ON DISJOINTLY SINGULAR CENTRALIZERS

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Abstract. We study "disjoint" versions of the notions of trivial, locally trivial, strictly singular and super-strictly singular quasi-linear maps in the context of Köthe function spaces. Among other results, we show: i) (locally) trivial and (locally) disjointly trivial notions coincide on reflexive spaces; ii) On non-atomic superreflexive Köthe spaces, no centralizer is singular, although most are disjointly singular. iii) No super singular quasi-linear maps exist between superreflexive spaces although Kalton-Peck centralizers are super disjointly singular; iv) Disjoint singularity does not imply super disjoint singularity.

1. Introduction

For all unexplained notation and terms, please keep reading. This paper has its roots in [10] where the authors introduced the notion of disjointly singular centralizer on Köthe function spaces, proved that disjoint singularity coincides with singularity on Banach spaces with unconditional basis and presented a technique to produce disjointly singular centralizers via complex interpolation.

A second equally important fact to consider is that the fundamental Kalton-Peck map [25] is disjointly singular on L_p [10, Proposition 5.4], but it is not singular [35]. In fact, as the last stroke one could wish to foster the study of disjoint singularity is the argument of Cabello [2] that no centralizer on L_p can be singular that we extend here by showing that no centralizer can be singular. It is thus obvious that while singularity is an important notion in the domain of Köthe sequence spaces, disjoint singularity is the core notion in Köthe function spaces. The purpose of this paper is then to study the *disjointly supported* versions of the basic (trivial, locally trivial, singular and supersingular) notions in the theory of centralizers and present several crucial examples.

2. Background

Most of the action in this paper will take place in the ambient of Köthe function spaces. We present Kalton's definition of Köthe function space since it is slightly different from the standard one [29]. Let (S, Σ, μ) be a σ-finite measure space. Let $L_0 \equiv L_0(\mu)$ be the space of all measurable complex functions on S endowed with the topology of convergence in measure on sets of finite measure and we apply the usual convention about identifying functions equal almost everywhere. According to [23, p. 482 and p. 486], a Köthe function space on S is a linear subspace $\mathcal{K}$ of L_0 with a norm $\|\cdot\|_{\mathcal{K}}$ which makes it into a Banach space such that

(1) The unit ball $B_{\mathcal{K}} = \{x \in \mathcal{K} : \|x\|_{\mathcal{K}} \leq 1\}$ of $\mathcal{K}$ is closed in L_0.

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(2) There exist strictly positive $h, k \in L_0$ so that $\|hf\|_1 \leq \|f\|_{\mathcal{K}} \leq \|kf\|_\infty$ for every $f \in L_0$. (with the convention that $\|f\|_{\mathcal{K}} = \infty$ for every $f \in L_0 \setminus \mathcal{K}$).
(3) For every $x, y \in L_0$, if $y \in \mathcal{K}$ and $|x| \leq |y|$ then $x \in \mathcal{K}$ and $\|x\|_{\mathcal{K}} \leq \|y\|_{\mathcal{K}}$.

A particular case of which is that of Banach spaces with a 1-unconditional basis (called Köthe sequence spaces in what follows) with their associated ℓ_∞-module structure.

2.1. **Exact sequences, quasi-linear maps and centralizers.** For a rather complete background on the theory of twisted sums see [11]. We recall that a twisted sum of two Banach spaces Y and Z is a quasi-Banach space X which has a closed subspace isomorphic to Y such that the quotient X/Y is isomorphic to Z. Equivalently, X is a twisted sum of Y, Z if there exists a short exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0.$$

According to Kalton and Peck [25], twisted sums can be identified with homogeneous maps $\Omega : Z \to Y$ for which there exists $C > 0$ such that

$$\|\Omega(z_1 + z_2) - \Omega(z_1) - \Omega(z_2)\| \leq C(\|z_1\| + \|z_2\|),$$

for every $z_1, z_2 \in Z$. Such maps are called *quasi-linear*. The smallest of the constants C satisfying the above inequality is called the *quasi-linearity constant* of Ω and is denoted $Z(\Omega)$.

Every quasi-linear map $\Omega : Z \to Y$ induces an equivalent quasi-norm on X (seen algebraically as $Y \times Z$) by

$$\|(y, z)\|_\Omega = \|y - \Omega z\| + \|z\|.$$

This space is usually denoted $Y \oplus_\Omega Z$. When Y and Z are, for example, Banach spaces of non-trivial type, the quasi-norm above is equivalent to a norm; therefore, the twisted sum obtained is a Banach space. Two exact sequences $0 \to Y \to X_1 \to Z \to 0$ and $0 \to Y \to X_2 \to Z \to 0$ are said to be *equivalent* if there exists an operator $T : X_1 \to X_2$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & X_1 & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X_2 & \longrightarrow & Z & \longrightarrow & 0. \end{array}$$

The classical 3-lemma (see [11, p. 3]) shows that T must be an isomorphism.

Definition 2.1. *An L_∞-centralizer (resp. an ℓ_∞-centralizer) on a Köthe function (resp. sequence) space $\mathcal{K}$ is a homogeneous map $\Omega : \mathcal{K} \to L_0$ such that there is a constant C satisfying that, for every $f \in L_\infty$ (resp. ℓ_∞) and for every $x \in \mathcal{K}$, the difference $\Omega(fx) - f\Omega(x)$ belongs to $\mathcal{K}$ and*

$$\|\Omega(fx) - f\Omega(x)\|_{\mathcal{K}} \leq C\|f\|_\infty \|x\|_{\mathcal{K}}.$$

The centralizer is called real when it sends real functions (sequences) to real functions (sequences).

When no confusion arises we will simply say: a centralizer. Observe that a centralizer Ω on $\mathcal{K}$ does not take values in $\mathcal{K}$, but in L_0, and still it induces an exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{\ \jmath\ } d_\Omega\mathcal{K} \xrightarrow{\ q_\Omega\ } \mathcal{K} \longrightarrow 0$$

as follows: $d_\Omega\mathcal{K} = \{(w, x) : w \in L_0, x \in \mathcal{K} : w - \Omega x \in \mathcal{K}\}$ endowed with the quasi-norm

$$\|(w, x)\|_{d_\Omega\mathcal{K}} = \|x\|_{\mathcal{K}} + \|w - \Omega x\|_{\mathcal{K}}$$

and with obvious inclusion $\jmath(x) = (x, 0)$ and quotient map $q_\Omega(w, x) = x$. The reason is that a centralizer "is" quasi-linear, in the sense that for all $x, y \in \mathcal{K}$ one has $\Omega(x+y)-\Omega(x)-\Omega(y) \in \mathcal{K}$ and $\|\Omega(x+y) - \Omega(x) - \Omega(y)\| \leq C(\|x\| + \|y\|)$ for some $C > 0$ and all $x, y \in \mathcal{K}$. To describe the fact that the centralizer acts $\Omega : \mathcal{K} \to L_0$ but defines a twisted sum of $\mathcal{K}$ with itself we will use sometimes the notation $\Omega : \mathcal{K} \curvearrowright \mathcal{K}$. Centralizers arise naturally by complex interpolation [1] as can be seen in [23].

2.2. **Trivial maps.** An exact sequence $0 \to Y \to X \to Z \to 0$ is trivial if and only if it is equivalent to $0 \to Y \to Y \oplus Z \to Z \to 0$, where $Y \oplus Z$ is endowed with the product norm. In this case we say that the exact sequence *splits.* Two quasi-linear maps $\Omega, \Omega' : Z \to Y$ are said to be equivalent, denoted $\Omega \equiv \Omega'$, if the difference $\Omega - \Omega'$ can be written as $B + L$, where $B : Z \to Y$ is a homogeneous bounded map (not necessarily linear) and $L : Z \to Y$ is a linear map (not necessarily bounded). Two quasi-linear maps are equivalent if and only if the associated exact sequences are equivalent. A quasi-linear map is trivial if it is equivalent to the 0 map, which also means that the associated exact sequence is trivial. Given two Banach spaces Y, Z we will denote by $\ell(Z, Y)$ the vector space of linear (not necessarily continuous) maps $Z \to Y$. The distance between two homogeneous maps T, R will be the usual operator norm (the supremum on the unit ball) of the difference; i.e., $\|T - R\|$, which can make sense even when R and T are unbounded. So a quasi-linear map $\Omega : Z \longrightarrow Y$ is trivial if and only if $d(\Omega, \ell(Z, Y)) \leq C < +\infty$, in which case we will say that Ω is C-trivial. A centralizer $\mathcal{K} \curvearrowright \mathcal{K}$ is trivial if and only if there is a linear map $L : \mathcal{K} \to L_0$ so that $\Omega - L : \mathcal{K} \to \mathcal{K}$ is bounded. In a more classical language, Ω is trivial if and only if Y is complemented in $Y \oplus_\Omega Z$.

