# THEOREMS OF BANACH-STONE TYPE FOR ALGEBRAS OF HOLOMORPHIC FUNCTIONS ON INFINITE DIMENSIONAL SPACES

Daniela M. Vieira*
Universidade Estadual de Campinas, Campinas, Brasil

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Abstract

We prove that two domains of holomorphy $U$ and $V$ in separable Fréchet spaces with the bounded approximation property are biholomorphically equivalent if and only if the topological algebras $(\mathcal{H}(U), \tau)$ and $(\mathcal{H}(V), \tau)$ are topologically isomorphic for $\tau = \tau_0$, $\tau_\omega$, $\tau_\delta$. We prove also that given two absolutely convex open subsets $U$ and $V$ of Tsirelson-like spaces, the algebras of holomorphic functions of bounded type $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$ are topologically isomorphic if and only if there is a biholomorphic mapping of a special type between $U$ and $V$. We obtain similar results for algebras of holomorphic germs $\mathcal{H}(K)$ and $\mathcal{H}(L)$, where $K$ and $L$ are two compact subsets of Tsirelson-like spaces.

## Introduction

In [3], S. Banach proved that two compact metric spaces $X$ and $Y$ are homeomorphic if and only if the Banach algebras $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are isometrically isomorphic. M.H. Stone, in [35], generalised this result to arbitrary compact Hausdorff topological spaces, the well-known Banach-Stone theorem.

Let $E$ and $F$ be locally convex spaces, $U \subseteq E$ and $V \subseteq F$ open subsets. Consider $\mathcal{H}(U)$ the algebra of all holomorphic functions $f : U \longrightarrow \mathbb{C}$, and likewise $\mathcal{H}(V)$. The main objective of the present work is to compare the relations between the open sets $U$ and $V$ and those between the topological algebras $(\mathcal{H}(U), \tau)$ and $(\mathcal{H}(V), \tau)$, when $\tau$ is $\tau_0, \tau_\omega$ or $\tau_\delta$, (or even between the Fréchet algebras $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$).

In Section 1 we prove that when $E$ and $F$ are Fréchet spaces with the approximation property and $U \subseteq E$ and $V \subseteq F$ are absolutely convex open subsets, then $U$ and $V$ are biholomorphically equivalent if and only if the algebras $(\mathcal{H}(U), \tau)$ and $(\mathcal{H}(V), \tau)$ are topologically isomorphic, for $\tau = \tau_0$, $\tau_\omega$ and $\tau_\delta$. In Section 2, we have a similar result for polynomially convex open sets, and in Section 3, we obtain similar results for pseudoconvex domains. However, in the case of Section 3, when the geometric properties of the open sets $U$ and $V$ are weaker, we need stronger properties on the locally convex spaces $E$ and $F$, namely, they need to be Fréchet spaces with the bounded approximation property. We give counterexamples showing that the hypotheses on the Fréchet spaces and the open subsets cannot be omitted.

In Section 4, if $E$ and $F$ are Tsirelson-like spaces and $U \subseteq E$ and $V \subseteq F$ are convex and balanced open subsets, we also prove that there is a special type of

*E-mail: danim@ime.unicamp.br

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biholomorphic mapping between $U$ and $V$ if and only if the Fréchet algebras $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$ are topologically isomorphic. In Section 5, we have similar results for algebras of holomorphic germs in Tsirelson-like spaces. We prove that if $K$ and $L$ are convex and balanced compact subspaces of Tsirelson-like spaces $E$ and $F$, respectively, then $K$ and $L$ are biholomorphically equivalent if and only if the algebras of germs $\mathcal{H}(K)$ and $\mathcal{H}(L)$ are topologically isomorphic.

We refer to [15] or [27] for background information on infinite dimensional complex analysis.

## 1. Holomorphic functions on absolutely convex domains

Let $E$ and $F$ be locally convex spaces, and $U \subseteq E$ an open subset. Let $\mathcal{H}(U, F)$ denote the vector space of all holomorphic mappings $f : U \longrightarrow F$. When $F = \mathbb{C}$, we write $\mathcal{H}(U)$ instead of $\mathcal{H}(U, \mathbb{C})$, and in this case $\mathcal{H}(U)$ is an algebra. Let us now endow the algebra $\mathcal{H}(U)$ with one of its natural topologies, $\tau$. The problem of characterising the spectra of such algebras has been considered by many authors, in particular by J.M. Isidro in [19] and by J. Mujica in [24; 25; 26]. Their results are crucial for the main theorem of this section.

Let $z \in U$. The mapping $\delta_z : \mathcal{H}(U) \longrightarrow \mathbb{C}$ defined by $\delta_z(f) = f(z)$, for all $f \in \mathcal{H}(U)$, will denote the *evaluation at z*. It is clear that $\delta_z$ is a complex homomorphism of $\mathcal{H}(U)$. Let $\varphi \in \mathcal{H}(V, E)$ such that $\varphi(V) \subseteq U$. The mapping $C_\varphi : \mathcal{H}(U) \longrightarrow \mathcal{H}(V)$ given by $C_\varphi(f) = f \circ \varphi$, for all $f \in \mathcal{H}(U)$, is called a *composition operator*. It is clear that $C_\varphi$ is a homomorphism between the algebras $\mathcal{H}(U)$ and $\mathcal{H}(V)$. We will denote by $\tau_0$ the topology on $\mathcal{H}(U)$ of the uniform convergence on the compact subsets of $U$. $\tau_\omega$ will denote the topology on $\mathcal{H}(U)$ defined by the family of the seminorms that are ported by compact subsets of $U$, and we will denote by $\tau_\delta$ the bornological topology on $\mathcal{H}(U)$. We have that each evaluation and each composition operator are continuous, for $\tau = \tau_0, \tau_\omega, \tau_\delta$.

Throughout this paper we will often use the next lemma. This lemma is an immediate consequence of the Chain Rule, when $E$ and $F$ are normed spaces, but a separate proof is needed when $E$ and $F$ are locally convex spaces.

**Lemma 1.** *Let $E$ and $F$ be Hausdorff locally convex spaces, $U \subseteq E$ and $V \subseteq F$ be open subsets. Let $f \in \mathcal{H}(U; F)$, $g \in \mathcal{H}(V, E)$ such that $g(V) \subseteq U$ and $f \circ g : V \longrightarrow F$ is the inclusion mapping. Then $F$ is topologically isomorphic to a complemented subspace of $E$. If $f : U \longrightarrow V$ is bijective and $g = f^{-1}$, then $E$ and $F$ are topologically isomorphic.*

Proof. Let $f : U \longrightarrow F$ and $g : V \longrightarrow U$ such that $f(g(w)) = w$, for all $w \in V$. Let $b \in V$, $a = g(b) \in U$ and $f(a) = f(g(b)) = b \in V$. Let $P^n = P^n f(a) \in \mathcal{P}(^nE; F)$ and $Q^m = Q^m g(b) \in \mathcal{P}(^mF; E)$ be the Taylor polynomials of $f$ at $a$ and of $g$ at $b$, respectively, for all $n, m \in \mathbb{N}$. We will show that $P^1 \circ Q^1 : F \longrightarrow F$ is the identity operator, and then conclude that $F$ is topologically isomorphic to a complemented subspace of $E$. Let us denote

$$f_n(x) = \sum_{i=0}^{n} P^i(x-a) \text{ for all } x \in E,\ n \in \mathbb{N}, \text{ and } g_m(y) = \sum_{j=0}^{m} Q^j(y-b),$$
$$\text{for all } y \in F,\ m \in \mathbb{N}.$$

