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ORIGINAL PAPER

On the Banach-Stone theorem for algebras of holomorphic germs

Domingo García[1] · **Manuel Maestre**[1] ·
Daniela M. Vieira[2]

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Abstract Given two balanced compact subsets K and L of two Banach spaces X and Y respectively such that every continuous m-homogeneous polynomial on X^{**} and on Y^{**} is approximable, for all $m \in \mathbb{N}$, we characterize when the algebras of holomorphic germs $\mathcal{H}(K)$ and $\mathcal{H}(L)$ are topologically algebra isomorphic. This happens if and only if the polynomial hulls of K and L on their respective biduals are biholomorphically equivalent.

Keywords Holomorphic germs · Holomorphic functions · Algebras · Banach spaces

Mathematics Subject Classification 46G20 · 46E25

1 Introduction

In 1932, Banach [4] proved that two compact metric spaces K and L are homeomorphic if and only if the Banach spaces $\mathcal{C}(K)$ and $\mathcal{C}(L)$ are isometrically isomorphic. Stone [16]

Dedicated to Professor José Bonet on his 60th birthday.

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✉ Domingo García
Domingo.Garcia@uv.es

Manuel Maestre
manuel.maestre@uv.es

Daniela M. Vieira
danim@ime.usp.br

1 Departamento de Análisis Matemático, Universidad de Valencia, Doctor Moliner 50, 46100 Burjasot, Valencia, Spain

2 Departamento de Matemática, Universidade de São Paulo, Rua do Matão, 1010, São Paulo CEP 05508-090, Brazil

generalized this result to arbitrary compact Hausdorff topological spaces, the well-known Banach-Stone theorem. This result found a large number of extensions, generalizations and variants in many different contexts, and some of them can be found in [12].

Recently, Carando and Muro [7] studied this theorem for the case of holomorphic functions of bounded type $\mathcal{H}_b(U)$, when every m-homogeneous polynomial on the bidual of one of the spaces involved is approximable (see below for the definition). More precisely, they proved that if the algebras $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$, for U and V balanced open subsets of some Banach spaces X and Y respectively, are algebra isomorphic then their dual spaces X^* and Y^* are isomorphic, the converse being true when the domains are the whole spaces, generalizing a result of [6]. This is a Banach-Stone type theorem for holomorphic functions of bounded type.

In [19], Vieira proved another Banach-Stone theorem for the algebras of holomorphic functions $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$. This result holds whenever U and V are balanced and m-polynomially convex subsets (see definition below) of reflexive Banach spaces X and Y, respectively, for which all continuous m-polynomials are approximable (for all m).

Other algebras of holomorphic functions are studied by Vieira in [18,20].

Also, in [19] is obtained the same result for the algebras of holomorphic germs $\mathcal{H}(K)$ and $\mathcal{H}(L)$, where K and L are balanced and m-polynomially convex compact subsets of reflexive Banach spaces X and Y, respectively, such that every continuous m-homogeneous polynomial on X and on Y is approximable, for all $m \in \mathbb{N}$. In this paper we give a full characterization of those algebra isomorphisms.

The result in [19] concerning $\mathcal{H}(K)$ and $\mathcal{H}(L)$ rest on the characterizations of homomorphisms between $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$ obtained in [19].

To obtain the result for nonreflexive Banach spaces we use new characterizations of homomorphisms between $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$ obtained in [7].

We refer to [11] or [15] for background information on infinite dimensional complex analysis.

2 Preliminaires

Throughout this paper X and Y will always denote complex Banach spaces. We denote by X^{**} the bidual of X, and $J_X : X \longrightarrow X^{**}$ denotes the canonical inclusion. We also denote by B_X the open unit ball of X and, for a given $x \in X$, $r > 0$, $B_X(x, r)$ denotes the open ball of radius r centered at x. Let $\mathcal{P}(X)$ denote the normed space of all continuous complex polynomials $P : X \longrightarrow \mathbb{C}$, endowed with the norm $\|P\| = \sup\{|P(x)| : \|x\| \leq 1\}$. For each $m \in \mathbb{N}$, $\mathcal{P}(^m X)$ denotes the subspace of $\mathcal{P}(X)$ of all m-homogeneous polynomials, and $\mathcal{P}_f(^m X)$ denotes the subspace of $\mathcal{P}(^m X)$ of all m-homogeneus polynomials of finite type, i. e. finite linear combinations of products of m continuous linear functionals on X. We recall that an m-homogeneous continuous polynomial on a Banach space X is called **approximable** if it is in the norm-closure of $\mathcal{P}_f(^m X)$.

