ACTIONS OF TAFT'S ALGEBRAS ON FINITE DIMENSIONAL ALGEBRAS

LUCIO CENTRONE AND FELIPE YASUMURA

Abstract. Let F be a field containing a primitive m-th root of the unit. We characterize the actions of a Taft's algebra H_m of a certain order m on finite dimensional arbitrary algebras. We describe the action in terms of gradings and actions by skew-derivations. Moreover we prove the associative algebra UT_2 of 2×2 upper triangular matrices with entries from F does not generate a variety of H_m-module algebras of almost polynomial growth.

1. Introduction

The H-module algebras, where H denotes a Hopf algebra, are important tool both in the theory of algebraic groups and quantum groups and not only. Motivated by algebraic structures appearing in Rational Conformal Field Theory (see [21]) it seems very useful finding constructions associating to an algebra in a monoidal category a commutative algebra in the monoidal centre. In particular, if you consider an H-module algebra A, then the corresponding commutative algebra in the braided category of Yetter-Drinfeld modules over H is given by the centralizer of A in the smash product $A\#H$ (see Corollary 5.4 of [7]).

Keeping in mind the above facts, in these notes we would like to characterize finite dimensional H_m-module algebras (not necessarily associative) in terms of their gradings and of a proper suitable element when, for a fixed m and over a field containing a primitive m-th root of unit, H_m is a Taft's Hopf algebra. We recall that given an algebra A over a field F, a skew-derivation is an F-linear map δ of A so that for every $a,b\in A$ we have $\delta(ab)=\delta(a)b+\alpha(a)\delta(b)$, where α is a suitable homomorphism of A. The skew-derivation δ is called inner if there exists an element $y\in A$ so that $\delta(a)=ya-\alpha(a)y$ and we write $\mathrm{ad}_\alpha(y)$ instead of δ. Moreover, if A is a graded algebra, we denote by $\deg(a)$ the homogeneous degree of $a\in A$ whenever a is homogeneous in the grading.

Here is the statement of our first general result (see Theorem 8 in the text) that can be seen as an improvement of one of the results obtained in [23].

If every skew-derivation of A is inner, then an H_m-module algebra structure on A is completely determined by a choice of a $\mathbb{Z}_m$-grading on A and a homogeneous element $y\in A$ so that $\deg y$ is non-trivial, and $\mathrm{ad}_\alpha(y)^m=0$.

Then an H_m-module algebra structure on UT_n, the algebra of $n\times n$ upper triangular matrices over a field F, can be effectively computed using one of the many results characterizing gradings on upper triangular matrices. On this purpose, we have to cite the papers [29] by Valenti and Zaicev, and [9] by Di Vincenzo, Koshlukov and Valenti.

The deep reason that lead us to find out relations between H-module algebra structures and gradings is given by the duality theorem in the case $H=FG$, where

G is a finite abelian group. In this case an FG-module algebra is a G-graded algebra and viceversa. Of course FG is a commutative cocommutative Hopf algebra which is semisimple if the characteristic of the field is 0 or does not divide the order of G whereas H_m is an m^2-dimensional algebra which is neither cocommutative nor semisimple. Moreover every non-semisimple Hopf algebra of dimension p^2 is isomorphic to H_p, if p is prime [24].

We also investigate H_m-identities of UT_2 seen as an associative algebra. We furnished a finite set of generators of H_m-identities and its sequence of codimension. We obtained the following result:

The H_m-exponent of UT_2 is 2.

We highlight the fact the H_m-exponent of UT_2 is an integer could not be recovered by the results of Gordienko [18] who proved the H_m-exponent of an H_m-simple algebra is an integer and Karasik [20] who proved the H-exponent of an algebra is an integer provided H being a finite dimensional semisimple Hopf algebra. In fact UT_2 is neither H_m-simple nor semisimple as mentioned above. We would also cite the papers [16] and [17] by Gordienko in which the author determines all the H_m-simple algebras showing they are not finite in number (up to H_m-isomorphisms) whereas if H is semisimple and finite dimensional the number of non-isomorphic H-simple algebra is finite (see [12]). We also obtained the next result about the growth of varieties (see Proposition 20 in the sequel).

The H_m-module algebra UT_2 does not generate a variety of almost polynomial growth.

Finally we wrote a short note on the action of pointed cocommutative Hopf algebras on UT_n in the associative case.

2. Preliminaries

2.1. Gradings on algebras.

Let F be a field so that A is a finite dimensional F-algebra (not necessarily associative) and α an automorphism of A of order m. We recall the *order* of an automorphism α of a given algebra A is the order of α as an element of the group $Aut(A)$ of the automorphisms of A.

Assume char F, the characteristic of F, does not divide m and F contains primitive m-th roots of the unit.

It is well known α induces a *$\mathbb{Z}_m$-grading* on A, that is,

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{m-1}, \tag{1}$$

where, for a fixed primitive m-root of unit γ,

$$A_k = \{a \in A \mid \alpha(a) = \gamma^k a\}$$

is a vector subspace of A and $A_iA_j \subseteq A_{i+j}$, for all $i, j \in \mathbb{Z}_m$. In this case, (1) is called a *$\mathbb{Z}_m$-grading* on A.

Group gradings have an extensive own theory. In general, a grading on an algebra can be defined by arbitrary groups, semigroups, or even sets without any structure. There is a one-to-one correspondence (*duality*) between gradings by a group and the action of a group of automorphisms under some restrictions. In particular we have the following classical result. We mention that if G is a group we shall denote by $\hat{G}$ its group of characters.

