Rev Mat Complut (2010) 23: 435–452
DOI 10.1007/s13163-009-0026-7

Weakly continuous holomorphic functions on pseudoconvex domains in Banach spaces

Jorge Mujica · Daniela M. Vieira

Received: 14 September 2009 / Accepted: 3 December 2009 / Published online: 11 December 2009

Abstract This paper is devoted to the study of weakly continuous holomorphic functions on pseudoconvex domains in Banach spaces with Schauder bases. We establish the identity of pseudoconvex domains and domains of existence of weakly continuous holomorphic functions. We show that holomorphic functions can be approximated by weakly continuous holomorphic functions. We study the spectra of certain algebras of weakly continuous holomorphic functions.

Keywords Holomorphic function · Banach space · Schauder basis · Pseudoconvex domain

Mathematics Subject Classification (2000) Primary 46G20 · 46B15 · 32E30 · 32E40

Introduction

Let E be a Banach space, and let U be an open subset of E. Let $\mathcal{H}(U)$ denote the algebra of all complex-valued holomorphic functions on U, and let $\mathcal{H}_{wu}(U)$ denote

Research of J. Mujica supported by FAPESP, Brazil, Project 06/02378-7.
Research of D.M. Vieira supported by FAPESP, Brazil, Project 04/07441-7 and Project 06/02378-7.

J. Mujica (✉) · D.M. Vieira
IMECC-UNICAMP, Caixa Postal 6065, 13083-970 Campinas, SP, Brazil
e-mail: mujica@ime.unicamp.br

D.M. Vieira
e-mail: danim@ime.unicamp.br

Present address:
D.M. Vieira
Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, 05315-970 São Paulo, SP, Brazil
e-mail: danim@ime.usp.br

the subalgebra of all $f \in \mathcal{H}(U)$ that are weakly uniformly continuous on each U-bounded set.

The study of weakly continuous holomorphic functions on Banach spaces was initiated by Aron in [2], and the papers of Aron and Prolla [4], and Aron, Hervés and Valdivia [3] have become the standard references to the subject.

Most papers in this direction have been devoted to the study of functions defined on the entire space, and only the most recent papers have been devoted to the study of functions defined on absolutely convex open sets. We refer for instance to the recent papers of Burlandy and Moraes [6] and Carando, García and Maestre [7].

This paper is devoted to the study of weakly continuous holomorphic functions defined on pseudoconvex open sets. More precisely we study the algebra $\mathcal{H}_{\alpha wud}(U)$ of all $f \in \mathcal{H}(U)$ that are weakly uniformly continuous on each ball $B(x; r)$, with $x \in U$ and $0 < r < \alpha d_U(x)$, where $d_U(x)$ denotes the distance from x to the boundary of U, and $0 < \alpha \leq 1$.

When studying a holomorphic function f defined on an open set U it is often necessary to approximate f by holomorphic functions that depend only of finitely many variables. If U is balanced, then one can first approximate f by the polynomials of its Taylor series, and then use some approximation property on the space to approximate polynomials by polynomials of finite type. If U is not balanced, then the situation is much more complicated. In this paper we use the pseudoconvexity of the open set, together with the hypothesis of the Schauder basis, to approximate holomorphic functions by holomorphic functions that depend only of finitely many variables. We essentially follow the approach of Gruman and Kiselman [12], when they solved the Levi problem in Banach spaces with a Schauder basis, and the refinements introduced by other authors to solve several related problems.

This paper is organized as follows. In Sect. 1 we show that every pseudoconvex open set U in a Banach space with a Schauder basis is the domain of existence of a function $f \in \mathcal{H}_{\alpha wud}(U)$, for a suitable α, which depends only on the Schauder basis.

In Sect. 2 we show that if U is a pseudoconvex open set in a Banach space with a Schauder basis, and (x_n) is a sequence of distinct points of U which tend to the boundary of U, then for each sequence (b_n) in a Banach space F, there is a function $f \in \mathcal{H}_{\alpha wud}(U; F)$ such that $f(x_n) = b_n$ for every n, for a suitable α, which depends only on the Schauder basis.

In Sect. 3 we obtain several theorems on holomorphic approximation. Among other results we show that if U is a pseudoconvex open set in a Banach space with a Schauder basis, then, for a suitable α, $\mathcal{H}_{\alpha wud}(U)$ is a sequentially dense subalgebra of $\mathcal{H}(U)$ for the compact-open topology.

Section 4 is devoted to a study of the spectrum of the Fréchet algebra $\mathcal{H}_{\alpha wud}(U)$, for a suitable α. We show that if U is a pseudoconvex open set in a Banach space with a Schauder basis, then each continuous complex homomorphism of $\mathcal{H}_{\alpha wud}(U)$ is the pointwise limit of a sequence of point evaluations.

For background information on the basic properties of holomorphic or plurisubharmonic functions on infinite dimensional spaces we refer to the books of Dineen [10] or Mujica [24].

1 The Levi problem

As far back as 1911 Levi [18] posed the problem as to whether every pseudoconvex domain in $\mathbb{C}^n$ is the domain of existence of a holomorphic function. In 1942 Oka [32] solved this problem for $n = 2$, and it was only in 1953–1954 that Oka [33], Bremermann [5] and Norguet [28] solved this problem for arbitrary n.

Since its solution, new versions of the Levi problem have naturally arisen. In 1972 Gruman and Kiselman [12] solved the Levi problem in the case of Banach spaces with a Schauder basis, and soon afterwards Noverraz [29] extended that result to the case of separable Banach spaces with the bounded approximation property. On the other hand Josefson [17] gave an example of a pseudoconvex domain in the nonseparable Banach space $c_0(I)$ (I uncountable) that is not the domain of existence of any holomorphic function.

In this section we are interested in determining whether every pseudoconvex domain in a Banach space with a Schauder basis is the domain of existence of a holomorphic function with some weak-continuity properties. We answer this question in the affirmative by proving that if U is a pseudoconvex open set in a Banach space with a Schauder basis, then U is the domain of existence of a function $f \in \mathcal{H}_{\alpha wud}(U)$ for a suitable α, which depends only on the Schauder basis.

Let E be a Banach space with a Schauder basis (e_n). Let E_n denote the subspace generated by $e_1, \ldots, e_n$, let $(\phi_n) \subset E'$ denote the sequence of coordinate functionals, and let $(T_n) \subset \mathcal{L}(E; E)$ denote the sequence of canonical projections, that is $T_n x = \sum_{i=1}^n \phi_i(x) e_i$ for every $x \in E$. Let $c \geq 1$ denote the *basis constant* and let $c_a \geq 1$ denote the *asymptotic basis constant*, that is,

$$c = \sup_{n \in \mathbb{N}} \|T_n\|, \qquad c_a = \limsup_{n \to \infty} \|T_n\|.$$

Observe that $1 \leq c_a \leq c < \infty$. The basis is said to be *monotone* if $c = 1$, and is said to be *asymptotically monotone* if $c_a = 1$. Clearly every monotone basis is asymptotically monotone, but while monotone bases are rather special, asymptotically monotone bases can be found everywhere. Indeed, an examination of the proof of a classical result of Mazur (see [19, Theorem 1.a.5]) shows the following theorem.

Theorem 1.1 *Every infinite dimensional Banach space contains a closed, infinite dimensional subspace with an asymptotically monotone Schauder basis.*

We remark that Theorem 1.1 was first proved by Day [9]. His proof is based on a generalization of the Borsuk-Ulam antipodal theorem.

If $f \in \mathcal{H}(U)$, then $r_c f(x)$ denotes the radius of convergence of the Taylor series of f at $x \in U$, and $r_b f(x)$ denotes the radius of boundedness of f at $x \in U$. Let $d_U(x)$ denote the distance from $x \in U$ to the boundary of U. Nachbin [27] has shown that $r_b f(x) = \min\{r_c f(x), d_U(x)\}$ for every $x \in U$ (see [27, Proposition 7.2] or [24, Theorem 7.13]). An open set U is said to be *pseudoconvex* if the function $-\log d_U$ is plurisubharmonic on U. An open set U is the *domain of existence of* f if and only if $r_c f(x) \leq d_U(x)$ for every $x \in U$.