Let U be an open subset of a Banach space X. We say that a set $A \subset U$ is U-bounded if A is bounded and there exists $\varepsilon > 0$ such that $A + B(0, \varepsilon) \subset U$. Every open set U admits a fundamental sequence of U-bounded sets, $(A_j)_{j \in \mathbb{N}}$, such that each U-bounded set is contained in some A_j. For example, we can take $A_j = \{x \in X : \|x\| \leq j\, ,\, d(x, X \backslash U) \geq \frac{1}{j}\}$.

We will denote by $\mathcal{H}_b(U)$ the algebra of all holomorphic functions $f : U \longrightarrow \mathbb{C}$ which are bounded on every U-bounded set. Such functions are called holomorphic functions of bounded type. We denote by τ_b the topology on $\mathcal{H}_b(U)$ of the uniform convergence on all U-bounded sets. Then $(\mathcal{H}_b(U), \tau_b)$ is a Fréchet algebra.

Let $A \subset X$ be a bounded set. We define

$$\widehat{A}''_{\mathcal{P}(X)} := \Big\{ z \in X^{**} \,:\, |AB(P)(z)| \le \sup_A |P|, \text{ for all } P \in \mathcal{P}(X) \Big\},$$

where $AB(P)$ denotes de Aron-Berner extension of P, see [2]. We also denote $\widehat{A}_{\mathcal{P}(X)} = \widehat{A}''_{\mathcal{P}(X)} \cap X$.

A compact subset K of a Banach space X is called polynomially convex if $\widehat{K}_{\mathcal{P}(X)} = K$. Changing $\mathcal{P}(X)$ to $\mathcal{P}(^m X)$ then it is said m-polynomially convex.

If U is an open subset of X, let $(A_n)_{n\in\mathbb{N}}$ be a fundamental sequence of U-bounded sets. Then we define

$$\widehat{U}''_{\mathcal{P}(X)} := \bigcup_{n\in\mathbb{N}} (\widehat{A_n})''_{\mathcal{P}(X)}.$$

In [7], it is shown that $\widehat{U}''_{\mathcal{P}(X)}$ is an open subset of X^{**}. Moreover, the following extension result is proved.

Theorem 1 [7, Proposition 3.3] *Let U be a bounded and balanced open subset of a symmetrically regular Banach space X. Then there exists an AB-extension operator from $\mathcal{H}_b(U)$ to $\mathcal{H}_b(\widehat{U}''_{\mathcal{P}(X)})$.*

By an *AB-extension operator* we mean a continuous homomorphism such that, for every $x \in U$, there exists $r > 0$ such that $AB(f)$ coincides with the Aron-Berner extension of f on $B_{X^{**}}(J_X(x), r)$. A Banach space X is said to be (symmetrically) regular if every continuous (symmetric) linear mapping $T : X \to X^*$ is weakly compact. Recall that T is symmetric if $< Tx, y >=< x, Ty >$ for all $x, y \in X$.

In the next proposition we present some properties of the set $\widehat{K}''_{\mathcal{P}(X)}$, when K is a compact subset of a Banach space X.

Proposition 1 *Let X be a Banach space, let $K \subset X$ be a compact set and let $U_k = K + \frac{1}{k} B_X$, for all $k \in \mathbb{N}$. Then*

1. $J_X\big(\widehat{K}_{\mathcal{P}(X)}\big) = \widehat{K}''_{\mathcal{P}(X)}$
2. $\widehat{K}''_{\mathcal{P}(X)}$ *is a compact subset of* $\big(\widehat{U_k}\big)''_{\mathcal{P}(X)}$
3. $\widehat{K}''_{\mathcal{P}(X)} = \bigcap_{k\in\mathbb{N}} \big(\widehat{U_k}\big)''_{\mathcal{P}(X)}$.