Theorem 1 (Duality Theorem)**.** *Let G be a finite abelian group and suppose that F contains a primitive $(\exp G)$-root of unit. Then any G-grading on an algebra A*

defines a $\hat{G}$-action on A by automorphisms and viceversa. In this action, a subspace $V \subseteq A$ is a graded subspace of A if and only if V is invariant under the $\hat{G}$-action. An element $a \in A$ is homogeneous in the G-grading if and only if a is an eigenvector for any $\chi \in \hat{G}$. □

Another generalization of the previous result can be obtained in the case of gradings by finite semilattices. The result is an easy consequence of Corollary 3.15 of [6] and uses the fact that *group-like elements* of a Hopf algebra can determine a grading under some special hypothesis.

Moreover, we can further generalize the previous result in the language of action of the so-called *automorphism group schemes*. In such context, we do not impose any restriction on the base field, and the grading group does not need to be finite. More information concerning graded algebras can be found in the monograph [11].

2.2. **Taft's Hopf algebras.** Let F be a field containing a primitive m-th root of the unit γ for some positive integer m. Let $(H_m, \Delta, \epsilon, S)$ be the Hopf algebra so that

$$H_m = F\langle c, x \mid c^m = 1, x^m = 0, xc = \gamma cx\rangle$$

as an algebra with comultiplication Δ such that

$$\Delta(c) = c \otimes c, \;\; \Delta(x) = x \otimes 1 + c \otimes x$$

and counit ϵ defined by

$$\epsilon(c) = 1, \;\; \epsilon(x) = 0.$$

Moreover the antipode S is such that

$$S(c) = c^{-1}, \; S(x) = -c^{-1}x.$$

Thus, H_m is an m^2-dimensional algebra which is neither commutative nor cocommutative. This algebra is known as the *Taft's Hopf algebra of order m.* A particular case of a Taft's algebra occurs when $m = 2$ and the latter algebra is known as the Sweedler's Hopf algebra.

2.3. **H-module algebras.** Let A be an F-algebra. We shall call A an *H-module algebra* if it is given an H-module structure on A with the additional condition

$$h(ab) = \sum h_{(1)}(a)h_{(2)}(b), \tag{2}$$

for every $a, b \in A$, $h \in H$ and $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ using Sweedler's notation.

As already mentioned the action of an abelian group G on an algebra A can be seen as an H-module algebra structure on A, where $H = FG$ is a Hopf algebra with canonical comultiplication and counit.

2.4. **Gradings on UT_n.** In this paper we shall study H-actions on algebras and we shall give an explicit description of H-actions on the F-algebra UT_n of upper triangular matrices with entries from the field F. We describe the actions in terms of gradings and actions by skew-derivations.

In [29] the authors give a complete classification of group gradings on UT_n considered as an associative algebra. In particular they prove that every group grading on this algebra is isomorphic to a so-called elementary grading. We would like to spend some words toward gradings on matrix algebras. If $A = M_n(F)$ (the algebra of $n \times n$ matrices with entries of F) and G is a group, let $\overline{g} = (g_1, \dots, g_n)$ be an n-tuple of elements of G, then A is G-graded by an *elementary grading* if we set $A^g = \mathrm{Span}\{e_{pq} \mid \|e_{pq}\| = g\}$, where e_{pq} are elementary matrices, $\|e_{pq}\| = g_q g_p^{-1}$

and $A = \bigoplus_{g\in G} A^g$. Of course these definitions apply to any subalgebra of A that is generated by matrix units and, in particular, to UT_n.

We state the main result of [29] in a different language. In order to do this, we should introduce some notations.

Let us give an alternative definition for the algebra UT_n of the upper triangular matrices with entries from the field F. Let $V = F^n$ be the n-dimensional vector space over F. Consider a *flag* of vector subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V, \tag{3}$$

where $\dim V_k = k$. We define

$$UT_n := \{u \in \mathrm{End}_F(V) \mid u(V_k) \subset V_k, \forall k\}.$$

Choose any basis of the flag (3), that is, an ordered basis $\{v_1, \dots, v_n\}$ of V so that $\{v_1, \dots, v_k\}$ is a basis of V_k, for all k. Thus the elements of UT_n have the following matrix form in the chosen basis:

$$UT_n = \left\{ \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \right\}.$$

As mentioned before, the complete classification of group gradings on UT_n was given in [29] by Valenti and Zaicev and this result can be restated in terms of automorphisms of UT_n as follows.

Theorem 2. *Let α be an automorphism of UT_n of order m, and consider the induced $\mathbb{Z}_m$-grading on $UT_n = \bigoplus_{k\in\mathbb{Z}_m} A_k$. Then there exists a basis of the flag* (3) *such that every A_k is spanned by matrix units.* □

However, the automorphism α acts as a multiplication by a scalar in each A_k. Hence another (and equivalent) way to state the result above is the following.

Theorem 3. *Let α be an automorphism of UT_n of order m and let γ be a primitive m-root of unit. Then there exists a basis of the flag* (3) *such that for each matrix unit e_{ij} we have $\alpha(e_{ij}) = \gamma^{a_{ij}} e_{ij}$ for some integer a_{ij}. Moreover $\alpha(e_{ii}) = e_{ii}$, for all i.* □

3. Finite dimensional H_m-module algebras

In this section we shall give some general results about finite dimensional H_m-module algebras. From now on let A be a finite dimensional algebra over a field F. We want to study the H_m-actions on A.