Consider the sets

$$A_j = A_j(U) = \left\{x \in U : \sup_{n \geq j} \|T_n x - x\| < d_U(x)\right\},$$

$$B_j = B_j(U) = \{x \in A_j : \|x\| < j \text{ and } d_{A_j}(x) > 1/j\},$$

$$C_j = C_j(U) = \left\{x \in B_j : \sup_{n \geq j} \|T_n x - x\| < d_{B_j}(x)\right\}.$$

Since we can readily prove that $\sup_{n \geq j} \|T_n x - x\|$ is a continuous function on E, it follows that the sets A_j, B_j and C_j are open. These sets were studied in the book of Mujica [24]. Fréchet space versions of these sets had appeared earlier in two articles of Mujica [21, 22], and have been studied in the book of Dineen [10].

When we are dealing with a fixed open set U, we will usually write A_j, B_j and C_j. But sometimes it is better to use the more precise notation $A_j(U)$, $B_j(U)$ and $C_j(U)$. Observe that $C_j = C_j(U) = A_j(B_j)$. This remark will be used later on.

Let $\mathcal{C}$ denote the sequence $(C_j)_{j=1}^{\infty}$, and let $\mathcal{H}_{wu}(\mathcal{C})$ denote the subalgebra of all $f \in \mathcal{H}(U)$ that are weakly uniformly continuous on each C_j. It follows from an argument of Aron and Prolla [4, Lemma 2.2] that each $f \in \mathcal{H}_{wu}(\mathcal{C})$ is bounded on each C_j. $\mathcal{H}_{wu}(\mathcal{C})$ is a Fréchet algebra for the topology of uniform convergence on the sets C_j.

The key result in this section is the following theorem.

Theorem 1.2 *Let E be a Banach space with a Schauder basis, and let U be a pseudo-convex open subset of E. Then U is the domain of existence of a function $f \in \mathcal{H}_{wu}(\mathcal{C})$.*

Before proving Theorem 1.2 we need some auxiliary lemmas. Before stating those lemmas we fix some additional notation and terminology.

Let $\mathcal{P}s(U)$ denote the set of all plurisubharmonic functions on U, and let $\mathcal{P}sc(U)$ denote the subset of all continuous plurisubharmonic functions on U. We recall that a function $f : U \to [-\infty, \infty)$ is said to be plurisubharmonic if f is upper semicontinous and

$$f(a) \leq \int_0^{2\pi} f(a + e^{i\theta} b) d\theta$$

for each $a \in U$ and $b \in E$ such that $a + \lambda b \in U$ whenever $|\lambda| \leq 1$.

Let $A \subset U$. We recall that, given a family $\mathcal{F} \subset \mathcal{H}(U)$, the set $\widehat{A}_{\mathcal{F}}$ is defined by

$$\widehat{A}_{\mathcal{F}} = \left\{x \in U : |f(x)| \leq \sup_A |f| \text{ for all } f \in \mathcal{F}\right\}.$$

Likewise, given a family $\mathcal{F} \subset \mathcal{P}s(U)$, the set $\widehat{A}_{\mathcal{F}}$ is defined by

$$\widehat{A}_{\mathcal{F}} = \left\{x \in U : f(x) \leq \sup_A f \text{ for all } f \in \mathcal{F}\right\}.$$

A set $A \subset U$ is said to be *U-bounded* if A is bounded in E and lies strictly inside U, that is $d_U(A) > 0$, where $d_U(A) = \inf_{x \in A} d_U(x)$.

Lemma 1.3 *Let E be a Banach space with a Schauder basis, and let U be a pseudoconvex open subset of E. Then, given $n \in \mathbb{N}$, $f_n \in H(U \cap E_n)$ and $\epsilon > 0$, there exists $f \in \mathcal{H}_{wu}(\mathcal{C})$ such that*

$$\sup_{C_n} |f - f_n \circ T_n| \leq \epsilon.$$

Proof The proof of [24, Lemma 45.5] shows the existence of a sequence $(f_j)_{j=n+1}^{\infty}$, with $f_j \in H(U \cap E_j)$ for every j, such that the sequence $(f_j \circ T_j)_{j=n}^{\infty}$ converges uniformly on each C_j to a function f. It is clear then that $f \in \mathcal{H}_{wu}(\mathcal{C})$. □

Lemma 1.4 *Let E be a Banach space with a Schauder basis, and let U be a pseudoconvex open subset of E. Then $(\widehat{C_j})_{\mathcal{H}_{wu}(\mathcal{C})} \subset (\widehat{B_j})_{Psc(U)}$, and in particular $d_U((\widehat{C_j})_{\mathcal{H}_{wu}(\mathcal{C})}) \geq 1/j$ for every j.*

Proof The proof of [24, Lemma 45.6] applies word by word, but using Lemma 1.3 instead of [24, Lemma 45.5]. □

We remark that Lemmas 45.5 and 45.6 in [24] are stated for Banach spaces with a monotone Schauder basis, but they are still true for Banach spaces with an arbitrary Schauder basis.

Proof of Theorem 1.2 Let D be a countable dense subset of U. Let $(x_j)_{j=1}^{\infty}$ be a sequence of points of D such that each point in D appears in the sequence $(x_j)_{j=1}^{\infty}$ infinitely many times. Consider the sequence $\mathcal{C} = (C_j)_{j=1}^{\infty}$. By Lemma 1.4 $d_U((\widehat{C_j})_{\mathcal{H}_{wu}(\mathcal{C})}) \geq 1/j$ for every j, and therefore $B(x; d_U(x)) \not\subset (\widehat{C_j})_{\mathcal{H}_{wu}(\mathcal{C})}$ for every $x \in U$ and $j \in \mathbb{N}$. After replacing $(C_j)_{j=1}^{\infty}$ by a subsequence, if necessary, we can find a sequence $(y_j)_{j=1}^{\infty} \subset U$ such that

$$y_j \in B(x_j, d_U(x_j)), \qquad y_j \notin (\widehat{C_j})_{\mathcal{H}_{wu}(\mathcal{C})} \quad \text{and} \quad y_j \in C_{j+1} \quad \text{for every } j.$$

Hence we can find a sequence $(g_j)_{j=1}^{\infty} \subset \mathcal{H}_{wu}(\mathcal{C})$ such that

$$|g_j(y_j)| > 1 > \sup_{C_j} |g_j| \quad \text{for every } j.$$

By taking powers of each g_j we can inductively find a sequence $(f_j)_{j=1}^{\infty} \subset \mathcal{H}_{wu}(\mathcal{C})$ such that

$$\sup_{C_j} |f_j| \leq 2^{-j} \quad \text{and} \quad |f_j(y_j)| \geq j + 1 + \left| \sum_{i<j} f_i(y_j) \right| \quad \text{for every } j.$$

Hence it follows that the series $\sum_{j=1}^{\infty} f_j$ converges uniformly on each C_j to a function $f \in \mathcal{H}_{wu}(\mathcal{C})$ such that $|f(y_j)| \geq j$ for every j. We will prove that U is the domain of existence of f. Given $x \in D$, let $(j_n)_{n=1}^{\infty}$ be a strictly increasing sequence in $\mathbb{N}$ such that $x_{j_n} = x$ for every n. Hence it follows that $y_{j_n} \in B(x; d_U(x))$ and $|f(y_{j_n})| \geq j_n$ for every n. This shows that f is unbounded on the ball $B(x; d_U(x))$, and therefore $r_c f(x) \leq d_U(x)$ for every $x \in D$. Since D is dense in U, it follows that $r_c f(x) \leq d_U(x)$ for every $x \in U$. Hence U is the domain of existence of f, as asserted. □