Proof 1. By the Hahn-Banach theorem, it follows that $\widehat{K}''_{\mathcal{P}(X)} \subset \overline{\Gamma}^{w^*}(K)$, where Γ denotes the absolutely convex hull and w^* is the weak-star topology $w(X^{**}, X^*)$. So if $z \in \widehat{K}''_{\mathcal{P}(X)}$, then there exists a net $(x_\alpha) \subset \Gamma(K)$ such that $x_\alpha \xrightarrow{w^*} z$. But since $\Gamma(K)$ is norm-compact, there exists $x \in X$ and a subnet (x_β) such that $x_\beta \xrightarrow{\|\cdot\|} x$. Then it follows that $z = J_X(x)$. The other inclusion is obvious.

2. Since K is compact, it follows by [15, Proposition 11.1] that $\widehat{K}_{\mathcal{P}(X)}$ is compact. Then the assertion follows by the previous item.
3. Recall that $U_k = K + \frac{1}{k} B_X$. Since $\widehat{K}''_{\mathcal{P}(X)} \subset (\widehat{U_k})''_{\mathcal{P}(X)}$, for all $k \in \mathbb{N}$, it follows that $\widehat{K}''_{\mathcal{P}(X)} \subset \bigcap_{k\in\mathbb{N}} \big(\widehat{U_k}\big)''_{\mathcal{P}(X)}$. To show the opposite inclusion, let us fix $P \in \mathcal{P}(X)$ and denote by $c = \sup_K |P|$ and by $c_k = \sup_{U_k} |P|$. Observe that $(c_k)_{k\in\mathbb{N}}$ is a bounded decreasing sequence. We will show that $c = \lim_{k\to\infty} c_k$. Given $\varepsilon > 0$, since P is uniformly continuous on K, there exists $k \in \mathbb{N}$ such that if $\|x - y\| < \frac{1}{k}$ then $|P(x) -$

$P(y)| < \varepsilon$. So, let $x \in K$ and $x_k \in B_X(0, \frac{1}{k})$. Then $y = x + x_k \in U_k$ and $\|x - y\| < \frac{1}{k}$ hence $|P(y)| < |P(x)| + \varepsilon$. This shows that $c \geq \inf_{k\in\mathbb{N}} c_k = \lim_{k\to\infty} c_k$. The other inequality is obvious. Now let $z \in \bigcap_{k\in\mathbb{N}} \left(\widehat{U_k}\right)''_{\mathcal{P}(X)}$. Then $|AB(P)(z)| \leq c_k$, for all $k \in \mathbb{N}$, which implies that $|AB(P)(z)| \leq c = \sup_K |P|$, and then $z \in \widehat{K}''_{\mathcal{P}(X)}$.

□

We thank Daniel Carando for pointing out item (1) of the previous proposition.

Let X be a Banach space, and $K \subset X$ a compact set. We define the algebra

$$h(K) = \bigcup\{\mathcal{H}(U) \, : \, U \supset K \text{ is open in } X\}.$$

Let $f_1, f_2 \in h(K)$ and U_1, U_2 be open subsets of X with $K \subset U_1$ and $K \subset U_2$, such that $f_1 \in \mathcal{H}(U_1)$ and $f_2 \in \mathcal{H}(U_2)$. We say that f_1 and f_2 are *equivalent* (and we write $f_1 \sim f_2$) if there is an open set $W \subseteq X$ with $K \subset W \subseteq U_1 \cap U_2$ such that $f_1 = f_2$ on W. Then $\sim$ is an equivalence relation in $h(K)$ and we denote $\mathcal{H}(K) = h(K)/\sim$. The elements of $\mathcal{H}(K)$ are called holomorphic germs. Finally, we endow $\mathcal{H}(K)$ with the locally convex inductive topology of the locally convex algebras $(\mathcal{H}(U), \tau_\omega)$, where U varies among the open subsets of X such that $K \subset U$, and we denote

$$(\mathcal{H}(K), \tau_\omega) = \underrightarrow{lim}_{U\supset K}(\mathcal{H}(U), \tau_\omega).$$

It has been proved [14, Theorem 7.1] that $\mathcal{H}(K)$ is a locally m-convex topological algebra. We recall that a topological algebra is called locally m-convex if there exists a basis for the neighborhoods of the origin consisting of convex sets with $V^2 \subset V$.