Notice that, by (2), the element c acts as a homomorphism of algebras on A. Moreover, since $c^m = 1$, we obtain that c acts as an automorphism of A of order m. Using the same idea, x acts as a c-derivation (also known as a *skew-derivation*), that is, it satisfies

$$x(ab) = x(a)b + c(a)x(b), \quad \forall a, b \in A.$$

Moreover, the actions of x and c are related by $xc = \gamma cx$. Conversely a choice of an automorphism α of A of order m and an α-derivation d satisfying $d^m = 0$, and $d\alpha = \gamma\alpha d$, defines an H_m-action on A. In fact it is sufficient to consider the F-algebra $F\langle\alpha, d\rangle$ which turns out to be a Hopf algebra isomorphic to H_m. Hence we get the next result.

Proposition 4. *Let A be a finite dimensional algebra over a field containing a m-th primitive root of unit γ, assume that its characteristic does not divide m, then an action of H_m on A is completely determined by a choice of:*

(i) *an automorphism α of A of order m,*
(ii) *an α-derivation d of A such that $d^m = 0$, and $\alpha d = \gamma d\alpha$.*

Equivalently, the structure of H_m-module algebra on A is uniquely determined by a choice of:

(i) *a $\mathbb{Z}_m$-grading $A = \bigoplus_{i \in \mathbb{Z}_m} A_i$,*
(ii) *an α-derivation d (where α defines the $\mathbb{Z}_m$-grading above) such that $d(A_i) \subseteq A_{i+1}$, and $d^m = 0$.*

□

3.1. **Properties of the skew-center.** In this subsection we show some properties of the so-called skew-center of a given algebra which intent will be shown later on. From now on we fix an automorphism α of A.

Definition 1. The *left skew-center* of A is defined by

$$Z_\alpha(A)^l := \{a \in A \mid ab = \alpha(b)a, \forall b \in A\}.$$

In the special case of A being associative, $\alpha = 1_A$ we get $Z_\alpha(A)^l = Z(A)$, where $Z(A)$ denotes the classical center of A.

We have the following easy result.

Lemma 5. *Consider the $\mathbb{Z}_m$-grading induced from the automorphism α of order m, then $Z_\alpha(A)^l$ is $\mathbb{Z}_m$-graded and $\alpha(Z_\alpha(A)^l) = Z_\alpha(A)^l$.*

Proof. The first part is an easy and straightforward computation. The second part is obtained apllying the final statement of Theorem 1 and we get the desired result. □

Of course we can define analogously the right skew-center of A namely $Z_\alpha(A)^r$ and in general the left and right skew-center of A do not coincide if α is any automorphism. In the case α is of order 2 the things are different. Suppose $a \in Z_\alpha(A)^l$, then for any $b \in A$ we have $ab = \alpha(b)a$ which implies $\alpha(a)\alpha(b) = b\alpha(a)$. Hence $\alpha(Z_\alpha(A)^l) \in Z_\alpha(A)^r$ but because of Lemma 5 $\alpha(Z_\alpha(A)^l) = Z_\alpha(A)^l$. We simply get the two definitions of left and right skew-center are equal in the case α is an automorphism of order 2 and we are allowed to write $Z_\alpha(A)$. Anyway from now on for the sake of simplicity we shall write $Z_\alpha(A)$ instead of $Z_\alpha(A)^l$.

The next result will be extremely useful in the sequel.

Lemma 6. *Let $y \in A$, $y \notin Z_\alpha(A)$, and assume that $\alpha(y) - \lambda y \in Z_\alpha(A)$, for some $\lambda \in F$. Then, $\lambda \neq 0$, and there exists $z \in Z_\alpha(A)$ such that*

$$\alpha(y + z) = \lambda(y + z).$$

Proof. By Lemma 5 we get $\lambda \neq 0$ because otherwise $\alpha(y) \in Z_\alpha(A) = \alpha(Z_\alpha(A))$ and due to the fact that α is an automorphism we obtain $y \in Z_\alpha(A)$ which is a contradiction. By the choice of y, the subspace $W = Fy + Z_\alpha(A)$ is $\mathbb{Z}_m$-graded too. In fact if $w = \gamma y + u$, where $\gamma \in F$ and $u \in Z_\alpha(A)$, then

$$\alpha(w) = \gamma\alpha(y) + \alpha(u) = \gamma\lambda y + \gamma u' + \alpha(u) \in Fy + Z_\alpha(A),$$

where $u' \in Z_\alpha(A)^l$. Consider a homogeneous basis of W as a vector space obtained from a completion of a homogeneous basis of $Z_\alpha(A)$, say $\{y+z\} \cup \{z_i\}_{i\in I}$, where $\{z_i\}_{i\in I}$ is a homogeneous basis of $Z_\alpha(A)^l$. Hence $F(y+z)$ is a graded subspace, thus $\alpha(y+z) = \lambda'(y+z)$, for some $\lambda' \in F$. On the other hand, $\alpha(y+z) = \lambda y + \alpha(z) + z'$, for some $z' \in Z_\alpha(A)^l$. However, since $Fy \cap Z_\alpha(A)^l = 0$, one has $\lambda = \lambda'$. This means, $\alpha(y+z) = \lambda(y+z)$ and we are done. □

We say that a skew-derivation d is *inner* if there exists $y \in A$ such that

$$d(a) = ya - \alpha(a)y, \quad \forall a \in A.$$

We denote by $\mathrm{ad}_\alpha(y)$ the map sending

$$\mathrm{ad}_\alpha(y) : a \mapsto ya - \alpha(a)y$$

which is of course a skew-derivation. With the notation above we are in position to state the next.