The algebra $\mathcal{H}_{wu}(\mathcal{C})$ has very nice properties, but it is not completely satisfactory, since its definition depends on the choice of the Schauder basis in E. We next consider an algebra of weakly continuous holomorphic functions which has an intrinsic definition. Let $0 < \alpha \leq 1$, and let $\mathcal{H}_{\alpha wud}(U)$ (resp. $\mathcal{H}_{\alpha d}(U)$) denote the subalgebra of all $f \in \mathcal{H}(U)$ that are weakly uniformly continuous (resp. bounded) on each ball $B(x; r)$, with $x \in U$ and $0 < r < \alpha d_U(x)$. It follows from an argument of Aron and Prolla [4, Lemma 2.2] that $\mathcal{H}_{\alpha wud}(U) \subset \mathcal{H}_{\alpha d}(U)$. If E is separable, then $\mathcal{H}_{\alpha d}(U)$ is a Fréchet algebra for the topology of uniform convergence on the balls $B(x; r)$, with $x \in U$ and $0 < r < \alpha d_U(x)$, and $\mathcal{H}_{\alpha wud}(U)$ is a closed subalgebra of $\mathcal{H}_{\alpha d}(U)$. The algebra $\mathcal{H}_{\alpha d}(U)$ was studied a long time ago by Matos [20], and has recently been rediscovered by Dineen and Venkova [11] when $\alpha = 1$. We remark that if $\mathcal{H}_b(U)$ denotes the classical Fréchet algebra of all $f \in \mathcal{H}(U)$ which are bounded on U-bounded sets, then $\mathcal{H}_b(U) \subset \mathcal{H}_d(U)$, and Dineen and Venkova [11] have given examples where there is equality, and where there is inequality.

Theorem 1.5 *Let E be a Banach space with a Schauder basis, and let U be a pseudoconvex open subset of E. If $\alpha = (3c_a)^{-2}$, then U is the domain of existence of a function $f \in \mathcal{H}_{\alpha wud}(U)$.*

To prove this theorem we need the following lemma, with additional properties of the open sets A_j, B_j and C_j.

Lemma 1.6 *Let E be a Banach space with a Schauder basis, and let U be an open subset of E. Then*:

(a) *Given $x \in U$ and $0 < r < (3c_a)^{-1} d_U(x)$, there exists $j_1 \in \mathbb{N}$ such that $B(x, r) \subset A_{j_1}$.*
(b) *Given $x \in U$ and $0 < r < (3c_a)^{-1} d_U(x)$, there exists $j_2 \in \mathbb{N}$ such that $B(x, r) \subset B_{j_2}$.*
(c) *Given $x \in U$ and $0 < r < (3c_a)^{-2} d_U(x)$, there exists $j_3 \in \mathbb{N}$ such that $B(x, r) \subset C_{j_3}$.*

Proof (a) Since $3c_a r < d_U(x)$, there exists $\theta > 1$ such that $3\theta c_a r < d_U(x)$. Since

$$c_a = \limsup_{n\to\infty} \|T_n\| = \lim_{j\to\infty} \sup_{n\geq j} \|T_n\|,$$

there exists $j_1 \in \mathbb{N}$ such that $\|T_n\| < \theta c_a$ for every $n \geq j_1$. Since $\|T_n x - x\| \to 0$, we may in addition assume that

$$\|T_n x - x\| < (\theta c_a - 1) r \quad \text{for every } n \geq j_1.$$

Let $y \in B(x,r)$ and $n \geq j_1$. Then

$$\|T_n y - y\| \leq \|T_n y - T_n x\| + \|T_n x - x\| + \|x - y\|$$
$$< \theta c_a r + (\theta c_a - 1) r + r = 2\theta c_a r < \frac{2}{3} d_U(x).$$

Since $y \in B(x,r) \subset B(x, \frac{1}{3} d_U(x))$, it follows that

$$\sup_{n \geq j_1} \|T_n y - y\| < \frac{2}{3} d_U(x) \leq d_U(y),$$

and therefore $y \in A_{j_1}$. Thus $B(x,r) \subset A_{j_1}$.

(b) Choose s such that $r < s < (3c_a)^{-1} d_U(x)$. By (a) there exists $j_1 \in \mathbb{N}$ such that $B(x,s) \subset A_{j_1}$, and hence

$$d_{A_{j_1}}(y) \geq s - r \quad \text{for every } y \in B(x,r).$$

Choose $j_2 \geq j_1$ such that $j_2 > \|x\| + r$ and $\frac{1}{j_2} < s - r$. Then it is clear that $B(x,r) \subset B_{j_2}$.

(c) To begin with we observe that $C_j = C_j(U) = A_j(B_j)$. If $U \subset V$, then it is clear that $A_j(U) \subset A_j(V)$. We next use these remarks to derive (c) from (a) and (b).

Since $3c_a r < (3c_a)^{-1} d_U(x)$, we may choose s such that $3c_a r < s < (3c_a)^{-1} d_U(x)$. By (b) there exists $j_2 > j_1$ such that $B(x,s) \subset B_{j_2}$. Thus $d_{B_{j_2}}(x) \geq s > 3c_a r$, and therefore $0 < r < (3c_a)^{-1} d_{B_{j_2}}(x)$. By (a) there exists $j_3 \in \mathbb{N}$ such that

$$B(x,r) \subset A_{j_3}(B_{j_2}) \subset A_{j_3}(B_{j_3}) = C_{j_3},$$

and the proof is complete. □

Proof of Theorem 1.5 By Lemma 1.6 $\mathcal{H}_{wu}(\mathcal{C}) \subset \mathcal{H}_{\alpha wud}(U)$, and the conclusion follows from Theorem 1.2. □

2 Interpolation sequences

Theorem 2.1 *Let E be a Banach space with a Schauder basis, and let U be a pseudoconvex open subset of E. Let (x_p) be a sequence of distinct points of U such that $\lim_{p\to\infty} d_U(x_p) = 0$. Then there exists $f \in \mathcal{H}_{wu}(\mathcal{C})$ such that*:

(a) $\lim_{p\to\infty} |f(x_p)| = \infty$.
(b) $f(x_p) \neq f(x_q)$ *whenever* $p \neq q$.

Before proving Theorem 2.1, we need the following auxiliary lemma.

Lemma 2.2 *Let $\mathcal{A}$ be a subalgebra of $\mathcal{H}(U)$ and let A be a subset of U such that $A = \widehat{A}_{\mathcal{A}}$. Then for each finite set $B \subset U \setminus A$ there exists $f \in \mathcal{A}$ such that $|f(y)| > \sup_A |f|$ for every $y \in B$.*

Proof Let $B = \{y_1, \dots, y_n\}$. The conclusion is obviously true if B has one element. Let $n > 1$ and assume that the conclusion is true for sets B containing at most $n-1$ elements. Then there are $\phi, \psi \in \mathcal{A}$ such that $|\phi(y_1)| > \sup_A |\phi|$ and $|\psi(y_j)| > \sup_A |\psi|$ for $j = 2, \dots, n$. If $|\psi(y_1)| > \sup_A |\psi|$, then ψ has the required properties. Hence we may assume that $|\psi(y_1)| \le \sup_A |\psi|$. After multiplying ϕ and ψ by suitable constants, we may assume that $|\phi(y_1)| > 1 > \sup_A |\phi|$ and $|\psi(y_j)| > 1 > \sup_A |\psi| \ge |\psi(y_1)|$ for $j = 2, \dots, n$. Let $f = \phi^p$, where $p \in \mathbb{N}$ is chosen so that $|f(y_1)| > \frac{3}{2} > \frac{1}{2} > \sup_A |f|$. Let $g = \psi^q$, where $q \in \mathbb{N}$ is chosen so that

$$|g(y_j)| > |f(y_j)| + 1 > \frac{1}{2} > \sup_A |g| \ge |g(y_1)| \quad \text{for } j = 2, \dots, n.$$

Then $\sup_A |f + g| \le \sup_A |f| + \sup_A |g| < \frac{1}{2} + \frac{1}{2} = 1$. Thus

$$|(f+g)(y_1)| \ge |f(y_1)| - |g(y_1)| > \frac{3}{2} - \frac{1}{2} = 1 > \sup_A |f+g|$$

and

$$|(f+g)(y_j)| \ge |g(y_j)| - |f(y_j)| > 1 > \sup_A |f+g| \quad \text{for } j = 2, \dots, n. \qquad \square$$