2.3. **Locally trivial maps.** A quasi-linear map $\Omega : Z \to Y$ is said to be *locally trivial* [21] if there exists $C > 0$ such that for any finite dimensional subspace F of Z, there exists a linear map L_F such that $\|\Omega_{|F} - L_F\| \leq C$. It is clear that a trivial map is locally trivial. The converse is not true, although locally trivial quasi-linear maps $\Omega : Z \to Y$ in which Y is reflexive are trivial, by [5]. We say that Y is locally complemented in $Y \oplus_\Omega Z$ if and only if Ω locally splits.

2.4. **Singular maps.** An operator between Banach spaces is said to be *strictly singular* if no restriction to an infinite dimensional closed subspace is an isomorphism. Analogously, a quasi-linear map (in particular, a centralizer) is said to be *singular* if its restriction to every infinite dimensional closed subspace is never trivial. An exact sequence induced by a singular quasi-linear map is called a *singular sequence.* A quasi-linear map is singular if and only if the associated exact sequence has strictly singular quotient map [13, Lemma 1]. Singular ℓ_∞-centralizers exist and the most natural example is the Kalton-Peck map $\mathcal{K}_p : \ell_p \curvearrowright \ell_p$, $0 < p < +\infty$, defined by $\mathcal{K}_p(x) = x \log \frac{|x|}{\|x\|_p}$. The proof that $\mathcal{K}_p$ is singular can be found in [25] for $1 < p < +\infty$, [13] for $p = 1$, and [8] for all $0 < p < +\infty$. A simple characterization of singular ℓ_∞-centralizers on Banach sequence spaces can be presented

Proposition 2.2. *Let X be a Banach space with an unconditional basis not containing c_0. Let $\Omega : X \curvearrowright X$ be an ℓ_∞-centralizer such that for every sequence (A_k) of consecutive subsets of $\mathbb{N}$ and every sequence (u_n) of consecutive normalized blocks of the basis, for which $\sup_k \|\sum_{n\in A_k} u_n\| \to +\infty$ one has*

$$\limsup_k \frac{\|\Omega(\sum_{n\in A_k} u_n) - \sum_{n\in A_k} \Omega(u_n)\|}{\|\sum_{n\in A_k} u_n\|} = +\infty.$$

Then Ω is singular.

Proof. If $\Omega : X \curvearrowright X$ is an ℓ_∞-centralizer verifying the condition above and, at the same time, trivial on some subspace H, by the blocking principle (see [8, Lemma 2]), it must be trivial on the subspace $[u_n]$ spanned by some consecutive blocks of the basis. Standard manipulations (see [8, 10]) show that the linear map $\ell(u_n) = \Omega(u_n)$ is at finite distance from Ω, which implies that $\limsup_k \|\sum_{n\in A_k} u_n\| < +\infty$ for all choices of (A_k), thus (u_n) is equivalent to the canonical basis of c_0 and consequently H contains c_0. □

In sharp contrast, Cabello [2] proved that no L_∞-centralizer is singular on $L_p[0,1]$. Let us observe that quite the same proof of Cabello provides the following proposition.

Proposition 2.3. *Let $\mathcal{K}$ be a superreflexive Köthe function space over a non-atomic σ-finite measure space and let Ω be a L_∞-centralizer on $\mathcal{K}$. Then Ω is not singular. Moreover, Ω is bounded on some copy of ℓ_2.*

Sketch of proof: Let $\mathcal{K}$ be a superreflexive Köthe function space with base space (S,μ) where μ is a non-atomic σ-finite measure and let $\Omega_{\mathcal{K}}$ be an L_∞-centralizer on $\mathcal{K}$. For a subset $I \subset S$ we will denote $\mathcal{K}(I)$ the subspace of $\mathcal{K}$ formed by those functions with support contained in I. By Kalton's theorem [23, Thm. 7.6] plus the comments in [2, Section 1.3] there are two Köthe spaces A, B so that $\mathcal{K} = (A,B)_{1/2}$; these spaces can be assumed to be superreflexive by reiteration and [24, Thm. 7.8]. The admissibility of the norm yields functions h_a, k_a such that $\|h_a f\|_1 \leq \|f\|_A \leq \|k_a f\|_\infty$ for every $f \in A$; and functions h_b, k_b such that $\|h_b f\|_1 \leq \|f\|_B \leq \|k_b f\|_\infty$ for every $f \in B$. Thus, one can find a positive measure set $I \subset S$ and a constant $M > 0$ such that $k_a, k_b \leq M$ and $h_a, h_b \geq M^{-1}$ on I. This provides continuous inclusions $L_\infty(I) \subset A(I) \subset L_1(I)$ and $L_\infty(I) \subset B(I) \subset L_1(I)$.

By super-reflexivity, both spaces A, B are p-convex and q-concave for some $1 < p, q < +\infty$ ([29, Thm 1.f.12 and Thm 1.f.7.]) So, using the Johnson-Maurey-Schechtman-Tzafriri remark [19, p.14] then also $L_q(I) \subset A(I) \subset L_p(I)$ and $L_q(I) \subset B(I) \subset L_p(I)$. Since μ is non-atomic, let $R(I)$ be the subspace generated by Rademacher functions supported in I. The L_p and L_q-norms are equivalent on $R(I)$ by Khintchine's inequality, and are also equivalent to $\|\cdot\|_A$ and to $\|\cdot\|_B$, and thus $R(I) \simeq \ell_2$. The equivalence of norms A and B on $R(I)$ makes the differential $\Omega_{1/2}$ bounded on $R(I)$, and since $\Omega_{\mathcal{K}}$ is boundedly equivalent to $\Omega_{1/2}$, it must be bounded too.

2.5. **Super-singular maps.** An operator $T : Z \to Y$ between two Banach spaces is said to be *super strictly singular* (in short, super-SS) if there does not exist a number $c > 0$ and a sequence of subspaces E_n of Z, with dim $E_n = n$, such that $\|Tx\| \geq c\|x\|$ for every $x \in \bigcup_n E_n$. Equivalently [14, Lemma 1.1.], if every ultrapower of T is strictly singular. Super strictly singular operators have also been called finitely strictly singular; they were first introduced in [30, 31], and form a closed ideal containing the ideal of compact and contained in the ideal of strictly singular operators. See also [14] for the study of such a notion in the context of twisted sums, as well as [9] where a few results are also mentioned in relation to complex structures on twisted sums.

It is a standard fact (see [11]) that given an exact sequence $0 \to Y \to X \to Z \to 0$ and an ultrafilter $\mathfrak{U}$ the ultrapowers form an exact sequence $0 \to Y_{\mathfrak{U}} \to X_{\mathfrak{U}} \to Z_{\mathfrak{U}} \to 0$. If Ω is a quasi-linear map associated to the former sequence we will call $\Omega_{\mathfrak{U}}$ any quasi-linear map associated to the later. We do not need for the moment to specify the construction of $\Omega_{\mathfrak{U}}$. We will say, following [9] that a quasi-linear map Ω is super-singular if every ultrapower $\Omega_{\mathfrak{U}}$ is singular. We need to state here two facts proved in [9]:

- Ω is super-singular if and only the quotient map q_Ω of the exact sequence it defines is super strictly singular.

- No super singular quasi-linear maps between B-convex Banach spaces exist. This follows from [32, Thm. 3], where it is proved that a super strictly singular operator on a B-convex space has super strictly singular adjoint. Since superreflexive spaces are B-convex, B-convexity is a 3-space property (see [11]) and the adjoint of a quotient map is an into isomorphism, the result follows.

After these prolegomena, we tackle the study of the "disjoint" versions of the preceding properties. It is worth to observe that all our forthcoming "disjoint" notions admits an immediate translation to general quasi-linear maps on Banach lattices.

3. Disjoint local triviality

Definition 3.1. *A quasi-linear map $\Omega : \mathcal{K} \to Y$ defined on a Banach lattice is said to be* disjointly trivial *if it is trivial on any subspace generated by a sequence of disjointly supported elements. It is said to be* locally disjointly trivial *if there exists $C > 0$ such that for any finite dimensional subspace F of $\mathcal{K}$ generated by disjointly supported vectors, there exists a linear map L_F such that $\|\Omega_{|F} - L_F\| \leq C$.*

We can show:

Proposition 3.2. *Let $\mathcal{K}$ be a Köthe function space over a σ-finite measure space and let $\Omega : \mathcal{K} \to Y$ be a quasi-linear map. Consider the following assertions:*

(i) *Ω is trivial.*
(ii) *Ω is disjointly trivial.*
(iii) *Ω is locally trivial.*
(iv) *Ω is locally disjointly trivial.*

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv)$. Moreover, if Y is complemented in its bidual, then all assertions are equivalent.