Lemma 7. *Let us consider an H_m-action on A and assume x acts as $\mathrm{ad}_\alpha(y)$, where $y \notin Z_\alpha(A)^l$. Then we can find $y' \in A$ such that $\mathrm{ad}_\alpha(y) = \mathrm{ad}_\alpha(y')$, and $\alpha(y') = \gamma^{-1}y'$.*

Proof. Denote by α the action of c, thus $\alpha^m = 1$. Since $x\alpha = \gamma\alpha x$, for any $a \in A$, we have

$$\begin{aligned} y\alpha(a) - \alpha^2(a)y &= \mathrm{ad}_\alpha(y)(\alpha(a)) \\ &= \gamma\alpha(\mathrm{ad}_\alpha(y)(a)) \\ &= \gamma\alpha(ya - \alpha(a)y) \\ &= \gamma(\alpha(y)\alpha(a) - \alpha^2(a)\alpha(y)). \end{aligned}$$

Thus $(y-\gamma\alpha(y))\alpha(a) = \alpha^2(a)(y-\gamma\alpha(y))$, for all $a \in A$. This means that $y-\gamma\alpha(y) \in Z_\alpha(A)^l$. By Lemma 6, we can find $z \in Z_\alpha(A)^l$ such that $\alpha(y+z) = \gamma^{-1}(y+z)$. Thus, $y' := y+z$ satisfies the conditions of the lemma. □

At the light of the previous result and under the hipothesis of Lemma 7, we can use without loss of generality either $\mathrm{ad}_\alpha(y)$ or $\mathrm{ad}_\alpha(y')$. Remark that y' satisfies $\mathrm{ad}_\alpha(y')^m = 0$. Then by Proposition 4 any choice of an automorphism of order m and such an element y' defines an action of H on A. This is the content of the following result.

Theorem 8. *Assume that every skew-derivation of A is inner. An H_m-module algebra structure on A is completely defined by a choice of:*

(i) *an automorphism α of A of order m,*
(ii) *an element $y \in A$ such that $\alpha(y) = \gamma^{-1}y$, and $\mathrm{ad}_\alpha(y)^m = 0$.*

Equivalently, an action of H_m on A is completely defined by:

(i) *a $\mathbb{Z}_m$-grading on A,*
(ii) *a homogeneous element $y \in A$, such that $\deg y = \gamma^{-1}$, and $\mathrm{ad}_\alpha(y)^m = 0$.*

□

Remark 9. It should be noted that, for a given algebra A, the identification of a structure of H_m-module algebra with a pair (Γ, y), where Γ is a grading by a group on A and $y \in A$ preserves morphisms in the following sense. Denote by A_1 and A_2 two structure of H_m-module algebras on A, and by (Γ_1, y_1) and (Γ_2, y_2) their respective pairs. Then a homomorphism of algebras $\varphi : A_1 \to A_2$ is a homomorphism of H_m-module algebras if and only if φ is a graded homomorphism such that $\varphi(y_1) - y_2 \in Z_{c_2}(\varphi(A_1))$, where we denote by c_1 and c_2 the H_m-module structure on A_1 and A_2, respectively. Indeed, one has $c_2\varphi = \varphi c_1$ if and only if φ is a graded homomorphism. Moreover, $\varphi \mathrm{ad}_{c_1}(y_1) = \mathrm{ad}_{c_2}(y_2)\varphi$ if and only if $(\varphi(y_1) - y_2)\varphi(x) = c_2(\varphi(x))(\varphi(y_1) - y_2)$, for all $x \in A$.

4. H_m-actions on upper triangular matrices

Now, we investigate toward the particular case of UT_n, the algebra of upper triangular matrices of order n, with entries from F. We will make use of the results of the last section and we assume further char $F \neq 2$.

Proposition 10. *Let α be an automorphism of UT_n of order m. Then every α-derivation of UT_n is inner.*

Proof. By Theorem 3, we can assume $\alpha(e_{ij}) = \lambda_{ij}e_{ij}$, for some $\lambda_{ij} \neq 0$, for all e_{ij}, and $\alpha(e_{ii}) = e_{ii}$, for all i. Let d be an α-derivation of UT_n. The proof will be performed by induction on n. Let $a = d(e_{11}) - 2e_{11}d(e_{11})$. So

$$\begin{aligned}
\mathrm{ad}_\alpha(a)(e_{11}) &= d(e_{11})e_{11} - 2e_{11}d(e_{11})e_{11} - (\alpha(e_{11})d(e_{11}) - 2\alpha(e_{11})e_{11}d(e_{11})) \\
&= d(e_{11})e_{11} - 2e_{11}d(e_{11})e_{11} + e_{11}d(e_{11}) \\
&= d(e_{11}^2) - 2e_{11}d(e_{11})e_{11} = d(e_{11}) - 2e_{11}d(e_{11})e_{11}.
\end{aligned}$$

However, $d(e_{11}) = d(e_{11}^2) = d(e_{11})e_{11} + \alpha(e_{11})d(e_{11})$. Thus,

$$e_{11}d(e_{11})e_{11} = e_{11}d(e_{11})e_{11}^2 + e_{11}^2d(e_{11})e_{11},$$

so $e_{11}d(e_{11})e_{11} = 0$. This gives $\mathrm{ad}_\alpha(a)(e_{11}) = d(e_{11})$.

Hence, $d' := d - \mathrm{ad}_\alpha(a)$ is a derivation such that $d'(e_{11}) = 0$. Let $f = e_{22} + \cdots + e_{nn}$. Then $fUT_nf \simeq UT_{n-1}$. Since $d'(e_{11}) = d'(1) = 0$, we have $d'(f) = 0$. Moreover, for any $fuf \in fUT_nf$, we have

$$d'(fuf) = \alpha(f)d'(u)f = fd'(u)f \in fUT_nf.$$

By induction hypothesis, we can find $y \in fUT_nf$ such that $d' \mid_{fUT_nf} = \mathrm{ad}_\alpha(y) \mid_{fUT_nf}$. Note that $\mathrm{ad}_\alpha(y)(e_{11}) = 0$. Let $d'' = d' - \mathrm{ad}_\alpha(y)$. So, $d''(fUT_nf) = 0$, and $d''(e_{11}) = 0$.