Proof of Theorem 2.1 (i) By modifying the proof of Theorem 1.2 we show the existence of $f \in \mathcal{H}_{wu}(\mathcal{C})$ satisfying (a). Since $\lim_{p\to\infty} d_U(x_p) = 0$, there is a strictly increasing sequence $(p_j) \subset \mathbb{N}$ such that $d_U(x_p) < 1/j$ for every $p \ge p_j$. Consider the sets F_j defined by $F_1 = \{x_p : d_U(x_p) \ge 1/2\}$ and

$$F_j = \{x_p : 1/(j+1) \le d_U(x_p) < 1/j\} \quad \text{for } j \ge 2.$$

Since $\bigcup_{i=1}^j F_i = \{x_p : d_U(x_p) \ge 1/(j+1)\}$, it follows that $p < p_{j+1}$ whenever $x_p \in \bigcup_{i=1}^j F_i$. Hence $|F_j| \le |\bigcup_{i=1}^j F_i| < p_{j+1}$. Thus (F_j) is a sequence of disjoint finite sets whose union is $\{x_p : p \in \mathbb{N}\}$. By Lemma 1.4 $d_U((\widehat{C_j})_{\mathcal{H}_{wu}(\mathcal{C})}) \ge 1/j$, and it follows that

$$F_j \subset U \setminus (\widehat{C_j})_{\mathcal{H}_{wu}(\mathcal{C})} \quad \text{for every } j \ge 2.$$

By using Lemma 2.2 we can find a sequence $(g_j)_{j=2}^\infty \subset \mathcal{H}_{wu}(\mathcal{C})$ such that

$$|g_j(y)| > 1 > \sup_{C_j} |g_j| \quad \text{for every } y \in F_j,\ j \ge 2.$$

By taking powers of each g_j we can inductively find a sequence $(f_j)_{j=2}^\infty \subset \mathcal{H}_{wu}(\mathcal{C})$ such that

$$\sup_{C_j} |f_j| \le 2^{-j} \quad \text{and} \quad |f_j(y)| \ge j + 1 + \|y\|^2 + \left|\sum_{i<j} f_i(y)\right| \quad \text{for all } y \in F_j,\ j \ge 2.$$

Hence it follows that the series $\sum_{j=2}^\infty f_j$ converges uniformly on each C_j to a function $f \in \mathcal{H}_{wu}(\mathcal{C})$ such that $|f(y)| \ge j + \|y\|^2$ for every $y \in F_j$, $j \ge 2$. In particular it follows that $\lim_{p\to\infty} |f(x_p)| = \infty$ and $|f(x_p)| \ge \|x_p\|^2$ for every $p \ge 2$.

(ii) If f verifies (b), there is nothing to prove. If f does not verify (b), we follow an argument of Hervier [13] to find $g \in \mathcal{H}_{wu}(\mathcal{C})$ that verifies (a) and (b). For $p \neq q$ the set

$$V_{pq} = \{\phi \in E' : \phi(x_p) - \phi(x_q) \neq f(x_p) - f(x_q)\}$$

is open and dense in E'. By the Baire theorem the intersection $\bigcap_{p\neq q} V_{pq}$ is non-empty. Thus there exists $\phi \in E'$ such that $\phi(x_p) - \phi(x_q) \neq f(x_p) - f(x_q)$ for all $p \neq q$. Thus the function $g = f - \phi$ verifies (b). If the sequence (x_p) is bounded, then it is clear that g verifies (a). If the sequence (x_p) is unbounded, then g also verifies (a), for

$$|g(x_p)| \geq |f(x_p)| - |\phi(x_p)| \geq \|x_p\|^2 - \|\phi\|\|x_p\| \to \infty. \qquad \square$$

From Theorem 2.1 and Lemma 1.6 we immediately obtain the following theorem.

Theorem 2.3 *Let E be a Banach space with a Schauder basis, and let U be a pseudoconvex open subset of U. Let (x_p) be a sequence of distinct points of U such that $\lim_{p\to\infty} d_U(x_p) = 0$. If $\alpha = (3c_a)^{-2}$, then there exists $f \in \mathcal{H}_{\alpha wud}(U)$ such that:*

(a) $\lim_{p\to\infty} |f(x_p)| = \infty$.
(b) *$f(x_p) \neq f(x_q)$ whenever $p \neq q$.*

In the preceding section we defined the algebras $\mathcal{H}_{wu}(\mathcal{C})$ and $\mathcal{H}_{\alpha wud}(U)$. If F is a Banach space, then the spaces $\mathcal{H}_{wu}(\mathcal{C}; F)$ and $\mathcal{H}_{\alpha wud}(U; F)$ are defined in the obvious way.

Theorem 2.4 *Let E be a Banach space with a Schauder basis, and let U be a pseudoconvex open subset of E. Let (x_p) be a sequence of distinct points of U such that $\lim_{p\to\infty} d_U(x_p) = 0$, and let (b_p) be an arbitrary sequence in a Banach space F. Then there exists $f \in \mathcal{H}_{wu}(\mathcal{C}; F)$ such that $f(x_p) = b_p$ for every p.*

Theorem 2.4 follows from Theorem 2.1 and the following lemma.

Lemma 2.5 *Let (a_p) be a sequence of distinct points in $\mathbb{C}$ such that $\lim_{p\to\infty} |a_p| = \infty$. Let (b_p) be an arbitrary sequence in a Banach space F. Then there exists $f \in \mathcal{H}(\mathbb{C}; F)$ such that $f(a_p) = b_p$ for every p.*

Proof When $F = \mathbb{C}$ this lemma is an exercise in Ahlfors' book [1], and the hint given there works equally in the vector-valued case. Indeed let $g \in \mathcal{H}(\mathbb{C})$ be a function with a simple zero at each a_p, and let $f \in \mathcal{H}(\mathbb{C}; F)$ be defined by

$$f(z) = \sum_{n=1}^{\infty} \frac{g(z)}{z - a_n} \frac{b_n}{g'(a_n)} e^{\rho_n \overline{a_n}(z-a_n)},$$

where (ρ_n) is a sequence of positive numbers to be chosen later (when $z = a_n$, we interpret $\frac{g(z)}{z-a_n}$ as $g'(a_n)$). Then for $|z| \leq \frac{|a_n|}{2}$ we have that

$$\left|\frac{e^{\rho_n \overline{a_n}(z-a_n)}}{z-a_n}\right| \leq \frac{2}{|a_n|} e^{\mathrm{Re}[\rho_n \overline{a_n}(z-a_n)]} \leq \frac{2}{|a_n|} e^{-\frac{1}{2}\rho_n |a_n|^2}.$$

There is $c_n > 0$ such that $|g(z)| \leq c_n$ for $|z| \leq \frac{|a_n|}{2}$. We may choose ρ_n such that

$$c_n \frac{\|b_n\|}{|g'(a_n)|} \frac{2}{|a_n|} e^{-\frac{1}{2}\rho_n |a_n|^2} \leq \frac{1}{n^2} \quad \text{for every } n.$$

Since $|a_n| \to \infty$, there is a strictly increasing sequence (n_p) in $\mathbb{N}$ such that $|a_n| > |a_p|$ for $n \geq n_p$. Then for $|z| \leq \frac{|a_p|}{2}$ we have that

$$\begin{aligned}\sum_{n=1}^{\infty} |g(z)| \frac{\|b_n\|}{|g'(a_n)|} \left|\frac{e^{\rho_n \overline{a_n}(z-a_n)}}{z-a_n}\right| &= \sum_{n=1}^{n_p-1} |g(z)| \frac{\|b_n\|}{|g'(a_n)|} \left|\frac{e^{\rho_n \overline{a_n}(z-a_n)}}{z-a_n}\right| \\ &\quad + \sum_{n=n_p}^{\infty} |c_n| \frac{\|b_n\|}{|g'(a_n)|} \frac{2}{|a_n|} e^{-\frac{1}{2}\rho_n |a_n|^2} \\ &\leq C_p + \sum_{n=n_p}^{\infty} \frac{1}{n^2},\end{aligned}$$

where C_p denotes the supremum of the first sum on the disc $|z| \leq \frac{|a_p|}{2}$. Thus the series converges uniformly on each disc and $f \in \mathcal{H}(\mathbb{C}; F)$. Clearly $f(a_p) = b_p$ for every p. □

Theorem 2.6 *Let E be a Banach space with a Schauder basis, and let U be a pseudo-convex open subset of E. Let (x_p) be a sequence of distinct points of U such that $\lim_{p\to\infty} d_U(x_p) = 0$, and let (b_p) be an arbitrary sequence in a Banach space F. If $\alpha = (3c_a)^{-2}$, then there exists $f \in \mathcal{H}_{\alpha wud}(U; F)$ such that $f(x_p) = b_p$ for every p.*

Theorem 2.6 follows from Theorem 2.4 and Lemma 1.6, or from Theorem 2.3 and Lemma 2.5.