Let $U_0 \subset E$ be a convex and balanced neighborhood of zero such that $a+U_0 \subseteq U$. Then there exists $V_0 \subset F$, a convex and balanced neighborhood of zero such that $b+V_0 \subseteq V$ and $g(b+t) \in a + \frac{1}{2}U_0$, for all $t \in V_0$. Let $t \in V_0$ be fixed. We will show that $f_n(g_n(b+\lambda t)) \longrightarrow f(g(b+\lambda t))$ uniformly for $|\lambda| \leq 1$. This means that for each continuous seminorm $\beta \in cs(F)$:

$$\lim_{n\to\infty} \beta(f_n(g_n(b+\lambda t)) - f(g(b+\lambda t))) = 0, \text{ uniformly for } |\lambda| \leq 1. \tag{1.1}$$

Let $\beta \in cs(F)$ and $\varepsilon > 0$ be fixed. Since $L = \{b+\lambda t : |\lambda| \leq 1\}$ is a compact subset of $b+V_0$, it follows by [4, proposition 27.2] that there exists $n_1 \in \mathbb{N}$ such that $g_m(v) - g(v) \in \frac{1}{2}U_0$, for all $m \geq n_1$ and $v \in L$. Since $g(L) \subset a + \frac{1}{2}U_0$, it follows that $g_m(v) \in a + U_0$, for all $m \geq n_1$ and $v \in L$. Since $K = \{g_m(v) : m \geq n_1, v \in L\} \cup g(L)$ is a compact subset of $a+U_0$, it follows by [4, proposition 27.2] that there exists $n_2 \in \mathbb{N}$ such that $\beta(f_n(u) - f(u)) \leq \frac{\varepsilon}{2}$ for all $n \geq n_2$ and $u \in K$. In particular,

$$\beta(f_n(g_m(v)) - f(g_m(v))) \leq \frac{\varepsilon}{2}, \text{ for all } n \geq n_2,\ m \geq n_1,\ v \in L.$$

Since $f|_K : K \longrightarrow F$ is uniformly continuous and $(g_m)_{m\geq n_1}$ converges to $g$ uniformly over $L$, it is not difficult to see that $f \circ g_m$ converges to $f \circ g$, uniformly over $L$. Then, there exists $n_3 \in \mathbb{N}$ $(n_3 \geq n_1)$ such that

$$\beta(f(g_m(v) - f(g(v))) \leq \frac{\varepsilon}{2}, \text{ for all } m \geq n_3,\ v \in L.$$

Let $n_0 = \max\{n_1, n_2, n_3\}$ and $n \geq n_0$. Then $\beta(f_n(g_n(v)) - f(g(v))) \leq \varepsilon$ uniformly for $v \in L$, proving (1.1).

For each $n \in \mathbb{N}$ and $y \in F$, we have that $f_n(g_n(y)) = b + P^1(Q^1(y-b)) + S_n(y-b)$, where $S_n : F \longrightarrow F$ is a finite sum of homogeneous polynomials of degree $\geq 2$. Then it follows by (1.1) that

$$\lim_{n\to\infty} \beta([P^1(Q^1(\lambda t)) - \lambda t] + S_n(\lambda t)) = 0,$$

uniformly for $|\lambda| \leq 1$, for all $\beta \in cs(F)$. Since $\{S_n(\lambda t) : n \in \mathbb{N}, |\lambda| \leq 1\} \cup \{\lambda(t - P^1(Q^1(t))) : |\lambda| \leq 1\}$ is a compact subset of $F$, it follows that for each $\beta \in cs(F)$ fixed, there exists $C_{t,\beta} = C_t > 0$ such that $\beta(S_n(\lambda t)) \leq C_t$, for all $n \in \mathbb{N}$, $|\lambda| \leq 1$. And since $\zeta \mapsto S_n(\zeta t)$ is a holomorphic mapping of one variable, for $|\zeta| \leq 1$, it follows by the Schwarz Lemma that $\beta(S_n(\zeta t)) \leq |\zeta|^2 C_t$, for all $|\zeta| \leq 1$, $n \in \mathbb{N}$. Hence

$$\beta(t - P^1(Q^1(t))) = \lim_{n\to\infty} \beta\Big(\frac{S_n(\zeta t)}{\zeta}\Big) \leq |\zeta| C_t, \text{ for all } 0 < |\zeta| \leq 1.$$

Now, letting $\zeta \longrightarrow 0$, we have that $\beta(t - P^1(Q^1(t))) = 0$, for all $\beta \in cs(F)$ and $t \in V_0$. Hence $P^1(Q^1(t)) = t$ for all $t \in F$. If $f$ is bijective and $g = f^{-1}$, by the same arguments we show that also $Q^1 \circ P^1 : E \longrightarrow E$ is the identity operator. ■

Let $E$ and $F$ be locally convex spaces. We say that two open subsets $U \subseteq E$ and $V \subseteq F$ are *biholomorphically equivalent* if there exists a mapping $\varphi : V \longrightarrow U$ that is biholomorphic, that is, $\varphi : V \longrightarrow U$ is a bijection, and both $\varphi$ and $\varphi^{-1}$ are holomorphic.

Now we are ready to prove the main result of this section.

**Theorem 2.** *Let $E$ and $F$ be Fréchet spaces, one of them with the approximation property, and $U \subseteq E$ and $V \subseteq F$ be convex and balanced open subsets. Consider the following conditions:*

(1) *$U$ and $V$ are biholomorphically equivalent.*
(2) *The algebras $(\mathcal{H}(U), \tau_0)$ and $(\mathcal{H}(V), \tau_0)$ are topologically isomorphic;*
(3) *the algebras $(\mathcal{H}(U), \tau_\omega)$ and $(\mathcal{H}(V), \tau_\omega)$ are topologically isomorphic;*
(4) *the algebras $(\mathcal{H}(U), \tau_\delta)$ and $(\mathcal{H}(V), \tau_\delta)$ are topologically isomorphic.*

*Then (1), (2) and (3) are equivalent and they imply (4). If $E$ or $F$ is separable then (1) – (4) are equivalent.*

Proof. (1) ⇒ (2) Let $\varphi : V \longrightarrow U$ be a biholomorphic mapping. Then the composition operator $C_\varphi : \mathcal{H}(U) \longrightarrow \mathcal{H}(V)$ is a topological isomorphism between $\mathcal{H}(U)$ and $\mathcal{H}(V)$.

(2) ⇒ (1) (a) We suppose initially that $E$ and $F$ both have the approximation property. Let $T : (\mathcal{H}(U), \tau_0) \longrightarrow (\mathcal{H}(V), \tau_0)$ be a topological isomorphism. For each $w \in V$ we have that $\delta_w \circ T : (\mathcal{H}(U), \tau_0) \longrightarrow \mathbb{C}$ is a continuous complex homomorphism of $(\mathcal{H}(U), \tau_0)$. By [19, proposition 4 ], there exists a unique $z \in U$ such that $\delta_w \circ T = \delta_z$. We define $\varphi : V \longrightarrow U$ by $\varphi(w) = z$ and show that $\varphi$ is holomorphic. We have that

$$(\delta_w \circ T)(f) = f(z) = f(\varphi(w)), \text{ for all } w \in V \text{ and } f \in \mathcal{H}(U),$$

that is, $T(f)(w) = f(\varphi(w))$, for all $w \in V$ and $f \in \mathcal{H}(U)$. Then $T(f) = f \circ \varphi$, for all $f \in \mathcal{H}(U)$.