Proof. Assertion (i) implies (iii) and it is well-known, see [5], that a locally trivial quasi-linear map taking values in a space complemented in its bidual is trivial. It is clear that (i) implies (ii) and that (iii) implies (iv). Let us show that (iv) implies (iii): Let Ω be a quasi-linear map verifying (iv) and let F be a finite dimensional subspace of $\mathcal{K}$. Approximating functions by characteristic functions we may find a nuclear operator N on $\mathcal{K}$ of arbitrary small norm so that $(Id + N)(F)$ is contained in the linear span $[u_n]$ of a finite sequence of disjointly supported vectors. The restriction $\Omega_{|[u_n]}$ is trivial with constant C, thus using [9, Lemma 5.6], we get that $\Omega = \Omega(I + N) - \Omega N$ is trivial with constant $C + \epsilon$ on F. Therefore (iii) holds.

It remains to show that (ii) implies (iii). We need to decompose first the measure $\nu = \nu_a + \mu$ in its atomic part ν_a and its purely non-atomic part μ [18, Theorem 2.1]. We observe that the result follows if one proves the implication assuming that the measure is either purely atomic or non-atomic. Let us prove that (ii) implies (iii) for the atomic part, which follows from the observation: *Let $\Omega : X \to Y$ be a quasilinear map and let $\delta : X \to X^{**}$ be the canonical isometric embedding. If $\delta\Omega$ is trivial then Ω is locally trivial.* Indeed, since X is locally complemented in X^{**} thanks to the Principle of Local Reflexivity and X^{**} is complemented in $X^{**} \oplus_{\delta\Omega} Y$ it turns out that X is locally complemented in $X^{**} \oplus_{\delta\Omega} Y$, hence in $X \oplus_{\Omega} Y$.

Thus, if Ω is not locally trivial then $\delta\Omega$ is not trivial, hence $\delta\Omega$ is not disjointly trivial since both notions are equivalent in the atomic case. Therefore Ω cannot be disjointly trivial.

We prove now (ii) implies (iii) for the non-atomic part. Let (S,μ) be the non-atomic σ-finite base space. Recall that for a subset $I \subset S$ we denote $\mathcal{K}(I)$ the subspace of $\mathcal{K}$ formed by those functions with support contained in I.

Claim 1. *If I and J are disjoint subsets of S and Ω is locally trivial on both $\mathcal{K}(I)$ and $\mathcal{K}(J)$ then it is locally trivial on $\mathcal{K}(I\cup J)$.* Let F be a finite dimensional subspace of $\mathcal{K}(I\cup J)$. Then $F \subseteq F_1 + F_2$ for finite dimensional subspaces $F_1 \subseteq \mathcal{K}(I)$ and $F_2 \subseteq \mathcal{K}(J)$. If $\|\Omega_{|F_1} - a\| \le c$ and $\|\Omega_{|F_2} - b\| \le d$, where a and b are linear, then $\|\Omega_{|F} - (a\oplus b)\| \le 2(Z(\Omega)+c+d)$, where $a\oplus b$ is the obvious linear map on F.

Claim 2. *If Ω is non-locally trivial on $\mathcal{K}$ then S can be split in two sets $S = I\cup J$ so that $\Omega_{|\mathcal{K}(I)}$ and $\Omega_{|\mathcal{K}(J)}$ are both non-locally trivial.* We first assume that S is a finite measure space. Assume the claim does not hold. Since μ is non-atomic, split $S = R_1 \cup I_1$ in two sets of the same measure and assume $\Omega_{|\mathcal{K}(I_1)}$ is locally trivial. Note that since the claim does not hold, given any $C \subset S$ and any splitting $C = I\cup J$ the map Ω is locally trivial on $\mathcal{K}(I)$ or $\mathcal{K}(J)$. So, split $R_1 = R_2 \cup I_2$ in two sets of equal measure and assume that $\Omega_{|\mathcal{K}(I_2)}$ is locally trivial, and so on. If Ω is λ-locally trivial on $\mathcal{K}(\cup_{j\le n} I_j)$ for $\lambda < +\infty$ and for all n then Ω is locally trivial on $\mathcal{K}$, and we get a contradiction. If $\lambda_n \to \infty$ is such that $\Omega_{|\mathcal{K}(\cup_{j\le n} I_j)}$ is λ_{n+1}-locally trivial but not λ_n-locally trivial for all n, then by Claim 1 we note that for $m<n$, Ω cannot be locally trivial with constant less than $\lambda_n/2 - Z(\Omega) - \lambda_m - 1$ on $\mathcal{K}(\cup_{m<j\le n} I_j)$. From this we find a partition of $\mathbb{N}$ as $N_1 \cup N_2$ so that if $I = \cup_{n\in N_1} I_n$ and $J = \cup_{n\in N_2} I_n$, then Ω is non-locally trivial on $\mathcal{K}(I)$ and $\mathcal{K}(J)$, another contradiction.

If S is σ-finite then the proof is essentially the same: either one can choose the sets I_n all having measure, say, 1 or at some step R_m is of finite measure, and we are in the previous case. This concludes the proof of the claim.

We pass to complete the proof that (ii) implies (iii). Assume that Ω is non-locally trivial on $\mathcal{K}$. By Claim 2, split $S = I_1 \cup J_1$ so that Ω is non-locally trivial on both $\mathcal{K}(I_1)$ and $\mathcal{K}(J_1)$. It cannot be locally disjointly trivial on them, so there is a finite number $\{u_n^1\}_{n\in F_1}$ of disjointly supported vectors on $\mathcal{K}(I_1)$ on which Ω is not 2-trivial. By the claim applied to $\mathcal{K}(J_1)$ split $J_1 = I_2 \cup J_2$ so that Ω is non-locally trivial neither on $\mathcal{K}(I_2)$ nor in $\mathcal{K}(J_2)$. It cannot be locally disjointly trivial on them, so there is a finite number of disjointly supported vectors $\{u_n^2\}_{n\in F_2}$ on $\mathcal{K}(I_2)$ on which Ω is not 4-trivial. Iterate the argument to produce a subspace Y generated by an infinite sequence

$$\{u_n^1\}_{n\in F_1}, \{u_n^2\}_{n\in F_2}, \ldots, \{u_n^k\}_{n\in F_k}, \ldots$$

of disjointly supported vectors, where Ω cannot be trivial. □

An immediate corollary of (the proof of) Proposition 3.2 is:

Corollary 3.3. *Let $\mathcal{K}$ be a Köthe space over a σ-finite measure space (S,μ) and let Ω be a non-locally trivial quasi-linear map on $\mathcal{K}$. Then there exists a sequence (I_n) of finite measure mutually disjoint subsets of S so that the restriction $\Omega_{|[1_{I_n}]}$ is non-locally trivial.*

In $L_p(S,\mu)$, given a sequence (I_n) of finite measure mutually disjoint subsets of S, the subspace $[1_{I_n}]$ is isomorphic to ℓ_p when $0<p<+\infty$ and to c_0 in $L_\infty(S,\mu)$. Hence we have

Corollary 3.4. *Let (S,μ) be a σ-finite measure space and let $0<p<\infty$. Then for every non-locally trivial quasi-linear map Ω defined on $L_p(S,\mu)$ (resp. $L_\infty(S,\mu)$) there is a copy of ℓ_p (resp. c_0) spanned by disjointly supported vectors on which the restriction of Ω is not locally trivial.*

It is not clear whether Proposition 3.2 can be translated to the domain of Banach lattices. $C(K)$-spaces are not, as a rule, Köthe spaces; however, the above essential part of Proposition 3.2 still survives for $C(K)$-spaces: every non-locally trivial quasi-linear map defined on a $C(K)$ must be non-trivial on a subspace isomorphic to c_0 [6, Theorem 2.1].

4. Disjoint singularity

Proposition 3.2 shows that (local) triviality and disjoint (local) triviality are essentially equivalent. We shall now see that the situation is much more complex regarding *singularity* notions. The following definition was introduced in [10].