Now,

$$d''(e_{12}) = d''(e_{11}e_{12}e_{22}) = \alpha(e_{11})d''(e_{12})e_{22},$$

so $d''(e_{12}) = \lambda e_{12}$, for some $\lambda \in F$. Let $D = d'' - \mathrm{ad}_\alpha(\lambda e_{11})$. Note that $D(e_{11}) = 0$, and $D(fUT_nf) = 0$. Moreover,

$$D(e_{12}) = \lambda e_{12} - \mathrm{ad}_\alpha(\lambda e_{11})(e_{12}) = 0.$$

Finally, for any $p > 2$, we have

$$D(e_{1p}) = D(e_{12}e_{2p}) = D(e_{12})e_{2p} + \alpha(e_{12})D(e_{2p}) = 0.$$

Thus $D = 0$ and d is an inner α-derivation. □

Combining Theorem 8 and Proposition 10, we obtain:

Theorem 11. *The action of the Taft's Hopf algebra H_m on UT_n is completely determined by a choice of an automorphism α of UT_n of order m, and an inner α-derivation by an element $y \in UT_n$ such that $\alpha(y) = \gamma^{-1}y$, and $\mathrm{ad}_\alpha(y)^m = 0$.*

Equivalently, the action of H on UT_n is completely determined by a choice of a $\mathbb{Z}_m$-grading on UT_n, and a homogeneous element $y \in UT_n$ of homogeneous degree γ^{-1} such that $\mathrm{ad}_\alpha(y)^m = 0$. □

Remark 12. Notice that the center of UT_n consists of scalar matrices and is therefore isomorphic to the base field. Moreover, given an automorphism α of UT_n, and $y \in UT_n$, if $\alpha(y) = \gamma^{-1}y$, then $y \in J(UT_n)$. Hence y is a nilpotent element. Moreover, $\mathrm{ad}_\alpha(y)^m = 0$ implies $y^m = 0$. For, denote by L_y and R_y the left and right multiplication by y, respectively. Then, for any $m > 0$,

$$\begin{aligned}\mathrm{ad}_\alpha(y)^m &= \sum_{i=0}^{m}\sum_{j=0}^{i}(-1)^{m-i}\gamma^{-j}L_y^i \circ R_y^{m-i} \circ \alpha^{m-i}\\ &= \sum_{i=0}^{m}(-1)^{m-i}\left(J_{m,i}(\gamma^{-1})\right)L_y^i \circ R_y^{m-i} \circ \alpha^{m-i},\end{aligned}$$

where $J_{m+1,i}(\gamma^{-1}) = \gamma^{-m}J_{m,i}(\gamma^{-1}) + J_{m,i-1}(\gamma^{-1}),\ J_{0,1}(\gamma^{-1}) = J_{1,1}(\gamma^{-1}) = 1$. Assume that $y^m \neq 0$, and let i be the minimum such that there exists a non-zero entry (i,j) of y^m (note that $j > i$). Then by the above formula, $\mathrm{ad}_\alpha(y)^m e_{ii} = R_y^m eii \neq 0$. The converse holds if $m = 2$, that is $y^2 = 0$ implies $\mathrm{ad}_\alpha(y)^2 = 0$.

Remark 13. We can explicitly compute the skew-center of UT_n. Let α be an inner automorphism of UT_n of order m. Then it is not hard to conclude that, after a choice of basis, $\alpha(x) = AxA^{-1}$, for all $x \in UT_n$, where $A \in UT_n$ is an invertible diagonal matrix. A direct computation shows that $Z_\alpha(UT_n) = \mathrm{Span}\{A\}$.

5. H_m-IDENTITIES OF UT_2

In this section we shall compute a finite set of identities in the sense of the action of H_m on UT_2 seen as an associative algebra. As a consequence we shall compute its codimension series and we find out its exponent exists and is an integer.

We alert the reader that in the sequel we shall write $A^{\phi\psi}$ instead of $\psi(\phi(A))$ if R is a ring acting on A in some sense and $\phi, \psi \in R$. Of course we shall use the right-to-left notation in order to denote the composition of homomorphisms too.

We shall construct a free object inside the class of H-module algebras. Let H be a Hopf algebra with unit 1 and let us consider a countable set of indeterminates $X := \{x_1, x_2, \ldots\}$; we set $x_j := x_j^1$. We choose a linear basis $(\gamma_\beta)_{\beta\in\Lambda}$ in H and we denote by $F\langle X|H\rangle$ the free associative algebra over F generated by the free formal generators $x_i^{\gamma_\beta}$ lying in the set $X^H = \{x_i^{\gamma_\beta} | i \in \mathbb{N}, \beta \in \Lambda, i \in \mathbb{N}\}$. If $h \in H$ let $x_i^h := \sum_{\beta\in\Lambda}\alpha_\beta x_i^{\gamma_\beta}$ for $h = \sum_{\beta\in\Lambda}\alpha_\beta\gamma_\beta$, $\alpha_\beta \in F$, where only a finite number of α_β's is non-zero. We refer to the elements of $F\langle X|H\rangle$ as *H-polynomials*. Note that here we do not consider any H-action on $F\langle X|H\rangle$.