In particular we have that $T(f) = f \circ \varphi \in \mathcal{H}(U)$, for all $f \in E'$, that is, $\varphi$ is $w$-holomorphic and consequently holomorphic, by [15, proposition 3.21]. Therefore $T = C_\varphi$.

The same argument yields the existence of a mapping $\psi \in \mathcal{H}(U, F)$ with $\psi(U) \subseteq V$ such that $T^{-1} = C_\psi$. Then $Id = C_\varphi \circ C_\psi = C_{\psi \circ \varphi}$, that is, $g = g \circ \psi \circ \varphi$, for all $g \in \mathcal{H}(V)$ and in particular for all $g \in F'$. By the Hahn-Banach Theorem, we conclude that $\psi \circ \varphi = id$, and by the same arguments we have that $\varphi \circ \psi = id$. Then $\varphi$ is bijective and $\varphi^{-1} = \psi \in \mathcal{H}(U, F)$, and therefore $\varphi$ is biholomorphic.

(b) We suppose now that $E$ has the approximation property and prove that $F$ has the approximation property. Let $\varphi$ be the holomorphic mapping constructed in part (a). For each $z \in U$ we have that $\delta_z \circ T^{-1} : (\mathcal{H}(V), \tau_0) \longrightarrow \mathbb{C}$ is a continuous

complex homomorphism. By the Mackey-Arens Theorem, there exists a unique $w \in F$ such that $\delta_z \circ T^{-1} = \delta_w$. We define $w = \psi(z)$, for all $z \in U$ and then we have that $T^{-1}(g) = g \circ \psi$ for all $g \in F'$, which shows that $\psi$ is holomorphic. We also have that $g = g \circ \psi \circ \varphi$ for all $g \in F'$, and then $\psi \circ \varphi : V \longrightarrow V$ is the identity mapping. Applying Lemma 1, we have that $F$ is topologically isomorphic to a complemented subspace of $E$, and then $F$ has the approximation property. Now the conclusion follows from part (a).
(1) ⇒ (3) Apply the same arguments used in (1) ⇒ (2).
(3) ⇒ (1) By [19, proposition 4], we have that $\mathcal{S}(\mathcal{H}(U), \tau_0) = \mathcal{S}(\mathcal{H}(U), \tau_\omega) = U$. Then apply the same arguments used in (2) ⇒ (1).
(1) ⇒ (4) Apply the same arguments used in (1) ⇒ (2).
(4) ⇒ (1) If $E$ and $F$ are both separable, we may use [25, theorem 3.1] and the same arguments of (2) ⇒ (1). If $E$ is separable, the argument of the proof of (2) ⇒ (1), part (b), shows that $F$ is separable as well. ■

An examination of the proof of Theorem 2 shows that the equivalence of (1), (2) and (3) is valid for a larger class of quasi-complete locally convex spaces, the *holomorphically Mackey* spaces. We say that a locally convex space $E$ is *holomorphically Mackey* if for every open subset $U$ of $E$ we have that $\mathcal{H}(U, F) = \mathcal{H}(U, F_\sigma)$, where $F_\sigma = (F, \sigma(F, F'))$, for all complete Hausdorff locally convex space $F$, which means that every $w$-holomorphic mapping in $U$ is holomorphic. Thus we have the following theorem:

**Theorem 3.** *Let $E$ and $F$ be quasi-complete holomorphically Mackey spaces, one of them with the approximation property, and $U \subseteq E$ and $V \subseteq F$ be convex and balanced open subsets. Then the following conditions are equivalent:*

(1) *$U$ and $V$ are biholomorphically equivalent;*
(2) *the algebras $(\mathcal{H}(U), \tau_0)$ and $(\mathcal{H}(V), \tau_0)$ are topologically isomorphic;*
(3) *the algebras $(\mathcal{H}(U), \tau_\omega)$ and $(\mathcal{H}(V), \tau_\omega)$ are topologically isomorphic.*

In [5], J. Barroso, M. Matos and L. Nachbin show that every Fréchet space and every $\mathcal{DFS}$-space is holomorphically Mackey, and in [12, corollary 10], S. Dineen shows that every $\mathcal{DFM}$-space is holomorphically Mackey. For other examples of holomorphically Mackey spaces we refer to S. Dineen [14], J. Bonet *et al.* [7] and P. Pérez Carrera and J. Bonet [31, chapter 12].

**Example 4.** We give an example showing that the hypotheses of holomorphically Mackey in Theorem 3 cannot be omitted. If $A$ is an uncountable set, we consider $\mathcal{C}_0(A)$ the set of all functions $f : A \longrightarrow \mathbb{C}$ such that for every $\epsilon > 0$ there exists a finite subset of $A$, $A(\epsilon)$ such that

$$\sup_{\alpha \in A \setminus A(\epsilon)} |f(\alpha)| \leq \epsilon.$$

We consider two topologies on $\mathcal{C}_0(A)$, one of them being the topology defined by the sup norm, and the other the projective topology with respect to the projections

$\mathcal{C}_0(A) \longrightarrow \mathcal{C}_0(B)$, where $B$ varies over the countable subsets of $A$. These two topological vector spaces has been considered in [13; 20; 30]. By $\mathcal{C}_0(A)$ we mean $\mathcal{C}_0(A)$ endowed with the norm, and by $\mathcal{C}_{0,p}(A)$ we mean $\mathcal{C}_0(A)$ endowed by the projective topology. It is well known that $\mathcal{C}_0(A)$ is a Banach space with the approximation property. In [20, §4, main proposition], Josefson shows that $\mathcal{H}(\mathcal{C}_0(A)) = \mathcal{H}(\mathcal{C}_{0,p}(A))$. Moreover, Noverraz shows in [30] that $(\mathcal{H}(\mathcal{C}_0(A)), \tau) = (\mathcal{H}(\mathcal{C}_{0,p}(A)), \tau)$, for $\tau = \tau_0, \tau_\omega, \tau_\delta$. However, if $\mathcal{C}_0(A)$ and $\mathcal{C}_{0,p}(A)$ are biholomorphically equivalent, then by Lemma 1 it follows that $\mathcal{C}_0(A)$ and $\mathcal{C}_{0,p}(A)$ are topologically isomorphic, which is not true since $\mathcal{C}_0(A)$ is a Banach space but $\mathcal{C}_{0,p}(A)$ is not even a Fréchet space. Finally, we show that $\mathcal{C}_{0,p}(A)$ is not holomorphically Mackey. In fact, for $f \in \mathcal{C}_0(A)'$, we have that $f \in \mathcal{H}(\mathcal{C}_0(A)) = \mathcal{H}(\mathcal{C}_{0,p}(A))$, that is, $f = f \circ Id : \mathcal{C}_{0,p}(A) \longrightarrow \mathbb{C}$ is holomorphic, but $Id : \mathcal{C}_{0,p}(A) \longrightarrow \mathcal{C}_0(A)$ is not holomorphic, since it is not even continuous. Therefore $\mathcal{C}_{0,p}(A)$ is not holomorphically Mackey.

From the main theorem of [21] and Theorem 2 we have the following corollary:

**Corollary 5.** *Let $E$ and $F$ be Banach spaces, one of them with the approximation property. Consider the following conditions:*

(1) *$E$ and $F$ are isometrically isomorphic;*
(2) *$B_E$ and $B_F$ are biholomorphically equivalent;*
(3) *the algebras $(\mathcal{H}(B_E), \tau_0)$ and $(\mathcal{H}(B_F), \tau_0)$ are topologically isomorphic;*
(4) *the algebras $(\mathcal{H}(B_E), \tau_\omega)$ and $(\mathcal{H}(B_F), \tau_\omega)$ are topologically isomorphic;*
(5) *the algebras $(\mathcal{H}(B_E), \tau_\delta)$ and $(\mathcal{H}(B_F), \tau_\delta)$ are topologically isomorphic.*

*Then (1), (2), (3) and (4) are equivalent and imply (5). If $E$ or $F$ is separable, then $(1) - (5)$ are equivalent.*

Let $E$ and $F$ be Banach spaces. We say that two open subsets $U \subseteq E$ and $V \subseteq F$ are *linearly equivalent* if there exists a topological isomorphism $\varphi : E \longrightarrow F$ such that $\varphi(U) = V$.