Definition 4.1. *A quasi-linear map on a Banach lattice is called* disjointly singular *if its restriction to every infinite dimensional subspace generated by a disjointly supported sequence is never trivial.*

Of course, a singular quasi-linear map is disjointly singular and a disjointly singular quasi-linear map on a Köthe sequence space is singular. An open question, to the best of our knowledge due to Félix Cabello, is about the existence of singular quasi-linear maps on Köthe function spaces; recall that no singular L_∞-centralizers exist on any reasonable Köthe space [2] (cf. Proposition 2.3); see also [35]).

4.1. **Examples.**

(1) As we mentioned at the introduction, the methods in [10] actually produce disjointly singular centralizers. In particular, it is shown [10, Proposition 5.4] that the Kalton-Peck centralizer
$$\mathcal{K}(x) = x \log \frac{|x|}{\|x\|}$$
is disjointly singular on any reflexive, p-convex Köthe function space, $p > 1$

(2) Given two Lorentz spaces L_{p_0,q_0}, L_{p_1,q_1}, it was proved in [4] that $(L_{p_0,q_0}, L_{p_1,q_1})_\theta = L_{p,q}$ for $p^{-1} = (1-\theta){p_0}^{-1} + \theta {p_1}^{-1}$ and $q^{-1} = (1-\theta){q_0}^{-1} + \theta {q_1}^{-1}$ with an associated derivation
$$\Omega(x) = q\left(\frac{1}{q_1} - \frac{1}{q_0}\right)\mathcal{K}(x) + \left(\frac{q}{p}\left(\frac{1}{q_0} - \frac{1}{q_1}\right) - \left(\frac{1}{p_0} - \frac{1}{p_1}\right)\right)\kappa(x)$$
Here $\mathcal{K}(\cdot)$ is the Kalton-Peck map earlier defined and $\kappa(\cdot)$ is the so-called Kalton map [20]; see also [4], given by $\kappa(x) = x\ r_x$ where r_x is the rank function $r_x(t) = \mu\{s : |x(s)| > |x(t)| \text{ or} |x(s)| = |x(t)| \text{ and } s \leq t\}$ (see [34]).

The map $\mathcal{K}$ is disjointly singular while κ has the property that every infinite dimensional subspace contains a further infinite dimensional subspace where it is trivial [4], so it is clear that Ω is disjointly singular.

(3) A different set of examples will be presented now in $C(K)$ or L_∞-spaces. These examples are relevant because in the category of Banach spaces no singular quasi-linear map is possible on a space containing ℓ_1. In fact, notice that every short exact sequence of Banach spaces $0 \to Y \to X \to \ell_1$ splits since the quotient map $X \to \ell_1$ admits a bounded and linear right inverse. In particular, the associated twisted sum for the Kalton-Peck quasi-linear map $\mathcal{K}_1 : \ell_1 \curvearrowright \ell_1$ is a non-locally convex quasi-Banach space. In [26], Kalton and Roberts proved that every twisted sum of a Banach space and an $\mathcal{L}_\infty$-space is locally convex (being thus isomorphic to a Banach space). We consider for the examples the spaces $C[0,1]$ and ℓ_∞. It is necessary to remark that $C[0,1]$ is not a Köthe space and thus the example lives in the domain of Banach lattices; see the comments after the Proposition 4.2.

Proposition 4.2. *There exist disjointly singular quasi-linear maps on $C[0,1]$ and ℓ_∞.*

Proof. Let us consider first the case of $C[0,1]$. As we have already remarked, one just needs to construct a c_0-singular map, that is, a quasi-linear map such that when restricted to every subspace isomorphic to c_0 is never trivial. Let $\omega : c_0 \to C[0,1]$ be a nontrivial quasi-linear map (see [7] for explicit examples). Let Γ be the set of all 2-isomorphic copies γ of c_0 inside $C[0,1]$, which are necessarily 4-complemented in $C[0,1]$ via some projection π_γ and let $\alpha_\gamma : \gamma \to c_0$ be a 2-isomorphism. Define a quasi-linear map $\Upsilon : C[0,1] \to \ell_\infty(\Gamma, C[0,1])$ by means of

$$\Upsilon(f)(\gamma) = \omega(\alpha_\gamma \pi_\gamma(f))$$

This map is c_0-singular because if there is a copy of c_0 in which Υ is trivial, that copy must contains some $\gamma \in \Gamma$, on which Υ must be trivial too. But if $f \in \gamma$ one has

$$\Upsilon_{|\gamma}(f)(\gamma) = \omega(\alpha_\gamma \pi_\gamma(f)) = \omega(\alpha_\gamma f)$$

thus, if $\delta_\gamma : \ell_\infty(\Gamma, C[0,1]) \to C[0,1]$ is the canonical evaluation at the coordinate γ we have obtained $\delta_\gamma \Upsilon_{|\gamma} = \omega\alpha_\gamma$. This map cannot be trivial since, otherwise, so it would be $\omega = \delta_\gamma \Upsilon_{|\gamma} \alpha_\gamma^{-1}$, which is not the case. But that means that $\Upsilon_{|}\gamma$ cannot be trivial because $\delta_\gamma \Upsilon_{|\gamma}$ is not trivial.

A standard reduction (see [13]) allows one to find an equivalent quasi-linear map $\Omega : C[0,1] \to \ell_\infty(\Gamma, C[0,1])$ having separable range. Since $\ell_\infty(\Gamma, C[0,1])$ is a Banach algebra, Ω can also be considered taking values in the closed subalgebra generated by $[\Omega(C[0,1])]$, which, being separable, is isometrically isomorphic to a $C(K)$ for some metrizable compact K by the classical Gelfand-Naimark theorem. When K is countable it is homeomorphic to an interval of ordinals $[0,\alpha]$ for some countable ordinal α, otherwise by Milutin's theorem $C(K)$ is isomorphic to $C[0,1]$. Thus, $\Omega : C[0,1] \to C[0,1]$ is a c_0-singular quasi-linear map, as desired.

The case of ℓ_∞ has to be treated differently because the projections π_γ do not exist now. Start with picking a nontrivial quasi-linear map $\omega : c_0 \to \ell_2$, which can be constructed as follows: pick the Kalton-Peck map $\mathcal{K} : \ell_2 \to \ell_2$ and a quotient map $Q : C[0,1] \to \ell_2$. The map $\mathcal{K}Q$ is not trivial (see [5, 7]). It cannot be locally trivial either since ℓ_2 is reflexive and Proposition 3.2 would make it trivial. Thus, using [6, Theorem 2.1] there is a copy of c_0 inside $C[0,1]$ via some isomorphic embedding j so that the restriction $\mathcal{K}Qj$ is not trivial. Let us simplify and call this map ω. Let Γ be the set of infinite sequences of finite subsets $\mathbb{N}$. Given such a sequence $\gamma = (A_n)$ we will call $\overline{\gamma} = \cup_{A_n \in \gamma} A$. Let also $\alpha_\gamma : [1_{A_n}] \to c_0$ be an isometry. Define a quasi-linear map $\Upsilon : c_0 \to \ell_\infty(\Gamma, \ell_2)$ as

$$\Upsilon(x)(\gamma) = \omega\alpha_\gamma(1_{\overline{\gamma}} x)$$

The bidual map $\Upsilon^{**} : \ell_\infty \to \ell_\infty(\Gamma, \ell_2)^{**}$ cannot be trivial either since $\ell_\infty(\Gamma, \ell_2)$ is complemented in its bidual. If π denotes a projection, the map $\pi\Upsilon^{**} : \ell_\infty \to \ell_\infty(\Gamma, \ell_2)$ cannot be trivial either. We define a new map $\Omega : \ell_\infty \to \ell_\infty(\Gamma \times \Gamma, \ell_2)$ in the form

$$\Omega(x)(\gamma, \gamma') = \pi\Upsilon^{**}(1_{\overline{\gamma} x})(\gamma')$$

This map Ω cannot be disjointly singular: if it becomes trivial on some γ then for $x \in \gamma$ one has

$$\Omega(x)(\gamma, \gamma) = \pi\Upsilon^{**}(1_{\overline{\gamma} x})(\gamma) = \Upsilon(x)(\gamma) = \omega\alpha_\gamma(1_{\overline{\gamma} x}) = \omega\alpha_\gamma(x)$$

This map cannot be trivial since α_γ is an isomorphism and ω is not trivial. □

It is an open problem posed in [2] whether there exists a singular quasi-linear map $\Omega : L_p \to L_p$ for $0 < p < 2$. Singular quasi-linear maps (not centralizers) $\Omega : L_p \to L_p$ exist for $2 \leq p < +\infty$ (see [8, Theorem 2(c)]); observe that in this case the Kadec-Pełczyński alternative immediately yields that a quasilinear map Ω on L_p that is both disjointly singular and ℓ_2-singular must be singular. Thus, we could also use a construction similar to that in Proposition 4.2 to obtain singular maps in L_p, $2 \leq p < +\infty$. None of these can be L_∞-centralizers, nonetheless.