Let $U_n = K + \frac{1}{n} B_X$, for all $n \in \mathbb{N}$. Then

$$(\mathcal{H}(K), \tau_\omega) = \underrightarrow{lim}_{n\in\mathbb{N}}\mathcal{H}_b(U_n). \tag{1}$$

Actually we are going to use (1) as definition of τ_ω.

We will denote by i_n the canonical inclusion $i_n : \mathcal{H}_b(U_n) \hookrightarrow \mathcal{H}(K)$, for each $n \in \mathbb{N}$. $[f]$ will denote the elements of the algebra $\mathcal{H}(K)$, i.e., $[f] \in \mathcal{H}(K)$ if and only if there exists $n \in \mathbb{N}$ such that $f \in \mathcal{H}_b(U_n)$.

We refer to [5,11] or [14] for background information on the algebras of holomorphic germs.

One of the keys of the main result of this paper is the following proposition.

Proposition 2 *Let X be a Banach space such that every polynomial on X^{**} is approximable, and let $K \subset X$ be a balanced compact set. Then the algebras $\mathcal{H}(K)$ and $\mathcal{H}(\widehat{K}''_{\mathcal{P}(X)})$ are topologically algebra isomorphic.*

Proof Given $[f] \in \mathcal{H}(K)$ there exists $k \in \mathbb{N}$ such that $f \in \mathcal{H}_b(U_k)$. Since every polynomial on X^{**} is approximable, then X must be symetrically regular [7]. Now consider $AB(f) \in \mathcal{H}_b\big((\widehat{U_k})''_{\mathcal{P}(X)}\big)$, given by Theorem 1. Then we define $T : \mathcal{H}(K) \longrightarrow \mathcal{H}\big(\widehat{K}''_{\mathcal{P}(X)}\big)$ by $T([f]) = [AB(f)]$. It is not difficult to see that T is well defined, linear and injective. To show that T is surjective, let $[g] \in \mathcal{H}\big(\widehat{K}''_{\mathcal{P}(X)}\big)$. Then there exists $k \in \mathbb{N}$ such that $g \in \mathcal{H}_b\big(\widehat{K}''_{\mathcal{P}(X)} + \frac{1}{k} B_{X^{**}}\big)$. Let $l \geq k$ be such that $\widehat{K}''_{\mathcal{P}(X)} + \frac{1}{l} B_{X^{**}} \subset (\widehat{U_k})''_{\mathcal{P}(X)}$. Since every polynomial on X^{**} is approximable, we have that $g \in \mathcal{H}_{w^*u}\big(\widehat{K}''_{\mathcal{P}(X)} + \frac{1}{l} B_{X^{**}}\big)$. For $z \in \widehat{K}''_{\mathcal{P}(X)} + \frac{1}{l} B_{X^{**}}$, we can apply Lemma 2.1 of [3] to the restriction of g to a suitable ball, to obtain that $d^n(g)(z)$ is a w^*-continuous polynomial, for every n. By [21, Theorem 2] we can conclude that $g = AB(f)$, where $f = g \circ J_X|_{U_l}$.

It remains to show that T is an algebra isomorphism. By the definition of T, we have that the following diagram commutes, for all $k \in \mathbb{N}$

$$\begin{array}{ccc} \mathcal{H}(K) & \xrightarrow{T} & \mathcal{H}\big(\widehat{K}''_{\mathcal{P}(X)}\big) \\ {\scriptstyle i_k}\uparrow & & \uparrow{\scriptstyle j_k} \\ \mathcal{H}_b(U_k) & \xrightarrow{AB} & \mathcal{H}_b\big((\widehat{U_k})''_{\mathcal{P}(X)}\big) \end{array}$$

where i_k and j_k denote the canonical inclusions. Now, since AB is continuous, it follows that T is continuous. To see that T^{-1} is continuous, just observe that $T^{-1}([g])=[g \circ J_X|_{U_l}]$. □

3 Banach-Stone theorems

Let X and Y be Banach spaces, $K \subset X$ and $L \subset Y$ be compact sets. We say that K and L are biholomorphically equivalent if there exist open sets $U \subset X$ and $V \subset Y$ with $K \subset U$ and $L \subset V$ and a biholomorphic mapping $\varphi : U \longrightarrow V$ such that $\varphi(K) = L$.