Theorems 2.1, 2.3, 2.4 and 2.6 sharpen results of Hervier [13]. By using cohomology theory Patyi [34] has recently obtained a version of Hervier's result.

Matos [20] has obtained a Cartan-Thullen theorem for the algebra $\mathcal{H}_d(U)$. His method of proof yields a similar theorem for the algebra $\mathcal{H}_{\alpha wud}(U)$. By combining this theorem with Theorem 1.5, and Theorems 2.3 and 2.6, we easily obtain the following theorem.

Theorem 2.7 *Let E be a Banach space with a Schauder basis, let $\alpha = (3c_a)^{-2}$ and let U be an open subset of E. Then the following conditions are equivalent.*

(a) *U is the domain of existence of a function $f \in \mathcal{H}_{\alpha wud}(U)$.*

(b) *U is an $\mathcal{H}_{\alpha wud}(U)$-domain of holomorphy.*
(c) *$d_U(\widehat{B}_{\mathcal{H}_{\alpha wud}(U)}) > 0$ for each ball $B = B(x; r)$, with $x \in U$ and $0 < r < \alpha d_U(x)$.*
(d) *$d_U(\widehat{K}_{\mathcal{H}_{\alpha wud}(U)}) > 0$ for each compact set $K \subset U$.*
(e) *$d_U(\widehat{K}_{\mathcal{P}s(U)}) > 0$ for each compact set $K \subset U$.*
(f) *U is pseudoconvex.*
(g) *For each sequence (x_p) of distinct points of U such that $d_U(x_p) \to 0$, there is $f \in \mathcal{H}_{\alpha wud}(U)$ such that $|f(x_p)| \to \infty$ and $f(x_p) \neq f(x_q)$ whenever $p \neq q$.*
(h) *For each sequence (x_p) of distinct points of U such that $d_U(x_p) \to 0$, and each sequence (b_p) in a Banach space F (or equivalently in $\mathbb{C}$), there exists $f \in \mathcal{H}_{\alpha wud}(U; F)$ such that $f(x_p) = b_p$ for every $p \in \mathbb{N}$.*

3 Holomorphic approximation

In 1965 Hörmander [14] used the $\overline{\partial}$ operator to give a new proof of the identity of pseudoconvex domains and domains of holomorphy in $\mathbb{C}^n$. His approach yielded also an approximation theorem that improved earlier results of Weil [38] and Oka [31]. Shortly after Gruman and Kiselman [12] solved the Levi problem in Banach spaces with a Schauder basis, Noverraz [30] followed the same approach to obtain similar approximation theorems. In this section we obtain approximation theorems that sharpen the results of Noverraz.

Definition 3.1 Let E be a Banach space and let U be a pseudoconvex open subset of E.

(a) We say that an open set $V \subset U$ is $\mathcal{P}s(U)$-convex if $d_V(\widehat{K}_{\mathcal{P}s(U)} \cap V) > 0$ for each compact set $K \subset V$.
(b) We say that V is strongly $\mathcal{P}s(U)$-convex if $\widehat{K}_{\mathcal{P}s(U)} \subset V$ and $d_V(\widehat{K}_{\mathcal{P}s(U)}) > 0$ for each compact set $K \subset V$.

We follow here the terminology in [23, Definition 4.2], which is slightly different from the terminology in [24, Definition 45.1] and [10, Definition 5.28].

The main result of this section is the following.

Theorem 3.2 *Let E be a Banach space with a Schauder basis, let U be a pseudoconvex open subset of E, let V be a strongly $\mathcal{P}s(U)$-convex open subset of U, and let $\alpha = (3c_a)^{-2}$. Then*:

(a) *$\mathcal{H}_{\alpha wud}(U)$ is sequentially dense in $(\mathcal{H}(V), \tau_0)$.*
(b) *If, in addition, the basis is shrinking, then, for each $f \in \mathcal{H}_{\alpha wud}(V)$, there is a sequence $(f_n) \subset \mathcal{H}_{\alpha wud}(U)$ which converges to f in $\mathcal{H}_{\beta wud}(V)$, where $\beta = \frac{\alpha}{c_a}$.*

Proof The proof of (a) is a straightforward adaptation of the proof of [24, Theorem 46.1], but we prefer to give the details, since some of the estimates obtained will be used to prove (b). Since V is strongly $\mathcal{P}s(U)$-convex, it follows that $\mathcal{H}(U \cap E_n)$ is dense in $(\mathcal{H}(V \cap E_n), \tau_0)$ for every n (see [24, Corollary 44.4]). Hence, given

$f \in \mathcal{H}(V)$, there is a sequence (g_n), with $g_n \in \mathcal{H}(U \cap E_n)$, such that $|g_n - f| < 1/n$ on $B_n(V) \cap E_n$, and therefore

$$|g_n \circ T_n - f \circ T_n| < 1/n \quad \text{on } C_n(V) \text{ for every } n.$$

By Lemma 1.3 there is a sequence $(f_n) \subset \mathcal{H}_{wu}(\mathcal{C}(U)) \subset \mathcal{H}_{\alpha wud}(U)$ such that

$$|f_n - g_n \circ T_n| < 1/n \quad \text{on } C_n(U) \text{ for every } n.$$

To complete the proof of (a) we show that (f_n) converges to f in $(\mathcal{H}(V), \tau_0)$. Indeed let L be a compact subset of V, and let $\epsilon > 0$. We first choose $\delta > 0$ such that $L + B(0;\delta) \subset V$ and $|f(y) - f(x)| < \epsilon$ whenever $x \in L$ and $\|y - x\| < \delta$. Choose $n_0 > 1/\epsilon$ such that $L \subset C_{n_0}(V)$ and $\|T_n x - x\| < \delta$ for all $x \in L$ and $n \geq n_0$. Then for $x \in L$ and $n \geq n_0$ it follows that

$$\begin{aligned}|f_n(x) - f(x)| &\leq |f_n(x) - g_n \circ T_n(x)| + |g_n \circ T_n(x) - f \circ T_n(x)| \\ &\quad + |f \circ T_n(x) - f(x)| < 3\epsilon.\end{aligned}$$

(b) If $f \in \mathcal{H}_{\alpha wud}(V)$, we will show that the sequence $(f_n) \subset \mathcal{H}_{\alpha wud}(U)$ constructed in (a) converges to f in $\mathcal{H}_{\beta wud}(V)$. Indeed let $a \in V$, $0 < r < \alpha d_V(a)$ and $\epsilon > 0$. Let $\theta > 1$ such that $\theta^2 r < \alpha d_V(a)$. By Lemma 1.6 there is $n_1 \in \mathbb{N}$ such that $B(a;\theta^2 r) \subset C_{n_1}(V)$, $\|T_n\| < \theta c_a$ and $\|T_n a - a\| < \theta^2 r - \theta r$ for all $n \geq n_1$. Hence it follows easily that

$$T_n\left(B\left(a; \frac{r}{c_a}\right)\right) \subset B(a;\theta^2 r) \quad \text{for all } n \geq n_1.$$

Since f is weakly uniformly continuous on $B(a;\theta^2 r)$, there is a balanced weakly open neighborhood of zero W in E such that $|f(y) - f(x)| < \epsilon$ whenever $x, y \in B(a,\theta^2 r)$ and $y - x \in W$. Since the basis is shrinking, there is $n_2 \geq n_1$ such that $n_2 > 1/\epsilon$ and $T_n x - x \in W$ for all $x \in B(a;\theta^2 r)$ and $n \geq n_2$ (see [37, Proposition 1.3]). It follows that

$$|f \circ T_n(x) - f(x)| < \epsilon \quad \text{for all } x \in B\left(a; \frac{r}{c_a}\right), n \geq n_2.$$