The papers [15, 16, 17] study the behaviour of strictly singular operators in Banach lattices by considering the more general notion of *lattice singular* operator (one for which no restriction to an infinite dimensional sublattice is an isomorphism). Obviously, strict singularity implies lattice singularity and this implies disjoint singularity. The authors obtain an interesting result [15]: *Let X, Y be Banach lattices such that X has finite cotype and Y admits a lower 2-estimate. Then an operator $T : X \to Y$ is strictly singular if and only if it is disjointly singular and ℓ_2-singular.* A non-vacuous centralizer version for this result is not possible since Proposition 2.3 establishes that no L_∞-centralizer can be ℓ_2-singular. It makes however sense the question about conditions ensuring that a quasi-linear map on a Köthe space that is simultaneously disjointly singular and ℓ_2-singular is necessarily singular.

4.2. **Characterizations.** Regarding characterizations, given a Köthe space $\mathcal{K}$ on S and a quasi-linear map $\Omega : \mathcal{K} \to \mathcal{K}$ the fact that the twisted sum space $d_\Omega\mathcal{K}$ is not necessarily a Köthe space complicates the characterization of disjointly singular maps in terms of the quotient operator. This difficulty can be overcome for centralizers, which always admit a version satisfying that $\operatorname{supp}\Omega x \subset \operatorname{supp} x$ for all $x \in \mathcal{K}$. Although, as we have just said, the space $d_\Omega\mathcal{K}$ is not a Köthe space, its elements are couples of functions of L_0; i.e., functions $S \to \mathbb{C} \times \mathbb{C}$. The following definition makes sense:

Definition 4.3. *A pair of nonzero elements $f = (w_0, x_0), g = (w_1, x_1)$ of $d_\Omega\mathcal{K}$ is said to be disjoint if the functions $f, g : S \to \mathbb{C} \times \mathbb{C}$ are disjointly supported. An operator $\tau : d_\Omega\mathcal{K} \to \mathcal{K}$ is said to be disjointly singular if the restriction of τ to any infinite dimensional subspace generated by a disjoint sequence of vectors is not an isomorphism.*

Recall that a Köthe function space $\mathcal{K}$ is maximal if whenever (f_n) is an increasing sequence of non-negative functions in $\mathcal{K}$ converging almost everywhere to f and $\sup_n \|f_n\|_\mathcal{K} < \infty$, then $f \in \mathcal{K}$ and $\|f\|_\mathcal{K} = \sup_n \|f_n\|_\mathcal{K}$. It was pointed out by Kalton [23, p. 487] (see also [10, p. 4683]) that his definition of Köthe space implies that maximality condition. We shall use this property in the proof of the next lemma to invoke a result from [10].

Lemma 4.4. *A centralizer Ω on a Köthe space $\mathcal{K}$ is disjointly singular if and only if q_Ω is disjointly singular.*

Proof. We choose the form $q_\Omega : \mathcal{K} \oplus_\Omega \mathcal{K} \to \mathcal{K}$ given by $q_\Omega(v, u) = u$. If Ω is trivial on the span $[u_n]$ of a disjointly supported sequence there is a linear map $L : [u_n] \to \mathcal{K}$ so that $\|\Omega - L\| \leq C$. Since $\mathcal{K}$ is maximal, by [10, Lemma 3.17] there exists a linear map $\Lambda : [u_n] \to \mathcal{K}$ such that $\operatorname{supp}\Lambda x \subset \operatorname{supp} x$ and $\|\Omega - \Lambda\| \leq C$ and thus $(\Lambda u_n, u_n)$ is a disjointly supported sequence for which the restriction of q_Ω to its closed linear span is an isomorphism since

$$\|\sum \lambda_n(\Lambda u_n, u_n)\| = \left\|\Omega\left(\sum \lambda_n u_n\right) - \sum \Lambda(\lambda_n u_n)\right\| + \|\sum \lambda_n u_n\| \leq (C+1)\|\sum \lambda_n u_n\|.$$

In this way, q_Ω disjointly singular implies Ω disjointly singular. To get the converse, assume that q_Ω is not disjointly singular, so there is a disjointly supported sequence (v_n, u_n) in $\mathcal{K} \oplus_\Omega \mathcal{K}$

such that q_Ω is an isomorphism on $[(v_n, u_n)]$. This means that

$$\left\|\sum \lambda_n v_n - \Omega\left(\sum \lambda_n u_n\right)\right\| \leq C \left\|\sum \lambda_n u_n\right\|$$

The linear map $L(u_n) = v_n$ verifies

$$\left\|L\left(\sum \lambda_n u_n\right) - \Omega\left(\sum \lambda_n u_n\right)\right\| = \left\|\sum \lambda_n v_n - \Omega\left(\sum \lambda_n u_n\right)\right\| \leq C \left\|\sum \lambda_n u_n\right\|$$

□

Now we want to mimicry Proposition 2.2. Let $\mathcal{K}$ be a Köthe space and let $\Omega : \mathcal{K} \to L_0$ be a centralizer. Given a finite sequence $b = (b_k)_{k=1}^n \subset \mathcal{K}$ we will follow [9] and define

$$\nabla_{[b]}\Omega = \mathrm{Ave}_\epsilon \left\|\Omega\left(\sum_{k=1}^n \epsilon_k b_k\right) - \sum_{k=1}^n \epsilon_k \Omega(b_k)\right\|,$$

where the average is taken over all choices of signs $\epsilon = (\epsilon_k)_{k=1}^n \in \{\pm 1\}^n$. The triangle inequality holds for $\nabla_{[b]}\Omega$: if Ω and Ψ are centralizers then $\nabla_{[b]}(\Omega + \Psi) \leq \nabla_{[b]}\Omega + \nabla_{[b]}\Psi$. If $\lambda = (\lambda_k)_k$ is a finite sequence of scalars and $x = (x_k)_k$ a sequence of vectors of $\mathcal{K}$, we write λx to denote the finite sequence obtained by the non-zero vectors of $(\lambda_1 x_1, \lambda_2 x_2, \ldots)$.

Recall from [10, Definiton 3.10] that a centralizer Ω on a Köthe function space $\mathcal{K}$ is *contractive* if $\operatorname{supp}\Omega(x) \subseteq \operatorname{supp} x$ for every $x \in \mathcal{K}$. Our next result provides a characterization of disjoint singularity for contractive centralizers in the L_p spaces. The contractive restriction is not so severe, since every centralizer Ω on a Köthe function space $\mathcal{K}$ admits a contractive centralizer ω such that $\Omega - \omega$ is bounded ([22, Proposition 4.1]). Also it is easy to see that the canonical centralizer induced by interpolation of Köthe spaces is contractive, see [10].

Proposition 4.5. *A contractive centralizer Ω defined on L_p is not disjointly singular if and only if there is a disjointly supported normalized sequence $u = (u_n)$ and a constant $C > 0$ such that for every $\lambda = (\lambda_k) \in c_{00}$ one has*

$$\nabla_{[\lambda u]}\Omega \leq C\|\lambda\|_p.$$

The proof follows from the following three lemmas.

Lemma 4.6. *Let Ω be a contractive centralizer on a Köthe space $\mathcal{K}$ satisfying an upper p-estimate, and let $u = (u_n)_n$ be a disjointly supported normalized sequence of vectors. Suppose that the restriction of Ω to the closed linear span $[u_n]$ is trivial. Then there is a constant $C > 0$ such that*

$$\nabla_{[\lambda u]}\Omega \leq C\|\lambda\|_p$$

for every $\lambda = (\lambda_k)_k \in c_{00}$.