Let A be an algebra with an H-module algebra structure. Any map $\psi : X \to A$ has a unique homomorphic extension $\overline{\psi} : F\langle X|H\rangle \to A$ such that $\overline{\psi}(x_i^h) = h\psi(x_i)$ for all $i \in \mathbb{N}$ and $h \in H$. An H-polynomial $f \in F\langle X|H\rangle$ is an *H-identity* of A if $\overline{\psi}(f) = 0$ for all maps $\psi : X \to A$ which are called *substitutions*. In other words,

$f(x_1, x_2, \dots, x_n)$ is an H-identity of A if and only if

$$\overline{\psi}(f) = f(\overline{x}_1, \dots, \overline{x}_n) = f(a_1, a_2, \dots, a_n) = 0$$

for any $a_i \in A$, where $\overline{x}_i := \psi(x_i)$. In this case we write $f \equiv 0$. The set $\mathrm{Id}^H(A)$ of all H-identities of A is an ideal of $F\langle X|H\rangle$ which is invariant under all endomorphisms of $F\langle X|H\rangle$, i.e., it is a T^H-ideal. Note that our definition of $F\langle X|H\rangle$ depends on the choice of the basis $(\gamma_\beta)_{\beta\in\Lambda}$ in H. However such algebras can be identified in a natural way, and $\mathrm{Id}^H(A)$ turns out to be the same.

It is worthy noticing that if we consider the trivial Hopf algebra $H = F$, then we are simply studying *ordinary polynomial identities* and we shall omit any index or super-index to refer to its H-identities or related stuffs. For further lectures about polynomail identities we strongly recommend the books [10] by Drensky and [14] by Giambruno and Zaicev.

Now, let us take a moment to analyze the H-polynomials. First, let us discuss the proof of Proposition 3.3.6 of [14]. The proof gives us a beautiful duality between G-gradings and G-actions if G is a finite abelian group (as we mentioned before). However one can define G-polynomials as FG-polynomials, where the group algebra FG is endowed with its canonical Hopf algebra structure. Notice also in that proof the authors take an opportune linear basis of FG (corresponding to "projections") such a way the FG-polynomials correspond to G-graded polynomials, where the G-grading is constructed adequately. Thus, given a finite abelian group G and a finite-dimensional algebra A with a G-action we obtain a G-grading on A and viceversa; furthermore, the G-polynomial identities and the G-graded identities coincide, that is

$$Id^{FG}(A) = Id^{gr}(A).$$

Notice also by [3] to every group grading we can associate a certain signature. We recall the definition of a signature. We say that a vector space $\mathcal{A}$ is an Ω-algebra and Ω is a *signature* of A, where $\Omega = \bigcup_{n\geq 0}\Omega_n$, if each $\omega_n \in \Omega_n$ defines an n-linear map $\omega_n : \mathcal{A}\times\cdots\times\mathcal{A} \to \mathcal{A}$. For instance, our definition of algebra is a Ω-algebra, where $|\Omega_2| = 1$, and $\Omega_n = \emptyset$, for $n \neq 2$. We can construct the free Ω-algebra, so we can talk about Ω-polynomials identities (see, for instance, [19, Chapter 2]). Let $\mathcal{A}$ be a G-graded algebra, where G is finite and define $\pi_g : \mathcal{A} \to \mathcal{A}$ as the projection with respect to the decomposition $\mathcal{A} = \bigoplus_{g\in G}\mathcal{A}_g$. Hence we can consider the signature $\Omega_G = \Omega_1 \cup \Omega_2$, where $|\Omega_2| = 1$, and $\Omega_1 = \{\pi_g \mid g \in G\}$. In [3], the authors prove that the elements $\pi_g(x)$ in the relatively free Ω_G-algebra correspond to graded variables of degree g. Thus

$$\mathrm{Id}_{\Omega_G}(\mathcal{A}) = \mathrm{Id}^{\mathrm{gr}}(\mathcal{A}).$$

Going back to the Taft's Hopf algebra H_m, note that if we consider a linear basis $\mathcal{B}_1$ of the subalgebra $\langle c\rangle$ of H_m generated by c, then $\{x^i\beta \mid \beta \in \mathcal{B}_1, i = 0, 1, \dots, m-1\}$ is a basis of H_m. Let A be an H_m-module algebra. The proof of [14, Proposition 3.3.6] gives us a basis $\{\chi_1, \dots, \chi_m\}$ of $\langle c\rangle$ such that each χ_i corresponds to a projection of a certain $\mathbb{Z}_m$-grading on A. So $\mathcal{B} = \{x^j\chi_i\}$ is a basis of H_m and $\Omega = \Omega_1 \cup \Omega_2$ is a signature, where $\Omega_1 = \mathcal{B}$, and $|\Omega_2| = 1$. Let $D_m = F\langle x\rangle = span_F\langle\{1, x, x^2, \dots, x^{m-1}\}\rangle$. By [3] again we have the variables $x^i(\chi_j(x))$ correspond to graded variables under the action of x^j. In few words the H_m-polynomials correspond to $\mathbb{Z}_m$-graded polynomials with the action of D_m and the polynomial identities coincide. Finally we have the next.