Then we can show as in (a) that

$$|f_n(x) - f(x)| < 3\epsilon \quad \text{for all } x \in B\left(a; \frac{r}{c_a}\right), n \geq n_2. \qquad \square$$

Corollary 3.3 *Let E be a Banach space with a Schauder basis, and let U be a pseudoconvex open subset of E, and let $\alpha = (3c_a)^{-2}$. Then $\mathcal{H}_{\alpha wud}(U)$ is sequentially dense in $(\mathcal{H}(U), \tau_0)$.*

Theorem 3.4 *Let E be a Banach space with a Schauder basis, let U be a pseudoconvex open subset of E, let K be a compact subset of U such that $\widehat{K}_{\mathcal{P}_s(U)} = K$, and let $\alpha = (3c_a)^{-2}$. Then:*

(a) *Given* $f \in \mathcal{H}(V)$, *where* V *is open and* $K \subset V \subset U$, *there exist an open set* W *with* $K \subset W \subset U$ *and a sequence* $(f_n) \subset \mathcal{H}_{\alpha wud}(U)$ *such that* $f \in \mathcal{H}(W)$ *and* (f_n) *converges to* f *in* $(\mathcal{H}(W), \tau_0)$.

(b) *If, in addition, the basis is shrinking, then given* $f \in \mathcal{H}_{\alpha wud}(V)$, *where* V *is open and* $K \subset V \subset U$, *there exist an open set* W *with* $K \subset W \subset U$ *and a sequence* $(f_n) \subset \mathcal{H}_{\alpha wud}(U)$ *such that* $f \in \mathcal{H}_{\alpha wud}(W)$ *and* (f_n) *converges to* f *in* $\mathcal{H}_{\beta wud}(W)$, *where* $\beta = \frac{\alpha}{c_a}$.

Proof Let $f \in \mathcal{H}(V)$ (resp. $f \in \mathcal{H}_{\alpha wud}(V)$). By [24, Theorem 46.2], there exists a strongly $\mathcal{P}s(U)$-convex open set W such that $K \subset W \subset V$. Now the conclusion follows from Theorem 3.2. □

Theorem 3.5 *Let* E *be a Banach space with a Schauder basis, let* U *be a pseudoconvex open subset of* E *and let* $\alpha = (3c_a)^{-2}$. *Then for each open set* $V \subset U$, *the following conditions are equivalent.*

(a) $d_V(\widehat{K}_{\mathcal{H}_{\alpha wud}(U)} \cap V) > 0$ *for each compact set* $K \subset V$.

(b) $\widehat{K}_{\mathcal{H}_{\alpha wud}(U)} \subset V$ *and* $d_V(\widehat{K}_{\mathcal{H}_{\alpha wud}(U)}) > 0$ *for each compact set* $K \subset V$.

(c) $d_V(\widehat{K}_{\mathcal{H}_{\alpha wud}(V)}) > 0$ *for each compact set* $K \subset V$, *and* $\mathcal{H}_{\alpha wud}(U)$ *is sequentially dense in* $(\mathcal{H}(V), \tau_0)$.

(d) $d_V(\widehat{K}_{\mathcal{H}_{\alpha wud}(V)}) > 0$ *for each compact set* $K \subset V$, *and* $\mathcal{H}_{\alpha wud}(U)$ *is dense in* $(\mathcal{H}(V), \tau_0)$.

Proof (a) ⇒ (b) Let K be a compact subset of V, and let us suppose that $\widehat{K}_{\mathcal{H}_{\alpha wud}(U)} \not\subset V$. Then $\widehat{K}_{\mathcal{H}_{\alpha wud}(U)}$ is a disjoint union of two compact sets A_0 and A_1, where $A_0 = \widehat{K}_{\mathcal{H}_{\alpha wud}(U)} \cap V$ and $A_1 = \widehat{K}_{\mathcal{H}_{\alpha wud}(U)} \setminus V$. Let $f \in \mathcal{H}(\widehat{K}_{\mathcal{H}_{\alpha wud}(U)})$ be equal to zero on a neighborhood of A_0 and equal to one on a neighborhood of A_1. By Theorem 3.4, there exists $g \in \mathcal{H}_{\alpha wud}(U)$ such that $\sup_{\widehat{K}_{\mathcal{H}_{\alpha wud}(U)}} |g - f| < \frac{1}{2}$. Then on the one hand we have that $|g| < \frac{1}{2}$ on K, and on the other hand we have that $|g| > \frac{1}{2}$ on A_1, which is a contradiction since $A_1 \subset \widehat{K}_{\mathcal{H}_{\alpha wud}(U)}$.

(b) ⇒ (c) Let K be a compact subset of V. Since $\widehat{K}_{\mathcal{H}_{\alpha wud}(V)} \subset \widehat{K}_{\mathcal{H}_{\alpha wud}(U)} \subset V$, the first assertion follows from (b). Since $\widehat{K}_{\mathcal{P}s(U)} \subset \widehat{K}_{\mathcal{H}_{\alpha wud}(U)}$, it follows from (b) that V is strongly $\mathcal{P}s(U)$-convex and then the second assertion follows from Theorem 3.2.

It is obvious that (c) ⇒ (d).

(d) ⇒ (a) Let K be a compact subset of V. Since $\mathcal{H}_{\alpha wud}(U)$ is dense in $(\mathcal{H}(V), \tau_0)$, one can easily verify that $\widehat{K}_{\mathcal{H}_{\alpha wud}(V)} = \widehat{K}_{\mathcal{H}_{\alpha wud}(U)} \cap V$, and (a) follows. □

Theorem 3.6 *Let* E *be a Banach space with a Schauder basis, let* U *be a pseudoconvex open subset of* E *and let* $\alpha = (3c_a)^{-2}$. *Then* $\widehat{K}_{\mathcal{P}s(U)} = \widehat{K}_{\mathcal{H}(U)} = \widehat{K}_{\mathcal{H}_{\alpha wud}(U)}$ *for each compact set* $K \subset U$.

Proof The inclusions $\widehat{K}_{\mathcal{P}s(U)} \subset \widehat{K}_{\mathcal{H}(U)} \subset \widehat{H}_{\mathcal{H}_{\alpha wud}(U)}$ are clear. To prove the reverse inclusion we adapt the proof of [24, Theorem 46.5]. Let V be any open set such that $\widehat{K}_{\mathcal{P}s(U)} \subset V \subset U$. By [24, Theorem 46.2] there is a strongly $\mathcal{P}s(U)$-convex open set W such that $\widehat{K}_{\mathcal{P}s(U)} \subset W \subset V$. Theorem 3.2 implies that $\mathcal{H}_{\alpha wud}(U)$ is dense in $(\mathcal{H}(W), \tau_0)$ and therefore $\widehat{K}_{\mathcal{H}_{\alpha wud}(U)} \cap W = \widehat{K}_{\mathcal{H}(W)}$. Theorem 2.7 guarantees that $d_W(\widehat{K}_{\mathcal{H}_{\alpha wud}(U)} \cap W) = d_W(\widehat{K}_{\mathcal{H}(W)}) > 0$. By Theorem 3.5 we conclude that $\widehat{K}_{\mathcal{H}_{\alpha wud}(U)} \subset W \subset V$. Since V was arbitrary, it follows that $\widehat{K}_{\mathcal{H}_{\alpha wud}(U)} \subset \widehat{K}_{\mathcal{P}s(U)}$. □

It follows from the preceding two theorems that the two notions in Definition 3.1 coincide when E has a Schauder basis, that is, V is $\mathcal{P}s(U)$-convex if and only if V is strongly $\mathcal{P}s(U)$-convex.

The results in this section sharpen results of Noverraz [30] and Mujica [23].

4 The spectrum

If A is a topological algebra of holomorphic functions on an open set U, then $\mathcal{S}(A)$ denotes the *spectrum* of A, that is the set of all nonzero continuous algebra homomorphisms $\Phi : A \to \mathbb{C}$.