We will need the next theorem, due to Grothendieck (see [13]).

Theorem 2 [13] *Let F be a Hausdorff locally convex space which is the union of an increasing sequence of Fréchet spaces $(F_n)_{n\in\mathbb{N}}$ and assume that each inclusion $i_n : F_n \longrightarrow F$ is continuous. Let $T : E \longrightarrow F$ be a continuous linear mapping of a Fréchet space E into F. Then there exists $n \in \mathbb{N}$ and a continuous linear mapping $T_n : E \longrightarrow F_n$ such that $i_n \circ T_n = T$.*

We will denote, as before, $U_n = K + \frac{1}{n}B_X$, for all $n \in \mathbb{N}$ and $V_m = L + \frac{1}{m}B_Y$ for all $m \in \mathbb{N}$. The mappings i_n and j_m will be the canonical inclusions $i_n : \mathcal{H}_b(U_n) \hookrightarrow \mathcal{H}(K)$ and $j_m : \mathcal{H}_b(V_m) \hookrightarrow \mathcal{H}(L)$, for each $n, m \in \mathbb{N}$.

Remark 3 Let X and Y be Banach spaces. Let $K \subset X$ and $L \subset Y$ be compact sets. Let us observe that if given a continuous homomorphism $A : \mathcal{H}(K) \longrightarrow \mathcal{H}(L)$ and an open set U with $K \subset U$ such that we can find an open set V with $L \subset V$ and a continuous homomorphism $B : \mathcal{H}_b(U) \longrightarrow \mathcal{H}_b(V)$ satisfying that the following diagram commutes

$$\begin{array}{ccc} \mathcal{H}(K) & \xrightarrow{A} & \mathcal{H}(L) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ \mathcal{H}_b(U) & \xrightarrow{B} & \mathcal{H}_b(V) \end{array}$$

where i and j are the corresponding canonical injections, then, given any other open set W such that $L \subset W \subset V$ and defined $\widetilde{B}(f) := B(f)|_W$ for $f \in \mathcal{H}_b(U)$, where $B(f)|_W$ is the restriction of $B(f)$ to W, the diagram

$$\begin{array}{ccc} \mathcal{H}(K) & \xrightarrow{A} & \mathcal{H}(L) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ \mathcal{H}_b(U) & \xrightarrow{\widetilde{B}} & \mathcal{H}_b(W) \end{array}$$

commutes too.

Next we present the main theorem of the article, which is a Banach-Stone type theorem for algebras of holomorphic germs.

Theorem 4 *Let X and Y be Banach spaces such that every continuous m-homogeneous polynomial on X^{**} and on Y^{**} is approximable, for all $m \in \mathbb{N}$. Let $K \subset X$ and $L \subset Y$ be balanced compact sets. Then the following conditions are equivalent.*

1. *$\mathcal{H}(K)$ and $\mathcal{H}(L)$ are topologically algebra isomorphic.*
2. *$\widehat{K}''_{\mathcal{P}(X)}$ and $\widehat{L}''_{\mathcal{P}(Y)}$ are biholomorphically equivalent.*
3. *$\mathcal{H}(\widehat{K}''_{\mathcal{P}(X)})$ and $\mathcal{H}(\widehat{L}''_{\mathcal{P}(Y)})$ are topologically algebra isomorphic.*

Proof (1)$\Rightarrow$ (2) Let $T : \mathcal{H}(K) \longrightarrow \mathcal{H}(L)$ be a topological algebra isomorphism. By Theorem 2 (see also [9, Theorem 3.1]), for each $k \in \mathbb{N}$ there exist $m_k \in \mathbb{N}$ and a continuous homomorphism $T_k : \mathcal{H}_b(U_k) \longrightarrow \mathcal{H}_b(V_{m_k})$ such that $T \circ i_k = j_{m_k} \circ T_k$ and actually, by Remark 3, the sequence (m_k) can be chosen to be strictly increasing. Then the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{H}(K) & \xrightarrow{T} & \mathcal{H}(L) \\ {\scriptstyle i_k}\uparrow & & \uparrow{\scriptstyle j_{m_k}} \\ \mathcal{H}_b(U_k) & \xrightarrow{T_k} & \mathcal{H}_b(V_{m_k}) \end{array}$$