Let us now suppose that $U$ is a convex, balanced and bounded open subset of a Banach space $E$. Then the space

$$E_U = \bigcup_{n \in \mathbb{N}} nU,$$

normed by the Minkowski functional of $U$, is a Banach space with open unit ball $U$. Furthermore, $E_U = E$ and the identity mapping $E_U \longrightarrow E$ is a topological isomorphism. Then we have the following corollary, which is a consequence of Corollary 5.

**Corollary 6.** *Let $E$ and $F$ be Banach spaces, one of them with the approximation property and $U \subseteq E$ and $V \subseteq F$ be convex, balanced and bounded open subsets. Consider the following conditions:*

(1) *$U$ and $V$ are linearly equivalent;*
(2) *$U$ and $V$ are biholomorphically equivalent;*

(3) *the algebras* $(\mathcal{H}(U), \tau_0)$ *and* $(\mathcal{H}(V), \tau_0)$ *are topologically isomorphic;*
(4) *the algebras* $(\mathcal{H}(U), \tau_\omega)$ *and* $(\mathcal{H}(V), \tau_\omega)$ *are topologically isomorphic;*
(5) *the algebras* $(\mathcal{H}(U), \tau_\delta)$ *and* $(\mathcal{H}(V), \tau_\delta)$ *are topologically isomorphic.*

*Then* (1)*,* (2)*,* (3) *and* (4) *are equivalent and imply* (5)*. If* $E$ *or* $F$ *is separable, then* $(1) - (5)$ *are equivalent.*

## 2. Holomorphic functions on polynomially convex domains

In Section 1 we saw that once the spectra of the topological algebra $(\mathcal{H}(U), \tau)$ is identified with $U$ ([19, proposition 4] and [25, theorem 3.1]), then we can prove Theorem 2. In this and in the next section, we will be able to generalise Theorem 2 to more general open sets, as soon as we have results on the spectra of the respective algebras of holomorphic functions. In this section we prove results for polynomially convex open sets. We say that an open subset $U$ of a locally convex space $E$ is *polynomially convex* if $\widehat{K}_{\mathcal{P}(E)} \cap U$ is compact for each compact set $K \subset U$. Here $\widehat{K}_{\mathcal{P}(E)}$ denotes the set

$$\Big\{x \in E \,:\, |P(x)| \leq \sup_K |P|, \text{ for all } P \in \mathcal{P}(E)\Big\}.$$

Using [26, theorem 5.5] and [25, theorem 4.1], both due to J. Mujica, we have the following theorem, whose proof is similar to the proof of Theorem 2.

**Theorem 7.** *Let* $E$ *and* $F$ *be Fréchet spaces, one of them with the approximation property and* $U \subseteq E$ *and* $V \subseteq F$ *be polynomially convex open subsets. Consider the following conditions:*

(1) $U$ *and* $V$ *are biholomorphically equivalent;*
(2) *the algebras* $(\mathcal{H}(U), \tau_0)$*, and* $(\mathcal{H}(V), \tau_0)$ *are topologically isomorphic;*
(3) *the algebras* $(\mathcal{H}(U), \tau_\omega)$ *and* $(\mathcal{H}(V), \tau_\omega)$ *are topologically isomorphic;*
(4) *the algebras* $(\mathcal{H}(U), \tau_\delta)$ *and* $(\mathcal{H}(V), \tau_\delta)$ *are topologically isomorphic.*

*Then* (1)*,* (2) *and* (3) *are equivalent and imply* (4)*. If* $E$ *or* $F$ *is separable and has the bounded approximation property, and* $U$ *and* $V$ *are connected, then* $(1) - (4)$ *are equivalent.*

## 3. Holomorphic functions on pseudoconvex domains

Finally, we can also obtain similar results for more general domains, the pseudoconvex domains. We say that an open subset $U$ of a locally convex space $E$ is *pseudoconvex* if $\widehat{K}_{\mathcal{P}s(U)}$ is relatively compact for each compact set $K \subset U$. Here, $\mathcal{P}s(U)$ denotes the family of all plurisubharmonic functions on $U$, and $\widehat{K}_{\mathcal{P}s(U)}$ denotes the set

$$\Big\{x \in U \,:\, f(x) \leq \sup_K f, \text{ for all } f \in \mathcal{P}s(U)\Big\}.$$

The main theorem of [34], due M. Schottenloher and [28, theorem 2.1], due to J. Mujica, together with the same ideas from Theorem 2, yield the following theorem:

**Theorem 8.** *Let $E$ and $F$ be Fréchet spaces, one of them separable with the bounded approximation property and $U \subseteq E$ and $V \subseteq F$ be connected pseudoconvex open subsets. Then the following conditions are equivalent:*

(1) *$U$ and $V$ are biholomorphically equivalent;*
(2) *the algebras $(\mathcal{H}(U), \tau_0)$ and $(\mathcal{H}(V), \tau_0)$ are topologically isomorphic;*
(3) *the algebras $(\mathcal{H}(U), \tau_\omega)$ and $(\mathcal{H}(V), \tau_\omega)$ are topologically isomorphic;*
(4) *the algebras $(\mathcal{H}(U), \tau_\delta)$ and $(\mathcal{H}(V), \tau_\delta)$ are topologically isomorphic.*

Under the hypotheses of Theorem 8, M. Schottenloher shows in [33, corollary 3.4] that $U$ and $V$ are actually domains of holomorphy. To end this section, we show that in every Banach space it is possible to construct a pair of open subsets $H$ and $D$ of a Banach space, one of them not a domain of holomorphy, which are not biholomorphically equivalent but for which the algebras $(\mathcal{H}(H), \tau)$ and $(\mathcal{H}(D), \tau)$ are topologically isomorphic, for $\tau = \tau_0$, $\tau_\omega$, $\tau_\delta$.

**Example 9.** *Let $E$ be a Banach space such that $dim(E) \geq 2$, and let us write $E = \mathbb{C}^2 \oplus N$, where $N$ is a Banach space. Let $D = \{z = (z_1, z_2, w) \in E : |z_1| < R_1, |z_2| < R_2 \text{ and } \|w\| < R\}$ and $H = \{z = (z_1, z_2, w) \in D : |z_1| > r_1 \text{ or } |z_2| < r_2\}$, where $0 < R \leq \infty$ and $0 < r_j < R_j \leq \infty$, for $j = 1, 2$. Then the algebras $(\mathcal{H}(H), \tau)$ and $(\mathcal{H}(D), \tau)$ are topologically isomorphic, for $\tau = \tau_0$, $\tau_\omega$, $\tau_\delta$, but $H$ and $D$ are not biholomorphically equivalent.*

Proof. First we will prove that each $f \in \mathcal{H}(H)$ has a unique extension $\tilde{f} \in \mathcal{H}(D)$. With that purpose, let $\rho_1 > 0$ such that $r_1 < \rho_1 < R_1$ and let $D' = \{z = (z_1, z_2, w) \in D : |z_1| < \rho_1\}$. Observe that $D = H \cup D'$. Given $f \in \mathcal{H}(H)$, let us define

$$g(z) = \frac{1}{2\pi i} \int_{|\zeta_1| = \rho_1} \frac{f(\zeta_1, z_2, w)}{\zeta_1 - z_1} d\zeta_1, \text{ for all } z \in D'. \tag{3.1}$$