There are examples of algebras for which every $\Phi \in \mathcal{S}(A)$ is a point evaluation, and there are other examples for which the point evaluations form a proper subset of $\mathcal{S}(A)$.

For instance, if E is a Banach space with a Schauder basis, and U is a pseudoconvex open subset of E, then a result of Schottenloher [36] asserts that every $\Phi \in \mathcal{S}(\mathcal{H}(U), \tau_0)$ is a point evaluation.

On the other hand, if $\mathcal{H}^\infty(\mathbb{D})$ denotes the Banach algebra of all bounded holomorphic functions on the open unit disc, then a result of Carleson [8] asserts that the point evaluations form a dense proper subset of $\mathcal{S}(H^\infty(\mathbb{D}))$ for the topology of pointwise convergence. This is the famous corona theorem.

In this section we prove that under certain conditions every $\Phi \in \mathcal{S}(\mathcal{H}_{wu}(\mathcal{C}))$ is the pointwise limit of a sequence of point evaluations. We also obtain a similar result for the algebra $\mathcal{H}_{\alpha wud}(U)$.

Theorem 4.1 *Let E be a Banach space with a shrinking Schauder basis, and let U be a pseudoconvex open subset of E. Then for each $\Phi \in \mathcal{S}(\mathcal{H}_{wu}(\mathcal{C}))$ there is a sequence $(a_n) \subset U$ such that*

$$\Phi(f) = \lim_{n\to\infty} f(a_n) \quad \textit{for every } f \in \mathcal{H}_{wu}(\mathcal{C}).$$

Proof We follow the approach of Schottenloher [36] and Mujica [25]. Given $\Phi \in \mathcal{S}(\mathcal{H}_{wu}(\mathcal{C}))$, there exist $j_0 \in \mathbb{N}$ and $\gamma > 0$ such that

$$|\Phi(f)| \leq \gamma \sup_{C_{j_0}} |f| \quad \text{for all } f \in \mathcal{H}_{wu}(\mathcal{C}). \tag{1}$$

Since Φ is a homomorphism, a standard argument shows that we may assume that $\gamma = 1$. For each $n \geq j_0$ we define $\Phi_n \in \mathcal{S}(\mathcal{H}(U \cap E_n), \tau_0)$ as follows. Given $f \in H(U \cap E_n)$, by Lemma 1.3 there is a sequence $(f_{nk}) \subset \mathcal{H}_{wu}(\mathcal{C})$ such that

$$\sup_{C_n} |f_{nk} - f \circ T_n| \leq 1/k \quad \text{for all } k. \tag{2}$$

It follows from (1) and (2) that $(\Phi(f_{nk}))$ is a Cauchy sequence in $\mathbb{C}$, and therefore converges. If we define

$$\Phi_n(f) = \lim_{k\to\infty} \Phi(f_{nk}), \tag{3}$$

then clearly Φ_n is well-defined, linear and multiplicative. It follows from (1), (2) and (3) that

$$|\Phi_n(f)| \le \sup_{C_{j_0}} |f \circ T_n| \le \sup_{B_{j_0}\cap E_n} |f|,$$

and therefore Φ_n is continuous. Since $U \cap E_n$ is a pseudoconvex open subset of E_n, it follows that Φ_n is a point evaluation (see [15, Theorem 7.2.10] or [24, Corollary 58.8]). Hence there exists $a_n \in U \cap E_n$ such that

$$\Phi_n(f) = f(a_n) \quad \text{for all } f \in H(U \cap E_n). \tag{4}$$

Fix $f \in \mathcal{H}_{wu}(\mathcal{C})$. By the preceding argument, there is a sequence $(f_{nk})_{k=1}^{\infty} \subset \mathcal{H}_{wu}(\mathcal{C})$ such that

$$\lim_{k\to\infty} \sup_{C_n} |f_{nk} - f \circ T_n| = 0 \quad \text{and} \quad \lim_{k\to\infty} \Phi(f_{nk}) = \Phi_n(f).$$

Hence we can find a sequence $(k_n) \subset \mathbb{N}$ such that

$$\sup_{C_n} |f_{nk_n} - f \circ T_n| < 1/n \tag{5}$$

and

$$|\Phi(f_{nk_n}) - \Phi_n(f)| < 1/n \tag{6}$$

for all $n \ge j_0$. We next show that

$$f = \lim_{n\to\infty} f_{nk_n} \quad \text{in } \mathcal{H}_{wu}(\mathcal{C}). \tag{7}$$

Let $j \ge j_0$ and $\epsilon > 0$ be given. Let W be a balanced weakly open neighborhood of zero en E such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in C_j$ and $x - y \in W$. Since the basis is shrinking, there is $n_j \ge 1/\epsilon$ such that $x - T_n x \in W$ for all $x \in C_j$ and $n \ge n_j$ (see [37, Proposition 1.3]). Hence

$$|f(x) - f \circ T_n(x)| < \epsilon \quad \text{for all } x \in C_j, n \ge n_j,$$

and therefore

$$\sup_{C_j} |f - f_{nk_n}| \le \sup_{C_j} |f - f \circ T_n| + \sup_{C_j} |f \circ T_n - f_{nk_n}| < \epsilon + 1/n < 2\epsilon$$

for every $n \ge n_j$. This shows (7). It follows from (7), (6) and (4) that

$$\Phi(f) = \lim_{n\to\infty} \Phi(f_{nk_n}) = \lim_{n\to\infty} \Phi_n(f) = \lim_{n\to\infty} f(a_n),$$

and the proof is complete. □

Theorem 4.2 *Let E be a Banach space with a shrinking Schauder basis, and let U be a pseudoconvex open subset of E. If $\alpha = (3c_a)^{-2}$, then for each $\Phi \in \mathcal{S}(\mathcal{H}_{\alpha wud}(U))$, there is a sequence $(a_n) \subset U$ such that*

$$\Phi(f) = \lim_{n\to\infty} f(a_n) \quad \text{for all } f \in \mathcal{H}_{\alpha wud}(U).$$

Proof The proof is a straightforward adaptation of the proof of Theorem 4.1. Given $\Phi \in \mathcal{S}(\mathcal{H}_{\alpha wud}(U))$, there exists a set B_0 of the form $B_0 = \bigcup_{i=1}^m B(x_i; r_i)$, with $x_i \in U$ and $0 < r_i < \alpha d_U(x_i)$, such that

$$|\Phi(f)| \le \sup_{B_0} |f| \quad \text{for all } f \in \mathcal{H}_{\alpha wud}(U).$$

By Lemma 1.6 there exists $j_0 \in \mathbb{N}$ such that $B_0 \subset C_{j_0}$. For each $n \ge j_0$ we define $\Phi_n \in \mathcal{S}(\mathcal{H}(U \cap E_n), \tau_0)$ as follows. Given $f \in \mathcal{H}(U \cap E_n)$, let $(f_k) \subset \mathcal{H}_{wu}(\mathcal{C}) \subset \mathcal{H}_{\alpha wud}(U)$ such that $\sup_{C_n} |f_k - f \circ T_n| \le 1/k$ for all k. Then the sequence $(\Phi(f_k))$ converges and we define $\Phi_n(f) = \lim_{k\to\infty} \Phi(f_k)$. For each $n \ge j_0$ there exists $a_n \in U \cap E_n$ such that $\Phi_n(f) = f(a_n)$ for all $f \in H(U \cap E_n)$. Fix $f \in \mathcal{H}_{\alpha wud}(U)$. Then for each $n \ge j_0$ there is $(f_{nk})_{k=1}^\infty$ such that $\lim_{k\to\infty} \sup_{C_n} |f_{nk} - f \circ T_n| = 0$ and $\lim_{k\to\infty} \Phi(f_{nk}) = \phi_n(f)$. Hence there is $(k_n) \subset \mathbb{N}$ such that $\sup_{C_n} |f_{nk_n} - f \circ T_n| < 1/n$ and $|\Phi(f_{nk_n}) - \Phi_n(f)| < 1/n$ for all $n \ge j_0$. We next show that

$$f = \lim_{n\to\infty} f_{nk_n} \quad \text{in } \mathcal{H}_{\alpha wud}(U).$$

Let $B = \bigcup_{i=1}^p B(x_i; r_i)$, with $x_i \in U$ and $0 < r_i < \alpha d_U(x_i)$, and let $j \ge j_0$ such that $B \subset C_j$. Given $\epsilon > 0$, let W be a balanced weakly open neighborhood of zero in E such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in B$ and $x - y \in W$. Let $n_j \ge 1/\epsilon$ such that $x - T_n x \in W$ for all $x \in C_j$ and $n \ge n_j$. Hence $|f(x) - f \circ T_n(x)| < \epsilon$ for all $x \in B$ and $n \ge n_j$, and therefore $\sup_B |f - f_{nk_n}| \le \sup_B |f - f \circ T_n| + \sup_{C_n} |f \circ T_n - f_{nk_n}| < \epsilon + 1/n < 2\epsilon$. It follows that

$$\Phi(f) = \lim_{n\to\infty} \Phi(f_{nk_n}) = \lim_{n\to\infty} \Phi_n(f) = \lim_{n\to\infty} f(a_n). \qquad \square$$

We have stated our results in Banach spaces with a Schauder basis, but it is clear that we obtain the best results when the Schauder basis is asymptotically monotone.