Proof. If the restriction of Ω to $[u_n]$ is trivial then there is a linear map $L : [u_n] \to \mathcal{K}$ so that $\|\Omega - L\| \leq C < +\infty$. From [10, Lemma 3.17] we can take such L so that $\operatorname{supp} L(x) \subset \operatorname{supp} x$. Then for every $\lambda \in c_{00}$

$$\left\|\Omega\left(\sum_{i=1}^n \lambda_i u_i\right) - L\left(\sum_{i=1}^n \lambda_i u_i\right)\right\| \leq C \left\|\sum_{i=1}^n \lambda_i u_i\right\|$$

which implies that

$$\begin{aligned}\left\|\Omega\left(\sum_{i=1}^{n}\lambda_i u_i\right)-\sum_{i=1}^{n}\Omega(\lambda_i u_i)\right\| &= \left\|\Omega\left(\sum_{i=1}^{n}\lambda_i u_i\right)-L(\sum_{i=1}^{n}\lambda_i u_i)+\sum_{i=1}^{n}\lambda_i L u_i-\sum_{i=1}^{n}\lambda_i\Omega u_i\right\| \\ &\leq C\left\|\sum_{i=1}^{n}\lambda_i u_i\right\|+\left\|\sum_{i=1}^{n}\lambda_i(\Omega-L)u_i\right\|,\end{aligned}$$

hence

$$\nabla_{[\lambda u]}\Omega \leq C'\|\lambda\|_p$$

□

Lemma 4.7. *Let Ω be a contractive centralizer on a Köthe space $\mathcal{K}$. Then there exists a constant $c > 0$ such that for every disjointly supported normalized sequence (v_i) in $\mathcal{K}$ and every $n \in \mathbb{N}$, we have*

$$\left\|\Omega\left(\sum_{i=1}^{n}v_i\right)-\sum_{i=1}^{n}\Omega(v_i)\right\| \leq c\left\|\sum_{i=1}^{n}v_i\right\|+\nabla_{[(v_i)_1^n]}\Omega.$$

Proof. Let (ϵ_i) be a sequence of signs. Let $v=\sum_{i=1}^{n}v_i$ and $\sum_{i=1}^{n}\epsilon_i v_i=\epsilon v$ for some function ϵ taking values ± 1.

$$\left\|\Omega(\epsilon v)-\epsilon\sum_{i=1}^{n}\Omega(v_i)\right\| \leq \|\Omega(\epsilon v)-\epsilon\Omega(v)\|+\left\|\epsilon\Omega(v)-\epsilon\sum_{i=1}^{n}\Omega(v_i)\right\|$$

Thus the centralizer Ω verifies for some constant $c > 0$,

$$\left\|\Omega(\epsilon v)-\epsilon\sum_{i=1}^{n}\Omega(v_i)\right\| \leq c\|v\|+\left\|\Omega(v)-\sum_{i=1}^{n}\Omega(v_i)\right\|$$

Since Ω is contractive, then also the $\Omega(v_i)$ are disjointly supported. Applying to $\epsilon_i v_i$ instead of v_i,

$$\left\|\Omega(v)-\sum_{i=1}^{n}\Omega(v_i)\right\| \leq c\|v\|+\left\|\Omega\left(\sum_{i=1}^{n}\epsilon_i v_i\right)-\sum_{i=1}^{n}\epsilon_i\Omega(v_i)\right\|$$

By taking the average, we obtain

$$\left\|\Omega(v)-\sum_{i=1}^{n}\Omega(v_i)\right\| \leq c\|v\|+\nabla_{[(v_i)_i^n]}\Omega$$

□

Lemma 4.8. *Let Ω be a contractive centralizer on a Köthe space $\mathcal{K}$ satisfying a lower q-estimate, and let $u=(u_n)_n$ be a disjointly supported normalized sequence of vectors. Suppose that there is a constant $C > 0$ such that*

$$\nabla_{[\lambda u]}\Omega \leq C\|\lambda\|_q$$

for every $\lambda=(\lambda_k)_k \in c_{00}$. Then the restriction of Ω to the closed linear span $[u_n]$ is trivial.

Proof. Let $\lambda = (\lambda_k)_k \in c_{00}$ and (ϵ_i) be a sequence of signs. It follows from the previous lemma that for some constant $c > 0$ and every $n \in \mathbb{N}$

$$\left\|\Omega\left(\sum_{i=1}^{n}\lambda_i u_i\right) - \sum_{i=1}^{n}\Omega(\lambda_i u_i)\right\| \;\leq\; c\left\|\sum_{i=1}^{n}\lambda_i u_i\right\| + \nabla_{[\lambda u]}\Omega \leq c\left\|\sum_{i=1}^{n}\lambda_i u_i\right\| + C\|\lambda\|_q$$

Since $\mathcal{K}$ satisfies a lower q-estimate

$$\left\|\Omega\left(\sum_{i=1}^{n}\lambda_i u_i\right) - \sum_{i=1}^{n}\Omega(\lambda_i u_i)\right\| \;\leq\; C'\left\|\sum_{i=1}^{n}\lambda_i u_i\right\|$$

Then $\|\Omega - L\| \leq C'$, where L is a linear map such that $L(u_i) = \Omega(u_i)$. □

5. Disjoint super singularity

It is part of the folklore that ultrapowers of Banach lattices are again Banach lattices. Thus, it makes sense to define an operator $T : \mathcal{K} \to Y$ to be *super-disjointly singular* if every ultrapower of T is disjointly singular; this means that for every sequence of subspaces $E_n \subseteq \mathcal{K}$ that are generated by disjointly supported elements and so that $\dim E_n = n$ there is a sequence (F_n) of subspaces, $F_n \subset E_n$ generated by disjointly supported elements such that $\dim F_n \to \infty$ and $\lim \|T_{|F_n}\| \to 0$. To transplant these ideas to the domain of quasi-linear maps Ω on Köthe function spaces it will be useful to define the *modulus of superdisjoint singularity* of a quasi-linear map Ω as

$$\psi_\Omega(n) = \inf \operatorname{dist}(\Omega|_{E_n}, L(E_n, Y)),$$

where the infimum is taken over all n-dimensional subspaces E_n of $\mathcal{K}$ generated by disjointly supported vectors. One has:

Lemma 5.1. *Let $\Omega : \mathcal{K} \to Y$ be a quasi-linear map defined on a Köthe space. The following are equivalent*

(1) *All ultrapowers of Ω are disjointly singular.*
(2) $\lim \psi_\Omega(n) = +\infty$.

If Ω is a centralizer, the conditions above are equivalent to

(3) *The quotient map q_Ω is super-disjointly singular.*

Proof. Condition (1) says that there does not exist $c > 0$ and a sequence of finite dimensional subspaces F_n of $Y \oplus_\Omega \mathcal{K}$ such that $E_n = q_\Omega(F_n)$ is generated by disjointly supported vectors so that $\|q_\Omega(x)\| \geq c\|x\|$ for every $x \in \cup F_n$. But if Ω is C-trivial on E_n, which is generated by the disjointly supported vectors $[u_i]_{i=1}^n$ then we claim that there is a linear map $L_n : E_n \to \mathcal{K}$ such that supp $L_n(x) \subseteq$ supp x for every $x \in E_n$ and $\|\Omega|_{E_n} - L_n\| \leq C$: Indeed assume L is linear such that $\|(\Omega - L)_{|E_n}\| \leq C$, and let G be the finite group of units generated by the vectors v_i that take value 1 on the support of u_i and -1 elsewhere); then it is enough to pick $L_n(x) = \mathrm{Ave}_{v\in G}\; vL(vx)$. The rest of the argument goes as in Lemma 4.4. Done that, $\Omega_{\mathfrak{U}}$ is trivial on $(E_n)_{\mathfrak{U}}$, which yields the equivalence between (1) and (2). The equivalence with (3) follows from Lemma 4.4. □

Definition 5.2. *A quasi-linear map (resp. a centralizer) $\Omega : \mathcal{K} \to Y$ on a Köthe space is said to be super disjointly singular if it satisfies the two (resp. three) equivalent conditions in the Lemma 5.1.*

It is clear that both singularity and super disjoint singularity imply disjoint singularity. We will present two proofs for the following fact.