Since

$$(\zeta_1 - z_1)^{-1} = \sum_{m=0}^{\infty} {z_1}^m {\zeta_1}^{-(m+1)},$$

it follows that for $z_2$ and $w$ fixed, $g$ is a holomorphic function of $z_1$, for all $|z_1| < \rho_1$. For $z_1$ and $w$ fixed, the function $\frac{f(\zeta_1, z_2, w)}{\zeta_1 - z_1}$ is a differentiable function of $z_2$, and therefore $g$ is differentiable function of $z_2$ ([27, proposition 13.14]). By the same argument we have that for $z_1$ and $z_2$ fixed $g$ is differentiable in $w$. Hence $g$ is separately holomorphic by [27, theorem 14.7], and therefore holomorphic by [27, theorem 36.8]. By the Cauchy Integral Formula for holomorphic functions of one variable, we have that $g(z) = f(z)$ for every $z = (z_1, z_2, w)$ such that $|z_1| < \rho_1$ and $|z_2| < r_2$, and therefore for every $z \in D' \cap H$, since $D' \cap H$ is connected. Then the function $\tilde{f}$ defined by $\tilde{f} = f$ on $H$ and $\tilde{f} = g$ on $D'$ is the desired extension. Hence $H$ cannot be a domain of holomorphy.

Let us define $T : \mathcal{H}(H) \longrightarrow \mathcal{H}(D)$ by $T(f) = \tilde{f}$, for every $f \in \mathcal{H}(H)$. Then $T$ is well defined, is surjective and is an isomorphism between algebras because $\tilde{f}$ is unique. It is easy to see that $T^{-1} : \mathcal{H}(D) \longrightarrow \mathcal{H}(H)$ is the restriction operator. Let us show that the algebras $(\mathcal{H}(H), \tau)$ and $(\mathcal{H}(D), \tau)$ are topologically isomorphic for $\tau = \tau_0$ and $\tau_\omega$. We have that $D \subset \mathcal{S}(\mathcal{H}(H), \tau_0)$. In fact, for $z \in D$ we define

$$h_z : \mathcal{H}(H) \longrightarrow \mathbb{C} \text{ by } h_z(f) = \tilde{f}(z) \text{ for all } f \in \mathcal{H}(H).$$

If $z \in H$, it is clear that $h_z$ is $\tau_0$-continuous. If $z \in D'$, then $h_z(f) = g(z)$, where $g$ is defined in (3.1). We see that there exists $C > 0$ such that

$$|g(z)| \leq C \sup_{|\zeta_1| = \rho_1} |f(\zeta_1, z_2, w)| = C \sup_{z \in K} |f(z)|, \quad \text{for every } f \in \mathcal{H}(H),$$

where $K = \{\zeta \in \mathbb{C} : |\zeta| = \rho_1\} \times \{z_2\} \times \{w\}$. Hence $h_z$ is $\tau_0$-continuous and therefore $D \subset \mathcal{S}(\mathcal{H}(H), \tau_0) = \mathcal{S}(\mathcal{H}(H), \tau_\omega) = \Sigma$. Let us consider the mapping $G : \mathcal{H}(H) \longrightarrow \mathcal{H}(\Sigma)$ given by $G(f) = \hat{f}$, where $\hat{f}(h) = h(f)$, for every $h \in \Sigma$. We have that $G$ is continuous for $\tau_0$ (see H. Alexander [2, sections 2 and 4]) and for $\tau_\omega$ (see M. Matos [22]). Since $H \subset D \subset \Sigma$ it follows that the following mappings are continuous:

$$(\mathcal{H}(\Sigma), \tau) \longrightarrow (\mathcal{H}(D), \tau) \xrightarrow{T^{-1}} (\mathcal{H}(H), \tau) \xrightarrow{G} (\mathcal{H}(\Sigma), \tau), \text{ for } \tau = \tau_0,\ \tau_\omega.$$

Consequently, we have that $(\mathcal{H}(H), \tau)$ and $(\mathcal{H}(D), \tau)$ are topologically isomorphic, for $\tau = \tau_0$, $\tau_\omega$. It remains to prove that $T : (\mathcal{H}(H), \tau_\delta) \longrightarrow (\mathcal{H}(D), \tau_\delta)$ is a topological isomorphism, but this is showed in [9] by G. Coeuré, in [18] by A. Hirschowitz and in [32] by M. Schottenloher.

Finally we will show that $H$ and $D$ are not biholomorphically equivalent. In fact, since $D$ is convex, we have that $D$ is domain of holomorphy, so if there exists a biholomorphic mapping between $H$ and $D$, we have that $H$ is a domain of holomorphy, by [32, theorem 1.8] or [18, theorem 2.15], but this is a contradiction. ■

When $E = \mathbb{C}^2$, the pair $(H, D)$ is called a *Hartogs figure* in $\mathbb{C}^2$.

## 4. Holomorphic functions of bounded type on Tsirelson-like spaces

In this section we present Banach-Stone theorems for algebras of holomorphic functions of bounded type, but in this case we will restrict ourselves to Banach spaces.

Let $E$ and $F$ be Banach spaces. Let $\mathcal{P}(^m E, F)$ denote the Banach space of all continuous $m$-homogeneous polynomials from $E$ into $F$. Let $\mathcal{P}_f(^m E, F)$ denote the subspace of $\mathcal{P}(^m E, F)$ generated by all polynomials of the form $P(x) = \varphi(x)^m b$, with $\varphi \in E'$ and $b \in F$. Finally, let $\mathcal{P}(E, F)$ denote the vector space of all continuous polynomials from $E$ into $F$. When $F = \mathbb{C}$, we write $\mathcal{P}(^m E)$, $\mathcal{P}_f(^m E)$ and $\mathcal{P}(E)$ instead of $\mathcal{P}(^m E, \mathbb{C})$, $\mathcal{P}_f(^m E, \mathbb{C})$ and $\mathcal{P}(E, \mathbb{C})$, respectively.

Let $U$ be an open subset of $E$. We say that an increasing sequence of open subsets $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ of $E$ is a *regular cover of $U$* if

$$U = \bigcup_{n=1}^{\infty} U_n \text{ and } d_{U_{n+1}}(U_n) > 0 \text{ for all } n \in \mathbb{N},$$

where $d_{U_{n+1}}(U_n)$ denotes the distance from $U_n$ to the boundary of $U_{n+1}$. If $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is a regular cover of $U$, we denote by $\mathcal{H}^\infty(\mathcal{U}, F)$ the vector space formed by all holomorphic mappings $f : U \longrightarrow F$ that are bounded on each $U_n$, for $n \in \mathbb{N}$. $\mathcal{H}^\infty(\mathcal{U}, F)$ is a Fréchet space for the topology of the uniform convergence on the sets $U_n$. When $F = \mathbb{C}$, we write $\mathcal{H}^\infty(\mathcal{U})$ instead of $\mathcal{H}^\infty(\mathcal{U}, F)$. In this case we have that $\mathcal{H}^\infty(\mathcal{U})$ is a Fréchet algebra. Defining composition operators between the algebras $\mathcal{H}^\infty(\mathcal{U})$ and $\mathcal{H}^\infty(\mathcal{V})$ is not as trivial as it is between the algebras $\mathcal{H}(U)$ and $\mathcal{H}(V)$. To illustrate this, we have the following theorem.