Proposition 14. *Let A be a finite-dimensional associative H_m-module algebra. Consider the corresponding $\mathbb{Z}_m$-grading and the skew-derivation d, as in Proposition 4, and let $D_m = F\langle x\rangle$. We consider the G-graded polynomials with the action of d, then*

$$Id^{H_m}(A) = Id^{gr,D_m}(A).$$

□

Denote by P_n^H the space of all multilinear H-polynomials in $x_1, ..., x_n$, $n \in \mathbb{N}$, i.e.,

$$P_n^H := \langle x_{\sigma(1)}^{h_1} x_{\sigma(2)}^{h_2} \cdots x_{\sigma(n)}^{h_n} | h_i \in H, \sigma \in S_n\rangle \subset F\langle X|H\rangle.$$

The symmetric group S_n acts on the left on the space P_n^H by $\sigma(x_i^h) = x_{\sigma(i)}^h$ if $\sigma \in S_n$. Notice that the vector space $P_n^H \cap \mathrm{Id}^H(A)$ is stable under this S_n action, hence $P_n^H(A) := P_n^H/(P_n^H \cap \mathrm{Id}^H(A))$ is a left S_n-module. This leads us to consider the S_n-character of $P_n^H(A)$, namely $\chi_n^H(A)$, which is called *n-th cocharacter of polynomial H-identities* or the *n-th H-cocharacter* of A. By the classical theory of representations of the symmetric group (see for instance the book by Sagan [25]), the irreducible S_n-characters are in one-to-one correspondence with the partitions of the non-negative integer n because the ground field is of characteristic 0. In particular, if χ_λ denotes the irreducible S_n-character corresponding to the partition λ, then we are allowed to write

$$\chi_n^H(A) = \sum_{\lambda \vdash n} m_\lambda^H \chi_\lambda,$$

where $m_\lambda^H \geq 0$ is the multiplicity of the irreducible character χ_λ in the decomposition of $\chi_n^H(A)$. Moreover the non-negative integer

$$c_n^H(A) := \dim_F(P_n^H(A))$$

is called the *n-th codimension of polynomial H-identities* or the *n-th H-codimension* of A. We shall also refer to the sequences $\{\chi_n^H(A)\}_{n\geq 0}$, $\{c_n^H(A)\}_{n\geq 0}$ as the *H-cocharacter sequence of A* and the *H-codimension sequence of A* respectively.

Given an H-module algebra A, if the limit

$$\lim_n \sqrt[n]{c_n^H(A)}$$

exists we shall call it *H PI-exponent* of A and we shall denote it by $\exp^H(A)$.

It is well known if we specialize H with the dual algebra of the group algebra FG, where G is a finite group, we get the notion of *G-graded identities, codimension, exponent*, etc. The existence of the exponent in the graded case has been studied by several authors as Giambruno and Zaicev in [15] when G is the trivial group, Benanti, Giambruno and Pipitone in [4] when $G = \mathbb{Z}_2$, by Aljadeff, Giambruno and La Mattina in [2] in the case A is affine and G is abelian, by Giambruno and La Mattina (see [13]) if A is any G-graded algebra and G is abelian and in general by Aljadeff and Giambruno in [1]. In the general case of an H-algebra only partial results are known about the existence of such exponent. If H is finite dimensional and semisimple, then Karasik proved in [20] the H-exponent exists and is an integer. It is easy to see Taft's algebras are not semisimple algebras. Hence the next result by Gordienko [18] is another good step in this direction.

Theorem 15. *Let A be a finite dimensional H_m-simple algebra over an algebraically closed field F of characteristic 0. Then $exp^{H_m}(A)$ exists and is an integer.*

Notice that neither UT_2 is an H_m-simple algebra nor H_m is semisimple, hence we cannot apply the arguments by Gordienko and Karasik in order to prove the existence of the H_m-exponent of UT_2.

Before starting the calculation of H_m-identities of UT_2 we recall the following result by Valenti about $\mathbb{Z}_2$-graded identities of UT_2 endowed with the elementary grading induced by the pair $(\overline{0},\overline{1})$. In this case we shall denote by $Y=\{y_1,y_2\dots\}$ the countable set of variables of degree $\overline{0}$ (even) and by $Z=\{z_1,z_2,\dots\}$ the countable set of variables of degree $\overline{1}$ (odd). We shall denote by R the algebra $F\langle Y\cup Z|\mathbb{Z}_2\rangle$.

Proposition 16 ([27, Theorem 2], [22, Theorem 2.1 and Theorem 4.1])**.** *The ideal of $\mathbb{Z}_2$-graded identities of UT_2 with the grading defined by the pair $(\overline{0},\overline{1})$ is generated by the following polynomials:*

$$[y_1,y_2],\ \ z_1z_2.$$

Moreover for every $n\geq 0$ a linear basis for the space $P_n^{\mathbb{Z}_2}(UT_2)$ is given by the following sets of polynomials:

- $u_S:=y_{i_1}\cdots y_{i_k}zy_{i_{k+1}}\cdots y_{i_{n-1}}$
- $u:=y_1\cdots y_n,$

where S denotes the ordered k-tuple $(i_1,\dots,i_k)$, $i_j\in\{1,\dots,n\}$ and all the other indexes are ordered. Finally

$$c_n^{\mathbb{Z}_2}(UT_2)=n2^{n-1}+1$$

for every $n\geq 1$.

Consider now an H_m-action on UT_2. By Theorem 11, a structure of H_m-module algebra on UT_2 is equivalent to a $\mathbb{Z}_m$-grading and an inner skew-derivation $d=\mathrm{ad}_\alpha(y)$, where y is homogeneous of non-trivial degree. From the results of [29] (see Section 2.4 above), there exist two gradings on UT_2 up to equivalence: the non-trivial one (where $\deg e_{12}$ is non-trivial), and the trivial grading. So, there are three cases to study:

(i) the grading is trivial (so d acts trivially): in this case, $\mathrm{Id}^{H_m}(UT_2)$ equals the ordinary polynomial identities of UT_2;
(ii) the grading is non-trivial and d acts trivially: in this case, $\mathrm{Id}^{H_m}(UT_2)$ equals the graded polynomial identities of UT_2, which are computed in [9] (see also Proposition 16);
(iii) the grading is non-trivial and d acts non-trivially: in this case, necessarily $d=\mathrm{ad}_\alpha(ae_{12})$, for some $a\in F$. We study this case below. Since this is the sole new case, from now on, an H_m-action on UT_2 means a non-trivial grading on UT_2 with a non-trivial action of d always on UT_2.