Proposition 5.3. *The Kalton-Peck map $\mathcal{K}$ on L_p is super disjointly singular for $1 < p < \infty$.*

Proof. Assume there exist a linear map $L : E_n \to L_p$, where E_n is spanned by a finite sequence $u = (u_i)$, and a constant $C > 0$ so that $||\mathcal{K}_{|E_n} - L|| \leq C$. By the proof of the previous lemma we may assume that supp $L(u_i) \subseteq$ supp u_i for all i. Put $\Omega' = \mathcal{K}_{|E_n} - L$ to get

$$p^{-1}n^{1/p}\log n \leq \nabla_{[u]}\mathcal{K} = \nabla_{[u]}\Omega' \leq \text{Ave}_\epsilon\|\Omega'(\sum \epsilon_i u_i)\| + \text{Ave}_\epsilon\|\sum \epsilon_i \Omega' u_i\| \leq 2Cn^{1/p},$$

which is impossible. □

Two functions $f, g : \mathbb{N} \to \mathbb{R}^+$ are called equivalent, and denoted $f \sim g$, if $0 < \liminf f(n)/g(n) \leq \limsup f(n)/g(n) < +\infty$. We recall from [10] the parameter

$$M_{\mathcal{K}}(n) = \sup\{\|x_1 + \ldots + x_n\| : x_1, \ldots, x_n \text{ disjoint in the unit ball of } \mathcal{K}\}.$$

The interest of this parameter lies in [10, Proposition 5.3]:

Proposition 5.4. *Let (X_0, X_1) be an interpolation couple of two Köthe function spaces so that M_{X_0} and M_{X_1} are not equivalent. Let $0 < \theta < 1$. Assume that X_θ is reflexive, that $M_W \sim M_{X_\theta}$ for every infinite-dimensional subspace generated by a disjoint sequence $W \subset X_\theta$, and $M_{X_\theta} \sim M_{X_0}^{1-\theta} M_{X_1}^{\theta}$. Then Ω_θ is disjointly singular.*

We observe that:

Lemma 5.5. *Let X be a Köthe space. Then $M_X \sim M_{X_\mathcal{U}}$*

Proof. Since $X \subset X_\mathcal{U}$ it is clear that $M_X \leq M_{X_\mathcal{U}}$. Given n, pick $u^k = [u_i^k] \in X_\mathcal{U}$ for $1 \leq k \leq n$, disjointly supported so that $M_{X_\mathcal{U}}(n) \sim \|u^1 + \cdots + u^n\|$ (we can freely assume that all u_i^k are norm one elements). Since u^t and u^s are disjointly supported this means that the set of all i so that u_i^s and u_i^t are disjointly supported belongs to $\mathcal{U}$. And the same for the set A of all i so that all $\{u_i^k, 1 \leq i \leq n\}$ are disjointly supported. Since $B = \{j : \|u^1(j) + \cdots + u^n(j)\| \geq M_{X_\mathcal{U}}(n) - \varepsilon\} \in \mathcal{U}$, also $A \cap B \in \mathcal{U}$. Thus, for any $i \in A \cap B$ we have $M_{X_\mathcal{U}}(n) - \varepsilon \leq \|u^1(i) + \cdots + u^n(i)\| \leq M_X(n)$. □

In the case of Köthe spaces, complex interpolation is actually simple factorization. Recall that given two Köthe function spaces Y, Z we define the space

$$YZ = \{yz : y \in Y, z \in Z\}$$

endowed with the quasi-norm $\|x\| = \inf \|y\|_Y \|z\|_Z$ where the infimum is taken on all factorizations as above. Now, assuming that one of the spaces X_0, X_1 has the Radon-Nikodym property, the Lozanovskii decomposition formula allows us to show (see [24, Theorem 4.6]) that the complex interpolation space X_θ is isometric to the space $X_0^{1-\theta}X_1^{\theta}$, with

$$\|x\|_\theta = \inf\{\|y\|_0^{1-\theta}\|z\|_1^{\theta} : y \in X_0, z \in X_1, |x| = |y|^{1-\theta}|z|^{\theta}\}.$$

If $a_0(x), a_1(x)$ is an $(1+\epsilon)$-optimal Lozanovskii decomposition for x then it is standard (see [10]) that

$$\Omega_\theta(x) = x \log \frac{|a_1(x)|}{|a_0(x)|}. \tag{1}$$

Lemma 5.6. *Let (X_0, X_1) be a couple of Köthe function spaces with non trivial concavity. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ then $(X_\theta)_\mathcal{U} = ((X_0)_\mathcal{U}, (X_1)_\mathcal{U})_\theta$ and $(\Omega_\theta)_\mathcal{U} = (\Omega_\mathcal{U})_\theta$.*

Proof. According to [33], given an interpolation couple (A, B) of Köthe spaces with non-trivial concavity their ultrapowers $(A_{\mathcal{U}}, B_{\mathcal{U}})$ form an interpolation couple. The point now is to show that $(X_0^{1-\theta}X_1^{\theta})_{\mathcal{U}} = (X_0)_{\mathcal{U}}^{1-\theta}(X_1)_{\mathcal{U}}^{\theta}$. Indeed, given $[x_i] \in (X_0^{1-\theta}X_1^{\theta})_{\mathcal{U}}$ pick an almost optimal factorization $x_i = y_i z_i$ and then $[x_i] = [y_i][z_i]$ is an almost optimal factorization. Conversely, if $x \in (X_0)_{\mathcal{U}}^{1-\theta}(X_1)_{\mathcal{U}}^{\theta}$ and set $x = [y_i][z_i]$ an almost optimal factorization then of course that $x_i = y_i z_i$ is not an almost optimal factorization for x_i, but it is so when the indices i belong to a certain element of $\mathcal{U}$, and thus $[x_i] \in (X_\theta)_{\mathcal{U}}$. The assertion about the induced centralizer follows from this. □

To apply the general criteria proved in [10] (see below) we need to analyze the estimate M_W associated to any subspace W generated by a sequence of disjoint vectors of $(X_\theta)_{\mathcal{U}}$. As a rule, it is false that the ultrapower of the interpolated space is the interpolated between ultrapowers. To overcome this we concentrate first on the test case in which X_θ is an $L_p(\mu)$-space. In this situation $M_W(n) \sim n^{1/p}$ for all subspaces W of $(X_\theta)_{\mathcal{U}}$ generated by disjointly supported vectors, and from Proposition 5.4 we deduce:

Proposition 5.7. *Let (X_0, X_1) be an interpolation couple of two Köthe function spaces and let $0 < \theta < 1$ so that X_θ is an $L_p(\mu)$-space. If*

$$\limsup \left(\left| \log \frac{M_{X_0}(n)}{M_{X_1}(n)} \right| \frac{n^{1/p}}{M_{X_0}(n)^{1-\theta} M_{X_1}(n)^{\theta}} \right) = +\infty, \tag{2}$$

then the induced centralizer Ω_θ on X_θ is super disjointly singular

Proof. Thanks to Lemma 5.5 we observe that the hypotheses and Proposition 5.4 imply that the centralizer $(\Omega_{\mathcal{U}})_\theta$ is disjointly singular. By Lemma 5.6 we conclude that Ω_θ is super disjointly singular. □

This provides the second proof that the Kalton-Peck centralizer is super disjointly singular on L_p-spaces. In the case of Köthe spaces on a discrete measure space (i.e. the unconditional basis case), as a consequence of the fact that (disjoint) singularity and "block" singularity are equivalent, the conclusion of Proposition 5.7 still holds if one replaces the parameter M_X by the parameter M_X^s, where the supremum is over successive vectors instead of disjointly supported. Therefore:

- If $\mathcal{S}$ denotes the Schlumprecht space then $(\mathcal{S}, \mathcal{S}^*)_{1/2} = \ell_2$ then the associated centralizer is super disjointly singular. Since these are Köthe sequence spaces, it is also singular. This follows from the estimates $M_{\mathcal{S}}^s(n) = n$ and $M_{\mathcal{S}^*}^s(n) \sim \log_2(n)$.
- Let L_{p_0,p_1}, L_{p_1,q_1} be Lorentz function spaces. Then, $(L_{p_0,p_1}, L_{p_1,q_1})_\theta = L_{p,q}$ as in Section 4.1 and the associated centralizer is singular when $q_0 \neq q_1$ and super disjointly singular when $\min\{p_0, p_1\} \neq \min\{p_1, q_1\}$ as it follows from the estimate $M_{L_{p,q}}(n) = n^{\frac{1}{\min\{p,q\}}}$. Observe that in this case we require a variation of Proposition 5.7: it is not true now that "X_θ is an $L_p(\mu)$-space"; rather "X_θ is an $L_{p,q}(\mu)$-space" and thus their subspaces generated by disjointly supported vectors are $\ell_{p,q}$, whose parameters are the same as those of $L_{p,q}$.