**Theorem 10.** *Let $E$ and $F$ be Banach spaces, $U \subseteq E$ and $V \subseteq F$ be open subsets and $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ and $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be regular covers of $U$ and $V$, respectively. Consider $\varphi \in \mathcal{H}(V, E)$, with $\varphi(V) \subseteq U$. Then the operator $C_\varphi : \mathcal{H}^\infty(\mathcal{U}) \longrightarrow \mathcal{H}^\infty(\mathcal{V})$ given by $C_\varphi(f) = f \circ \varphi$ for all $f \in \mathcal{H}^\infty(\mathcal{U})$ is well defined and is a continuous homomorphism if and only if for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $\varphi(V_k) \subseteq (\widehat{U_{n_k}})_{\mathcal{H}^\infty(\mathcal{U})}$.*

Proof. Let us suppose that $C_\varphi$ is well defined and is a continuous homomorphism; and let $k \in \mathbb{N}$. Since $C_\varphi$ is continuous, there exists $c > 0$ and $n_k \in \mathbb{N}$ such that

$$\sup_{V_k} |C_\varphi(f)| \le c \sup_{U_{n_k}} |f|, \text{ for all } f \in \mathcal{H}^\infty(\mathcal{U}).$$

Replacing $f$ by $f^n$, taking $n$th roots and letting $n \longrightarrow \infty$, we have that

$$\sup_{V_k} |C_\varphi(f)| \le \sup_{U_{n_k}} |f|, \text{ for all } f \in \mathcal{H}^\infty(\mathcal{U}),$$

which shows that $\varphi(V_k) \subseteq (\widehat{U_{n_k}})_{\mathcal{H}^\infty(\mathcal{U})}$.

Conversely, we have to show that $f \circ \varphi \in \mathcal{H}^\infty(\mathcal{V})$ for all $f \in \mathcal{H}^\infty(\mathcal{U})$. To see this, let $k \in \mathbb{N}$. By hypothesis there exists $n_k \in \mathbb{N}$ such that $\varphi(V_k) \subseteq (\widehat{U_{n_k}})_{\mathcal{H}^\infty(\mathcal{U})}$. For $y \in V_k$ we have that

$$|f(\varphi(y))| \le \sup_{U_{n_k}} |f| < \infty, \text{ for all } f \in \mathcal{H}^\infty(\mathcal{U}).$$

Since $y$ is an arbitrary element of $V_k$, it follows that $f \circ \varphi$ is bounded on $V_k$, for all $k \in \mathbb{N}$ and $f \in \mathcal{H}^\infty(\mathcal{U})$, proving that $f \circ \varphi \in \mathcal{H}^\infty(\mathcal{V})$ for all $f \in \mathcal{H}^\infty(\mathcal{U})$. Now it is easy to check that $C_\varphi$ is a continuous homomorphism between $\mathcal{H}^\infty(\mathcal{U})$ and $\mathcal{H}^\infty(\mathcal{V})$. ■

Now we can define composition operators between two algebras $\mathcal{H}^\infty(\mathcal{U})$ and $\mathcal{H}^\infty(\mathcal{V})$.

**Definition 11.** *Let $U \subseteq E$ and $V \subseteq F$ be open subsets and $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ and $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be regular covers of $U$ and $V$, respectively.*

(1) *We denote by $\mathcal{H}^\infty(\mathcal{V}, \mathcal{U})$ the set of all mappings $\varphi \in \mathcal{H}(V, E)$ such that for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $\varphi(V_k) \subseteq (\widehat{U_{n_k}})_{\mathcal{H}^\infty(\mathcal{U})}$.*

(2) *Let $\varphi \in \mathcal{H}^\infty(\mathcal{V},\mathcal{U})$. Then the operator $C_\varphi : \mathcal{H}^\infty(\mathcal{U}) \longrightarrow \mathcal{H}^\infty(\mathcal{V})$ defined by $C_\varphi(f) = f \circ \varphi$ for all $f \in \mathcal{H}^\infty(\mathcal{U})$, is called a composition operator.*

In [36], B. Tsirelson constructed a reflexive Banach space $X$, with an unconditional Schauder basis, that does not contain any subspace that is isomorphic to $c_0$ or to any $l_p$. R. Alencar, R. Aron and S. Dineen proved in [1] that $\mathcal{P}_f(^mX)$ is norm-dense in $\mathcal{P}(^mX)$ for all $m \in \mathbb{N}$. Inspired by this result, we will say that a Banach space $E$ is a *Tsirelson-like space* if $E$ is reflexive and $\mathcal{P}_f(^mE)$ is norm-dense in $\mathcal{P}(^mE)$ for all $m \in \mathbb{N}$.

Next we have the main theorem of this section.

**Theorem 12.** *Let $E$ and $F$ be Banach spaces, one of them a Tsirelson-like space. Let $U \subseteq E$ and $V \subseteq F$ be convex and balanced open subsets, and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ and $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be regular covers for $U$ and $V$ respectively, such that each $U_n$ and $V_n$ are bounded and circular. Then the following conditions are equivalent:*

(1) *There exists a bijective mapping $\varphi : V \longrightarrow U$ such that $\varphi \in \mathcal{H}^\infty(\mathcal{V},\mathcal{U})$ and $\varphi^{-1} \in \mathcal{H}^\infty(\mathcal{U},\mathcal{V})$;*
(2) *the algebras $\mathcal{H}^\infty(\mathcal{U})$ and $\mathcal{H}^\infty(\mathcal{V})$ are topologically isomorphic.*

PROOF. (1) $\Rightarrow$ (2) Use Theorem 10 and the ideas of the proof of Theorem 2.

(2) $\Rightarrow$ (1) If $E$ and $F$ are both Tsirelson-like spaces, we follow the same ideas of the proof of Theorem 2, using [29, lemma 2.1], instead of [19, propostion 4]. If only $E$ is a Tsirelson-like space, following the arguments of Theorem 2, (2) $\Rightarrow$ (1), part (b), we have that $F$ is a complemented subspace of $E$, and it is easy to see that in this case $F$ is a Tsirelson-like space as well. ■

By combining Theorems 12 and 2, we have the following extension result:

**Corollary 13.** *Let $E$ and $F$ be Banach spaces, one of them a Tsirelson-like space. Let $U \subseteq E$ and $V \subseteq F$ be convex and balanced open subsets, and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ and $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be regular covers for $U$ and $V$ respectively, such that each $U_n$ and $V_n$ are bounded and circular. If the algebras $\mathcal{H}^\infty(\mathcal{U})$ and $\mathcal{H}^\infty(\mathcal{V})$ are topologically isomorphic, then the algebras $(\mathcal{H}(U),\tau)$ and $(\mathcal{H}(V),\tau)$ are topologically isomorphic, for $\tau=\tau_0$, $\tau_\omega$ and $\tau_\delta$.*