Since T_k is multiplicative, it follows from [7, Lemma 4.6] that there is a holomorphic mapping $\varphi_k : \big(\widehat{V_{m_k}}\big)''_{\mathcal{P}(Y)} \longrightarrow \big(\widehat{U_k}\big)''_{\mathcal{P}(X)}$ such that $\overline{T_k(f)} = \bar{f} \circ \varphi_k$, for all $f \in \mathcal{H}_b(U_k)$, where the bar denotes the extension given by Theorem 1. Applying the same argument for $S = T^{-1}$, beginning with $\mathcal{H}_b(V_{m_k})$, we find an integer n_k, that again by Remark 3, it can be chosen strictly bigger than n_{k-1} and a holomorphic mapping $\psi_k : \big(\widehat{U_{n_k}}\big)''_{\mathcal{P}(X)} \longrightarrow \big(\widehat{V_{m_k}}\big)''_{\mathcal{P}(Y)}$ such that

$$\begin{array}{ccc} \mathcal{H}(L) & \xrightarrow{S} & \mathcal{H}(K) \\ {\scriptstyle j_{m_k}}\uparrow & & \uparrow{\scriptstyle i_{n_k}} \\ \mathcal{H}_b(V_{m_k}) & \xrightarrow{S_k} & \mathcal{H}_b(U_{n_k}) \end{array}$$

is commutative, where $S_k : \mathcal{H}_b(V_{m_k}) \longrightarrow \mathcal{H}_b(U_{n_k})$ is such that $\overline{S_k(g)} = \bar{g} \circ \psi_k$, for all $g \in \mathcal{H}_b(V_{m_k})$.

Since both diagrams are commutative, it follows that $i_k = i_{n_k} \circ S_k \circ T_k$, for all $k \in \mathbb{N}$. Then for $f \in X^* \subseteq \mathcal{H}_b(U_k)$ we have that $[f] = [S_k(T_k(f))]$, and therefore by the Identity Principle we have the equality $f = S_k(T_k(f))$ on U_{n_k}. Hence $\bar{f} = \overline{S_k(T_k(f))}$ on $\big(\widehat{U_{n_k}}\big)''_{\mathcal{P}(X)}$, that is, $\bar{f} = \bar{f} \circ \varphi_k \circ \psi_k$ on $\big(\widehat{U_{n_k}}\big)''_{\mathcal{P}(X)}$, for all $f \in X^*$. This shows that for each $z \in \big(\widehat{U_{n_k}}\big)''_{\mathcal{P}(X)}$, we have that $z(f) = (\varphi_k \circ \psi_k)(z)(f)$, for all $f \in X^*$, that is, $\varphi_k \circ \psi_k : \big(\widehat{U_{n_k}}\big)''_{\mathcal{P}(X)} \longrightarrow \big(\widehat{U_k}\big)''_{\mathcal{P}(X)}$ is the inclusion mapping.

Observe now that $T([f]) = [T_1(f)] = [T_k(f)]$, for all $f \in X^*$. Since V_{m_k} is connected, we conclude that $T_1(f) = T_k(f)$ on V_{m_k}, and then $\bar{f} \circ \varphi_1 = \overline{T_1(f)} = \overline{T_k(f)} = \bar{f} \circ \varphi_k$, for all $f \in X^*$. Hence $\varphi_1 = \varphi_k$ on $\big(\widehat{V_{m_k}}\big)''_{\mathcal{P}(Y)}$. By the same arguments we prove that $\psi_1 = \psi_k$ on each $\big(\widehat{U_{n_k}}\big)''_{\mathcal{P}(X)}$.

