONE$(Y, n)$
    $P_1 \leftarrow$ STRASSEN$(A, F - H, n/2)$
    $P_2 \leftarrow$ STRASSEN$(A + B, H, n/2)$
    $P_3 \leftarrow$ STRASSEN$(C + D, E, n/2)$
    $P_4 \leftarrow$ STRASSEN$(D, G - E, n/2)$
    $P_5 \leftarrow$ STRASSEN$(A + D, E + H, n/2)$
    $P_6 \leftarrow$ STRASSEN$(B - D, G + H, n/2)$
    $P_7 \leftarrow$ STRASSEN$(A - C, E + F, n/2)$
    $R \leftarrow P_5 + P_4 - P_2 + P_6$
    $S \leftarrow P_1 + P_2$
    $T \leftarrow P_3 + P_4$
    $U \leftarrow P_5 + P_1 - P_3 - P_7$
    devolva $P \leftarrow$ CONSTRÓI-MAT$(R, S, T, U)$
```

Consumo de tempo

linha		todas as execuções da linha
1	$=$	$\Theta(1)$
2-3	$=$	$\Theta(n^2)$
4-10	$=$	$7\,T(n/2) + \Theta(n^2)$
11-14	$=$	$\Theta(n^2)$
15	$=$	$\Theta(n^2)$
total	$=$	$7\,T(n/2) + \Theta(n^2)$

Consumo de tempo

As dicas no nosso estudo de recorrências sugeri que a solução da recorrência

$$\begin{aligned} T(1) &= \Theta(1) \\ T(n) &= 7\,T(n/2) + \Theta(n^2) \quad \text{para } n = 2, 3, 4, \ldots \end{aligned}$$

está na mesma classe Θ que a solução de

$$T'(n) = 7\,T'(n/2) + n^2$$

n	1	2	4	8	16	32	64	128	256
$T'(n)$	1	11	93	715	5261	37851	269053	1899755	13363821

Conclusões

$T'(n)$ é $\Theta(n^{\lg 7})$.

Logo $T(n)$ é $\Theta(n^{\lg 7})$.

O consumo de tempo do algoritmo STRASSEN é $\Theta(n^{\lg 7})$ $(2{,}80 < \lg 7 < 2{,}81)$.

Mais conclusões

Consumo de tempo de algoritmos para multiplicação de matrizes:

Ensino fundamental	$\Theta(n^3)$
Strassen (1969)	$\mathrm{O}(n^{2.81})$
⋮	⋮
Coppersmith e Winograd (1987)	$\mathrm{O}(n^{2.3755})$
Stothers (2010)	$\mathrm{O}(n^{2.3736})$
⋮	⋮
Alman e Williams (2020)	$\mathrm{O}(n^{2.3729})$
⋮	⋮
Williams, Xu, Xu, Zhou (2024)	$\mathrm{O}(n^{2.3715})$