To end this section, let us define the Fréchet space $\mathcal{H}_b(U,F)$. If $d_U(x)$ denotes the distance from $x$ to the boundary of $U$, then the open sets

$$U_n = \{x \in U \;:\; \|x\| < n \text{ and } d_U(x) > 2^{-n}\}$$

form a sequence of bounded open sets that cover $U$ and $d_{U_{n+1}}(U_n) > 0$ for each $n \in \mathbb{N}$. In the particular case when $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$, $U_n$ defined above, the space $\mathcal{H}^\infty(\mathcal{U},F)$ is denoted by $\mathcal{H}_b(U,F)$ and its elements are called *holomorphic mappings of bounded type.* If $F = \mathbb{C}$, we write $\mathcal{H}_b(U)$ instead of $\mathcal{H}_b(U,F)$. Let $V \subseteq F$ be an open subset, and $\mathcal{V} = (V_n)$ is defined in the similar way as above. Then the set $\mathcal{H}^\infty(\mathcal{V},\mathcal{U})$ will be denoted by $\mathcal{H}_b(V,U)$. We say that a subset $A \subseteq U$ is *$U$-bounded* if $A$ is bounded and there exists $\epsilon > 0$ such that $A + B(0,\epsilon) \subseteq U$. It is easy to see that $\mathcal{H}_b(U)$ is the set of all holomorphic functions $f : U \longrightarrow \mathbb{C}$ that are bounded in each $U$-bounded subset. Then we have the following corollaries:

**Corollary 14.** *Let $E$ and $F$ be Banach spaces, one of them a Tsirelson-like space. Let $U \subseteq E$ and $V \subseteq F$ be convex and balanced open subsets. Then the following conditions are equivalent:*

(1) *There exists a bijective mapping $\varphi : V \longrightarrow U$ such that $\varphi \in \mathcal{H}_b(V,U)$ and $\varphi^{-1} \in \mathcal{H}_b(U,V)$;*
(2) *the algebras $\mathcal{H}_b(U)$ e $\mathcal{H}_b(V)$ are topologically isomorphic.*

Corollary 14 has been obtained independently by D. Carando, D. García and M. Maestre in [8].

**Corollary 15.** *Let $E$ and $F$ be Banach spaces, one of them a Tsirelson-like space. Let $U \subseteq E$ and $V \subseteq F$ be convex and balanced open subsets. If the algebras $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$ are topologically isomorphic, then the algebras $(\mathcal{H}(U),\tau)$ and $(\mathcal{H}(V),\tau)$ are topologically isomorphic for $\tau{=}\tau_0$, $\tau_\omega$ and $\tau_\delta$.*

## 5. Germs of holomorphic functions on Tsirelson-like spaces

In this section we present similar results for algebras of germs of holomorphic functions defined on Tsirelson-like spaces.

Let $E$ be a Banach space and $K \subset E$ be a compact subset. We define the algebra $h(K) = \cup\{\mathcal{H}(U) \,:\, U \supset K \text{ is open in } E\}$. Let $f_1,\ f_2 \in h(K)$ and $U_1$, $U_2$ be open subsets of $E$ with $K \subset U_1$ and $K \subset U_2$ such that $f_1 \in \mathcal{H}(U_1)$ and $f_2 \in \mathcal{H}(U_2)$. We say that $f_1$ and $f_2$ are *equivalent* (and we denote $f_1 \sim f_2$) if there is an open subset $W \subseteq E$ with $K \subset W \subseteq U_1 \cap U_2$ such that $f_1 = f_2$ in $W$. Then $\sim$ is an equivalence relation in $h(K)$, and we denote $\mathcal{H}(K) = h(K)/\sim$, and the elements of $\mathcal{H}(K)$ are called *germs of holomorphic functions*. Finally, we endow $\mathcal{H}(K)$ with the locally convex inductive limit of the locally convex algebras $(\mathcal{H}(U),\tau_\omega)$, where $U$ varies among the open subsets of $E$ such that $K \subset U$, and we denote

$$(\mathcal{H}(K),\tau_\omega) = \varinjlim_{U\supseteq K}(\mathcal{H}(U),\tau_\omega). \tag{5.1}$$

We will denote by $i_U$ the canonical inclusion $i_U : \mathcal{H}(U) \longrightarrow \mathcal{H}(K)$, for each $U \supset K$.

Let $U_n = K + B(0,\frac{1}{n})$ for all $n \in \mathbb{N}$. Then it is easy to see that

$$(\mathcal{H}(K),\tau_\omega) = \varinjlim_{n\in\mathbb{N}}\mathcal{H}_b(U_n). \tag{5.2}$$

We will denote by $i_n$ the canonical inclusion $i_n : \mathcal{H}_b(U_n) \longrightarrow \mathcal{H}(K)$, for each $n \in \mathbb{N}$.

In this section we will denote by $\bar{f}$ the elements of the algebra $\mathcal{H}(K)$, i.e. $\bar{f} \in \mathcal{H}(K)$ if and only if there exists $n \in \mathbb{N}$ such that $f \in \mathcal{H}_b(U_n)$ and we will arbitrarily use characterisations (5.1) and (5.2) above, according to the conveniences. We refer to [6; 15; 23] for background information on algebras of germs of holomorphic functions.

Let $E$ and $F$ be Banach spaces, and $K \subset E$ and $L \subset F$ be compact subsets. We say that $K$ and $L$ are *biholomorphically equivalent* if there exist open subsets $U \subseteq E$ and $V \subseteq F$ with $K \subset U$ and $L \subset V$ and a biholomorphic mapping $\varphi : V \longrightarrow U$ such that $\varphi(L) = K$.

**Theorem 16.** *Let $E$ and $F$ be Tsirelson-like spaces. Let $K \subset E$ and $L \subset F$ be convex and balanced compact subsets. Then the following conditions are equivalent:*

(1) *$K$ and $L$ are biholomorphically equivalent;*
(2) *the algebras $\mathcal{H}(K)$ and $\mathcal{H}(L)$ are topologically isomorphic.*

PROOF. (1) $\Rightarrow$ (2) Let $V \supset L$, $U \supset K$ be open subsets, and $\varphi : V \longrightarrow U$ be a biholomorphic mapping such that $\varphi(L) = K$. Let us define $T : \mathcal{H}(K) \longrightarrow \mathcal{H}(L)$ as follows. Given $\bar{f} \in \mathcal{H}(K)$, choose a representative $f \in \mathcal{H}(U_0)$, with $K \subset U_0$, and let $V_0 = \varphi^{-1}(U \cap U_0)$. Then define $T(\bar{f}) = \overline{f \circ \left(\varphi|_{V_0}\right)}$. It is easy to see that $T$ is well defined and is a homomorphism. To show that $T$ is continuous, let $U_\alpha \supset K$ be an open subset and $V_\alpha = \varphi^{-1}(U \cap U_\alpha) \subseteq V$. Let $\varphi_\alpha = \varphi_{|V_\alpha} : V_\alpha \longrightarrow E$, with $\varphi_\alpha(V_\alpha) \subseteq U_\alpha$. Then the following diagram is commutative.

$$\begin{array}{ccc} (\mathcal{H}(K), \tau_\omega) & \xrightarrow{T} & (\mathcal{H}(L), \tau_\omega) \\ {\scriptstyle i_{U_\alpha}}\big\uparrow & & \big\uparrow{\scriptstyle i_{V_\alpha}} \\ (\mathcal{H}(U_\alpha), \tau_\omega) & \xrightarrow{C_{\varphi_\alpha}} & (\mathcal{H}(V_\alpha), \tau_\omega) \end{array}$$

Since $C_{\varphi_\alpha}$ is continuous, we conclude that $T$ is continuous as well. Defining $T^{-1}$ by the same way, using in this case $\varphi^{-1}$ instead of $\varphi$, we conclude that the algebras $\mathcal{H}(K)$ and $\mathcal{H}(L)$ are topologically isomorphic.