Although it is easy to believe that disjoint singularity implies super disjoint singularity, it is not so:

Proposition 5.8. *There are singular centralizers on ℓ_2 that are not super disjointly singular.*

Proof. Let $1 \le p_1 < p_0 \le \infty$ and $0 < \theta < 1$. For $p^{-1} = (1-\theta)p_1^{-1} + \theta p_0^{-1}$. It follows from [12, Theorem 3.4] that $(\ell_{p_0}(\bigoplus \ell_2^n), \ell_{p_1}(\bigoplus \ell_2^n))_\theta = \ell_p(\bigoplus \ell_2^k)$ with associated centralizer

$$\Omega(x) = \left(\left(\frac{p}{p_1} - \frac{p}{p_0} \right) \log \left(\frac{\|x^k\|_2}{\|x\|} \right) x^k \right)_k .$$

The map Ω is not super disjointly singular since $\Omega(x) = 0$ for every $x \in \ell_2^k$ and every $k \in \mathbb{N}$. Let us take $p = 2$ at $\theta = 1/2$, and denote by Ω the respective centralizer on the Hilbert space. We show that Ω is singular. Suppose that Ω is trivial on an infinite dimensional subspace $W \subseteq \ell_2(\bigoplus \ell_2^n)$. By a perturbation argument (see [9, Lemma 5.6]) we may assume that $W = [y_n]$ for a normalized successive vectors with respect to the natural FDD. For every $n \in \mathbb{N}$, we write $y_n = \sum_{j \in J_n} y_{nj}$, where each y_{nj} belongs to a summand and they are successive. Hence for a norm one vector $\sum_n \alpha_n y_n$ (i.e. when $(\alpha_n) \in \ell_2$)

$$\begin{aligned}
\Omega\left(\sum_n \alpha_n y_n\right) &= \Omega\left(\sum_n \sum_{j\in J_n} \alpha_n y_{nj}\right) \\
&= c \sum_n \sum_{j \in J_n} \log(\|\alpha_n y_{nj}\|_2) \alpha_n y_{nj} \\
&= c \sum_n \alpha_n \log(|\alpha_n|) y_n + c \sum_n \sum_{j\in J_n} \alpha_n \log(\|y_{nj}\|_2) y_{nj}.
\end{aligned}$$

The right-hand side is a linear map on W and then trivial. Since we are considering the ℓ_2-sum of the vectors (y_n), the first summand is just the Kalton-Peck map defined on (y_n) instead of the canonical basis. Hence the left-hand side is not trivial, and so the sum above. We then get a contradiction by assuming Ω trivial on W. □

Remark 5.9. *Also for $p_1 < p_0 < 2$ and $0 < \theta < 1$ the centralizer Ω of Proposition 5.8 is disjointly singular by Proposition 5.4*

A few more precise estimates can be presented. Observe first that given a centralizer $\Omega : \mathcal{K} \curvearrowright \mathcal{K}$ on a Köthe function space then for n disjoint vectors $(u_i)_{i=1}^n$ in the unit sphere one has

$$\left\| \Omega(\sum_{i=1}^n u_i) - \sum_{i=1}^n \Omega(u_i) \right\| \le \|\Omega_{|[u_1,\dots,u_n]} - L\| \left\| \sum_{i=1}^n u_i \right\|$$

for any linear map $L : [u_n] \to \mathcal{K}$. Let us invoke now the estimate [10, Proposition 5.1]: given two Köthe spaces (X_0, X_1) (on the same base space), fixing $0 < \theta < 1$ and considering Ω_θ the induced centralizer on $X_\theta = X_0^{1-\theta} X_1^\theta$ then

$$\left\| \Omega_\theta(\sum_{i=1}^n u_i) - \sum_{i=1}^n \Omega_\theta(u_i) - \log \frac{M_{X_0}(n)}{M_{X_1}(n)} \Big(\sum_{i=1}^n u_i \Big) \right\| \le 3 \frac{M_{X_\theta}(n)}{\max\{\theta, 1-\theta\}}.$$

Therefore, in the same conditions as above, we get

$$\left| \log \frac{M_{X_0}(n)}{M_{X_1}(n)} \right| \left\| \sum_{i=1}^n u_i \right\| \le \|(\Omega_\theta)_{|[u_1,\dots,u_n]} - L\| \left\| \sum_{i=1}^n u_i \right\| + 3 \frac{M_{X_\theta}(n)}{\max\{\theta, 1-\theta\}}.$$

Let us define a new parameter

$$m_{\mathcal{K}}(n) = \inf\{\|x_1 + \ldots + x_n\| : x_1, \dots, x_n \text{ disjoint in the unit sphere of } \mathcal{K}\}.$$

We obtain

$$\left|\log \frac{M_{X_0}(n)}{M_{X_1}(n)}\right| m_{X_\theta}(n) \leq \left\|(\Omega_\theta)_{|[u_1,\dots,u_n]} - L\right\| M_{X_\theta}(n) + 3\frac{M_{X_\theta}(n)}{\max\{\theta, 1-\theta\}}.$$

Since this holds for all disjoint vectors $u_1, \dots, u_n$ in the sphere, we deduce

$$\left|\log \frac{M_{X_0}(n)}{M_{X_1}(n)}\right| m_{X_\theta}(n) \leq \psi_{\Omega_\theta}(n) M_{X_\theta}(n) + 3\frac{M_{X_\theta}(n)}{\max\{\theta, 1-\theta\}},$$

which yields

$$\left|\log \frac{M_{X_0}(n)}{M_{X_1}(n)}\right| \frac{m_{X_\theta}(n)}{M_{X_\theta}(n)} - \frac{3}{\max\{\theta, 1-\theta\}} \leq \psi_{\Omega_\theta}(n).$$

Observe that $m_{\mathcal{K}}(n) M_{\mathcal{K}^*}(n) \geq n$ for every Köthe function space. In particular:

- For $\Omega_{1/2}$ the centralizer obtained on $\ell_2 = (\mathcal{S}, \mathcal{S}^*)_{1/2}$ when $\mathcal{S}$ is the Schreier space, one gets
$$|\log n - \log\log n| - 6 \leq \psi_{\Omega_{1/2}}(n)$$
which shows that also the centralizers $\Omega_{1/2}$ are super-disjointly singular.
- In general, under a few minimal conditions on a Köthe function space $\mathcal{K}$ with base space S one has $(\mathcal{K}, \mathcal{K}^*)_{1/2} = L_2(S)$ (see [10, Proposition 6.2]), and thus if $M_{\mathcal{K}}$ and $M_{\mathcal{K}^*}$ are not equivalent then the induced centralizer on $L_2(S)$ is super disjointly singular.
- For $\mathcal{K}_p$ the Kalton-Peck map on L_p obtained from $X_0 = L_1, X_1 = L_\infty$ and $\theta = 1/p^*$ one gets
$$\log n - \frac{3}{\min\{\theta, 1-\theta\}} \leq \psi_{\mathcal{K}_p}(n)$$
which, as promised, shows again that $\mathcal{K}_p$ is super disjointly singular.
- Assume more generally that $p > 1$ and $\mathcal{K}$ is a p-convex Köthe space with base space S. Then $\mathcal{K} = (L_\infty(S), \mathcal{K}^p)_{1/p}$, where $\mathcal{K}^p$ denotes the p-concavification of $\mathcal{K}$, and [10, Proposition 3.7], this induces as centralizer the map $p\mathcal{K}$, where $\mathcal{K}(f) = f\log(|f|/\|f\|)$ is the Kalton-Peck map on $\mathcal{K}$. Since $M_{\mathcal{K}^p}(n) = M_{\mathcal{K}}(n)^p$, we obtain the following criteria for the super DSS property of Kalton-Peck map:
$$|\log M_{\mathcal{K}}(n)| \frac{m_{\mathcal{K}}(n)}{M_{\mathcal{K}}(n)} - \frac{3/p}{\max\{1/p, 1/p'\}} \leq \psi_{\mathcal{K}}(n)$$
- If for example $\mathcal{S}^{(p)}$ is the p-convexification of Schreier space then since $M_{\mathcal{S}^{(p)}}(n) = M_{\mathcal{S}}(n)^{1/p} = n^{1/p}$ and $m^s_{\mathcal{S}^{(p)}}(n) = m_{\mathcal{S}}(n)^{1/p} = (n/\log n)^{1/p}$ we obtain
$$\frac{1}{p}|\log n|^{1/p'} - \frac{3/p}{\max\{1/p, 1/p'\}} \leq \psi_{\mathcal{K}}(n)$$
and deduce that Kalton-Peck map is super disjointly singular on $\mathcal{S}^{(p)}$.
- The same estimates hold, in the case of Köthe sequence spaces, using the successive vectors versions $M^s_X(n)$, $m^s_X(n)$ and $\psi^s_X(n)$ of the parameters and of the modulus. So for example, if $\mathcal{S}^{(p)}$ is the p-convexification of Schlumprecht space then since $M^s_{\mathcal{S}^{(p)}}(n) = n^{1/p}$ and $m^s_{\mathcal{S}^{(p)}}(n) = (n/\log n)^{1/p}$ we also obtain
$$\frac{1}{p}|\log n|^{1/p'} - \frac{3/p}{\max 1/p, 1/p'} \leq \psi^s_{\mathcal{K}}(n)$$
and deduce that Kalton-Peck map is "super successively singular", therefore disjointly singular, hence singular on $\mathcal{S}^{(p)}$.

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