Next we are going to show that $\varphi_1(\widehat{L}''_{\mathcal{P}(Y)}) \subset \widehat{K}''_{\mathcal{P}(X)}$. By Proposition 1(2), we have that $\widehat{L}''_{\mathcal{P}(Y)} \subset \big(\widehat{V_{m_k}}\big)''_{\mathcal{P}(Y)}$ and hence $\varphi_1(\widehat{L}''_{\mathcal{P}(Y)}) \subset \varphi_1(\big(\widehat{V_{m_k}}\big)''_{\mathcal{P}(Y)}) = \varphi_k(\big(\widehat{V_{m_k}}\big)''_{\mathcal{P}(Y)}) \subset \big(\widehat{U_k}\big)''_{\mathcal{P}(X)}$, for all $k \in \mathbb{N}$. Then it follows that $\varphi_1(\widehat{L}''_{\mathcal{P}(Y)}) \subset \bigcap_{k\in\mathbb{N}} \big(\widehat{U_k}\big)''_{\mathcal{P}(X)} = \widehat{K}''_{\mathcal{P}(X)}$, where the last equality follows by Proposition 1(3). And by the same arguments we show that $\psi_1(\widehat{K}''_{\mathcal{P}(X)}) \subset \widehat{L}''_{\mathcal{P}(Y)}$.

If we set $V = \varphi_{n_1}^{-1}\big((\widehat{U_{n_1}})''_{\mathcal{P}(X)}\big)$, $U = \psi_1^{-1}\big((\widehat{V_{m_{n_1}}})''_{\mathcal{P}(Y)}\big)$, $\varphi = \varphi_{n_1}|_V : V \longrightarrow U$ and $\psi = \psi_1|_U : U \longrightarrow V$, then we have that φ and ψ are bijective holomorphic functions such that $\varphi^{-1} = \psi$, and $\varphi(\widehat{L}''_{\mathcal{P}(Y)}) = \widehat{K}''_{\mathcal{P}(X)}$.

(2)⇒ (3) Since the compact sets $\widehat{K}''_{\mathcal{P}(X)}$ and $\widehat{L}''_{\mathcal{P}(Y)}$ are biholomorphically equivalent, by the same arguments of [18, Theorem 16], we can show that the algebras $\mathcal{H}\big(\widehat{K}''_{\mathcal{P}(X)}\big)$ and $\mathcal{H}(\widehat{L}''_{\mathcal{P}(Y)})$ are topologically isomorphic.

(3)⇒ (1) Follows by Proposition 2. □

By using the notion of polynomially convex we can state the following corollary.

Corollary 1 *Let X and Y be reflexive Banach spaces such that every continuous m-homogeneous polynomial on these spaces is approximable, for all $m \in \mathbb{N}$, and let $K \subset X$ and $L \subset Y$ be balanced and polynomially convex compact sets. Then the following conditions are equivalent.*

1. *The algebras $\mathcal{H}(K)$ and $\mathcal{H}(L)$ are topologically isomorphic.*
2. *The compact sets K and L are biholomorphically equivalent.*

Corollary 1 is a generalization of Theorem 3.8 of [19].

Finally we are going to give examples of Banach spaces satisfying conditions of Theorem 4.

Examples 5 1. In [17], Tsirelson constructed a reflexive Banach space X, with an unconditional Schauder basis, that does not contain any subspace which is isomorphic to c_0 or to any ℓ_p, $1 \leq p \leq \infty$. R. Alencar, R. Aron and S. Dineen proved in [1] that $\mathcal{P}_f(^m X)$ is norm-dense in $\mathcal{P}(^m X)$, for all $m \in \mathbb{N}$. This space is known as Tsirelson space.
2. In [8], Casazza et al. constructed a nonreflexive Banach space X that does not contain any subspace which is isomorphic to c_0 or to any ℓ_p, (see also [11, Example 2.43]). This space is known as Tsirelson-James space. In [10, Lemma 19], it is shown that every continuous m-homogeneous polynomial on X^{**} is approximable.

The Tsirelson space, Examples 5(1), is the main example of a space satisfying conditions of Corollary 1. We see that Theorem 4 improves [19, Theorem 3.8] not only in the geometric aspect of the compact set K, but is valid for a greater class of Banach spaces, as Examples 5 shows.

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