In [26] we extend some of the results in this paper to the realm of separable Banach spaces with the bounded approximation property. To achieve that purpose we prove that every separable Banach space with the λ-bounded approximation property is isometrically isomorphic to a 1-complemented subspace of a Banach space with a Schauder basis whose asymptotic constant is not greater than λ. This is a sharp quantitative version of a classical result obtained independently by Pelczynski [35] and by Johnson, Rosenthal and Zippin [16].

We are indebted to the referee for his (or her) careful examination of the manuscript. Besides detecting a number of misprints, he (or she) made several suggestions that have helped to improve the presentation.

References

1. Ahlfors, L.: Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable. McGraw-Hill, New York (1960)
2. Aron, R.: Weakly uniformly continuous and weakly sequentially continuous entire functions. In: Barroso, J.A. (ed.) Advances in Holomorphy. North-Holland Math. Stud., vol. 34, pp. 47–66. North-Holland, Amsterdam (1979)
3. Aron, R., Hervés, C., Valdivia, M.: Weakly continuous mappings on Banach spaces. J. Funct. Anal. **52**, 189–204 (1983)
4. Aron, R., Prolla, J.B.: Polynomial approximation of differentiable functions on Banach spaces. J. Reine Angew. Math. **313**, 195–216 (1980)
5. Bremermann, H.: Über die Äquivalenz der pseudokonvexen Gebiete und der Holomorphiegebiete im Raum von n komplexen Veränderlichen. Math. Ann. **128**, 63–91 (1954)
6. Burlandy, P., Moraes, L.A.: The spectrum of an algebra of weakly continuous holomorphic mappings. Indag. Math. (N.S.) **11**, 525–532 (2000)
7. Carando, D., García, D., Maestre, M.: Homomorphisms and composition operators on algebras of analytic functions of bounded type. Adv. Math. **197**, 607–629 (2005)
8. Carleson, L.: Interpolation by bounded analytic functions and the corona problem. Ann. Math. **76**, 542–559 (1962)
9. Day, M.M.: On the basis problem in normed spaces. Proc. Am. Math. Soc. **13**, 655–658 (1962)
10. Dineen, S.: Complex Analysis on Infinite Dimensional Spaces. Springer, Berlin (1999)
11. Dineen, S., Venkova, M.: Extending bounded type holomorphic mappings on a Banach space. J. Math. Anal. Appl. **297**, 645–658 (2004). Special issue dedicated to John Horváth
12. Gruman, L., Kiselman, C.: Le problème de Levi dans les espaces de Banach à base. C. R. Acad. Sci. Paris **274**, 1296–1299 (1972)
13. Hervier, Y.: On the Weierstrass problem in Banach spaces. In: Hayden, T., Suffridge, T. (eds.) Proceedings on Infinite Dimensional Holomorphy. Lecture Notes in Math, vol. 364, pp. 157–167. Springer, Berlin (1974)
14. Hörmander, L.: L^2 estimates and existence theorems for the $\overline{\partial}$ operator. Acta Math. **113**, 89–152 (1965)
15. Hörmander, L.: An Introduction to Complex Analysis in Several Variables. North-Holland Mathematical Library, vol. 7. North-Holland, Amsterdam (1973)
16. Johnson, W.B., Rosenthal, H., Zippin, M.: On bases, finite dimensional decompositions and weaker structures in Banach spaces. Isr. J. Math. **9**, 488–506 (1971)
17. Josefson, B.: A counterexample in the Levi problem. In: Hayden, T., Suffridge, T. (eds.) Proceedings on Infinite Dimensional Holomorphy. Lecture Notes in Math., vol. 364, pp. 168–177. Springer, Berlin (1974)
18. Levi, E.: Sulle ipersuperfici dello spazio a 4 dimensioni che possono essere frontiera del campo di esistenza di una funzioni analitica di due variabili complesse. Ann. Mat. Pura Appl. **18**(3), 69–79 (1911)
19. Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces I. Springer, Berlin (1977)
20. Matos, M.: Domains of τ-holomorphy in a separable Banach space. Math. Ann. **195**, 273–278 (1972)
21. Mujica, J.: Holomorphic approximation in Fréchet spaces with basis. J. Lond. Math. Soc. **29**, 113–126 (1984)
22. Mujica, J.: Holomorphic approximation in infinite-dimensional Riemann domains. Stud. Math. **82**, 107–134 (1985)
23. Mujica, J.: Domains of holomorphy in Banach spaces. In: Chuaqui, R. (ed.) Analysis, Geometry and Probability, pp. 181–193. Dekker, New York (1985)
24. Mujica, J.: Complex Analysis in Banach Spaces. North-Holland Math. Stud., vol. 120. North-Holland, Amsterdam (1986)
25. Mujica, J.: Spectra of algebras of holomorphic functions on infinite dimensional Riemann domains. Math. Ann. **276**, 317–322 (1987)
26. Mujica, J., Vieira, D.M.: Schauder bases and the bounded approximation property in separable Banach spaces. Stud. Math. **196**, 1–12 (2010)
27. Nachbin, L.: Topology on Spaces of Holomorphic Mappings. Springer, New York (1969)
28. Norguet, F.: Sur les domaines d'holomorphie des functions uniformes de plusieurs variables complexes. Bull. Soc. Math. Fr. **82**, 137–159 (1954)
29. Noverraz, P.: Pseudo-Convexité, Convexité Polynomiale et Domaines d'Holomorphie en Dimension Infinie. North-Holland Math. Studies, vol. 3. North-Holland, Amsterdam (1973)

30. Noverraz, P.: Approximation of holomorphic or plurisubharmonic functions in certain Banach spaces. In: Hayden, T., Suffridge, T. (eds.) Proceedings on Infinite Dimensional Holomorphy. Lecture Notes in Mathematics, vol. 364, pp. 178–185. Springer, Berlin (1974)
31. Oka, K.: Sur les fonctions de plusieurs variables, II. Domaines d'holomorphie. J. Sci. Hiroshima Univ. **7**, 115–130 (1937)
32. Oka, K.: Sur les fonctions analytiques de plusieurs variables, VI. Domaines pseudoconvexes. Tohoku Math. J. **49**, 15–52 (1942)
33. Oka, K.: Sur les fonctions de plusieurs variables, IX. Domaines finis sans point critique intérieur. Jpn. J. Math. **27**, 97–155 (1953)
34. Patyi, I.: Cohomological characterization of pseudoconvexity in a Banach space. Math. Z. **245**, 371–386 (2003)
35. Pelczynski, A.: Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis. Stud. Math. **40**, 239–243 (1971)
36. Schottenloher, M.: Spectrum and envelope of holomorphy for infinite dimensional Riemann domains. Math. Ann. **263**, 213–119 (1983)
37. Vieira, D.M.: Polynomial approximation in Banach spaces. J. Math. Anal. Appl. **328**, 984–994 (2007)
38. Weil, A.: L'integral de Cauchy et les fonctions de plusieurs variables. Math. Ann. **111**, 178–182 (1935)