(2) $\Rightarrow$ (1) Let $T : \mathcal{H}(K) \longrightarrow \mathcal{H}(L)$ be a topological isomorphism, and denote $S = T^{-1}$. Let us fix an integer $k \in \mathbb{N}$. By [16, theorem A] or [17, theorem 1], there is $m_k \in \mathbb{N}$ and a continuous linear operator $T_k$ such that the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{H}(K) & \xrightarrow{T} & \mathcal{H}(L) \\ {\scriptstyle i_k}\big\uparrow & & \big\uparrow{\scriptstyle i_{m_k}} \\ \mathcal{H}_b(U_k) & \xrightarrow{T_k} & \mathcal{H}_b(V_{m_k}) \end{array}$$

Since $T \circ i_k = i_{m_k} \circ T_k$ and $T$ is multiplicative, we also have that $T_k$ is multiplicative, and since $U_k$ is convex and balanced, it follows from Corollary 14 that there is a holomorphic mapping $\varphi_k : V_{m_k} \longrightarrow E$ with $\varphi_k(V_{m_k}) \subseteq U_k$ such that $T_k = C_{\varphi_k}$. By the same argument, we can find an integer $n_k$ and a holomorphic mapping

$\psi_k : U_{n_k} \longrightarrow F$ with $\psi(U_{n_k}) \subseteq V_{m_k}$ such that the following diagram is commutative.

$$\begin{array}{ccc} \mathcal{H}(L) & \xrightarrow{\;S\;} & \mathcal{H}(K) \\ {\scriptstyle i_{m_k}}\Big\uparrow & & \Big\uparrow{\scriptstyle i_{n_k}} \\ \mathcal{H}_b(V_{m_k}) & \xrightarrow{\;C_{\psi_k}\;} & \mathcal{H}_b(U_{n_k}) \end{array}$$

Since $S \circ i_{m_k} = i_{n_k} \circ C_{\psi_k}$ and $S \circ T : \mathcal{H}(K) \longrightarrow \mathcal{H}(K)$ is the identity operator, we have that $\varphi_k \circ \psi_k : U_{n_k} \longrightarrow U_k$ is actually the inclusion mapping. Thus, we have increasing sequences $(n_k)_{k\in\mathbb{N}}$, $(m_k)_{k\in\mathbb{N}}$ and holomorphic mappings

$$\varphi_k : V_{m_k} \longrightarrow U_k \text{ and } \psi_k : U_{n_k} \longrightarrow V_{m_k}$$

such that $\varphi_k \circ \psi_k : U_{n_k} \longrightarrow U_k$ is the inclusion mapping.

**Claim 1:** $\varphi_1 = \varphi_k$ over each $V_{m_k}$. In fact, since $E' \subseteq \mathcal{H}_b(U_k)$, for each $k \in \mathbb{N}$, we have that $T(\bar{f}) = \overline{f \circ \varphi_1} = \overline{f \circ \varphi_k}$, for each $f \in E'$ and each $k \in \mathbb{N}$. By the definition of the equivalence classes $\overline{f \circ \varphi_1}$ and $\overline{f \circ \varphi_k}$ and by the Identity Principle for holomorphic functions we have that $f \circ \varphi_1 = f \circ \varphi_k$ on $V_{m_k}$ for all $f \in E'$. Now, by the Hahn-Banach Theorem, it follows that $\varphi_1 = \varphi_k$ on $V_{m_k}$. By the same arguments we have that $\psi_1 = \psi_k$ over each $U_{n_k}$.

Let $\bar{f} \in \mathcal{H}(K)$. Then $f \in \mathcal{H}_b(U_k)$, for some $k \in \mathbb{N}$. From Claim 1 it follows that $T(\bar{f}) = \overline{f \circ \varphi_k} = \overline{f \circ \big(\varphi_1|_{V_{m_k}}\big)}$; and for each $g \in \mathcal{H}_b(V_{m_k})$ we have that $S(\bar{g}) = \overline{g \circ \psi_k} = \overline{g \circ \big(\psi_1|_{U_{n_k}}\big)}$.

**Claim 2:** $\varphi_1(L) \subseteq K$. In fact, since $L \subset V_{m_k}$, for every $k \in \mathbb{N}$, it follows from Claim 1 that $\varphi_1(L) = \varphi_k(L) \subseteq U_k = K + B(0, \frac{1}{k})$ for all $k \in \mathbb{N}$. Since $K$ is closed it follows that $\varphi_1(L) \subseteq K$. By the same arguments we have that $\psi_1(K) \subseteq L$.

For each $g \in F' \subseteq \mathcal{H}_b(V_{m_1})$ we have that $S(\bar{g}) = \overline{g \circ \psi_1}$. Since $g \circ \psi_1 \in \mathcal{H}_b(U_{n_1})$ it follows that $T \circ S(\bar{g}) = \overline{(g \circ \psi_1) \circ \varphi_{n_1}} = \overline{g \circ \psi_1 \circ \varphi_1|_{V_{m_{n_1}}}}$. On the other hand, $T \circ S(\bar{g}) = \bar{g}$, and since $V_{m_{n_1}}$ is convex, it follows by the Identity Principle that $g = g \circ \psi_1 \circ \varphi_1$ on $V_{m_{n_1}}$ for all $g \in F'$. It follows by the Hahn-Banach Theorem that $\psi_1 \circ \varphi_1 : V \longrightarrow V$ is the identity mapping, where $V = V_{m_{n_1}}$. If we denote $U = \psi_1^{-1}(V)$, we have that $U \subset U_{n_1}$ and therefore $\varphi_1 \circ \psi_1 : U \longrightarrow U$ is the identity mapping. If we define $\varphi = \varphi_1|_V : V \longrightarrow U$ and $\psi = \psi_1|_U : U \longrightarrow V$, it follows that $\varphi^{-1} = \psi$ and by Claim 2 that $\varphi(L) = K$, and then we conclude that $K$ and $L$ are biholomorphically equivalent. ■

Our proof of Theorem 16 is inspired by the ideas of [10] and [11]. We do not know if Theorem 16 remains true if only one of the Banach spaces $E$ or $F$ is a Tsirelson-like space.

It is clear that all finite dimensional spaces are Tsirelson-like spaces, but in this case Theorem 16 is valid for a larger class of connected compact subsets, namely the *Oka-Weil compact subsets*. We say that a compact subset $K$ of a Banach space is an *Oka-Weil compact set* if there is a pseudoconvex open set $U$ containing $K$ such that $K = \widehat{K}_{\mathcal{P}s(U)}$. The next theorem sharpens Theorem 16 for the finite dimensional case.

**Theorem 17.** *Let $E$ and $F$ be finite dimensional spaces and let $K \subset E$ and $L \subset F$ be connected Oka-Weil compact subsets. Then the following conditions are equivalent:*

(1) *$K$ and $L$ are biholomorphically equivalent;*
(2) *the algebras $\mathcal{H}(K)$ and $\mathcal{H}(L)$ are topologically isomorphic.*

PROOF. (1) $\Rightarrow$ (2) is identical to the proof of Theorem 16.

(2) $\Rightarrow$ (1) We claim that $\mathcal{H}(K)$ is the inductive limit of a sequence of Fréchet spaces $(\mathcal{H}(W_n), \tau_\omega)$, where each $W_n$ is connected and pseudoconvex (and the same for $\mathcal{H}(L)$). Indeed, let $U_n = K + B(0; \frac{1}{n})$ for all $n \in \mathbb{N}$. Then by [27, theorem 46.2], for every $n \in \mathbb{N}$ there exists a connected pseudoconvex open subset $W_n$ such that $K \subseteq W_n \subseteq U \cap U_n$, so the claim is proved. Now, applying the same arguments of the proof of Theorem 16, using now Theorem 8 instead of Corollary 14, the conclusion follows. ■

Preliminary versions of the results in this paper were announced in [37; 38; 39; 40].

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