Notice that by Theorem 11, if α is an automorphism of UT_2 of degree m and γ is a primitive m-th root of unity, then

$$\alpha\left(\begin{pmatrix} x_{11} & x_{12}\\ 0 & x_{22}\end{pmatrix}\right)=\begin{pmatrix} x_{11} & \gamma^{-1}x_{12}\\ 0 & x_{22}\end{pmatrix},$$

whereas

$$ad_\alpha(y)\left(\begin{pmatrix} x_{11} & x_{12}\\ 0 & x_{22}\end{pmatrix}\right)=\begin{pmatrix} 0 & a(x_{22}-x_{11})\\ 0 & 0\end{pmatrix}$$

for any $\begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix} \in UT_2(F)$ and for a suitable $y = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ satisfying the hipothesis of Theorem 11.

Now we are ready to state our result about H_m-polynomial identities of UT_2.

Theorem 17. *For each $j = 0, 1, 2, \ldots, m-1$, let $\beta_j = \sum_{\ell=0}^{m-1} \gamma^{j\ell} c^\ell$. Denote $y_i = x_i^{\beta_0}$, $z_i = x_i^{\beta_1}$. Then the T^{H_m}-ideal of UT_2 is generated by the following polynomials:*

$$[y_1, y_2],\ z_1 x^h z_2,\ z^d,\ x^{d^2},\ y_1^d x^h y_2^d,\ x^{\beta_j},$$

where $h \in H_m$, and $j = 2, \ldots, m-1$.

Moreover, for every $n \geq 0$ a linear basis for the space $P_n^{H_m}(UT_2)$ is given by the following sets of polynomials:

- $u_S := y_{i_1} \cdots y_{i_k} z y_{i_{k+1}} \cdots y_{i_{n-1}}$
- $u := y_1 \cdots y_n$
- $w_S := y_{i_1} \cdots y_{i_k} y^d y_{i_{k+1}} \cdots y_{i_{n-1}}$,

where S denotes the ordered k-tuple $(i_1, \ldots, i_k)$, $i_j \in \{1, \ldots, n\}$ and all the other indexes are ordered. Finally

$$c_n^{H_m}(UT_2) = n2^n + 1$$

for every $n \geq 1$.

Proof. For the sake of simplicity we shall only show the proof for $m = 2$ because the other cases can be treated similarly. Moreover until the end of the proof we will denote $H := H_2$. Direct computations show the polynomials above are H-identities for UT_2. Let I be the T^H-ideal generated by those polynomials and let us prove it equals $\mathrm{Id}^H(UT_2)$. Let $f = f(x_1, \ldots, x_n)$ be an H-polynomial. Because the characteristic of the field is 0 we may suppose f being multilinear of degree n.

Because of Proposition 14 we have

$$F\langle X|H_2\rangle = F\langle Y \cup Z|D_2\rangle,$$

where $Y = \{y_1, y_2, \ldots\}$, $\{z_1, z_2, \ldots\}$ are $\mathbb{Z}_2$-graded variables and for every $i \in \mathbb{N}$ we have $y_i = x_i + x_i^c$ whereas $z_j = x_j - x_j^c$. From now on we shall consider f as an element of $F\langle Y \cup Z|D_2\rangle/I$. Immediately we get $f = \alpha u + \sum_S \alpha_S u_S + \sum_T \beta_T w_S$ with $\alpha, \alpha_S, \beta_T \in F$ and S and T are defined as above. Now we consider the substitution map $y \mapsto I_2$, where I_2 denotes the identity of UT_2, which annihilates w_S for every w_S. Hence by Proposition 16 and its proof we get $\alpha = \alpha_S = 0$ for every S. Now it remains to study the polynomials w_S. We define a deg-lex order on the set of ordered k-tuples S and let S' being maximal among those so that its coefficient in the decomposition of f is non-zero. Then we consider the substitution map

$$y_i \mapsto e_{11}, y_j \mapsto e_{22}, y \mapsto A,$$

where i is an entry of S', $j \in \{1, \ldots, n\} - S'$ and A is so that $A^d \neq 0$. This substitution annihilates every w_S such that $S \neq S'$ which means $\alpha_{S'} = 0$, then repeat until all coefficients are 0 and we are done.

Let us consider now $n \geq 1$, then the number of polynomials u_S and w_S is given by $\sum_{k=0}^{n-1} \binom{n-1}{k}$ so

$$c_n^{H_2}(UT_2) = 1 + 2n \sum_{k=0}^{n-1} \binom{n-1}{k} = n2^n + 1$$

and this completes the proof. □

Corollary 18. *The H_m-exponent of UT_2 is 2.*

We recall that given an H-module algebra A, the *variety generated by* A is the class

$$\mathcal{V}^H(A) = \{B \ H\text{-module algebra} | \mathrm{Id}^H(A) \subseteq \mathrm{Id}^H(B)\}.$$

We can speak of H-identities, codimensions, exponent, etc. of a variety $\mathcal{V}$ simply referring to the H-module algebra generating $\mathcal{V}$. Any variety which exponent is 2 and so that any proper subvariety has exponent 1 is said to be of *almost polynomial growth*. Varieties of almost polynomial growth are important objects of study in the theory of algebras with polynomial identities. For our intents we would only like to cite the paper [28] by Valenti in which varieties of graded algebras of almost polynomial growth are characterized. In particular we get UT_2 with any elementary G-grading generates a variety of almost polynomial growth. The next is an important remark on the purpose.

Remark 19. Let F be a field containing a primitive m-th root of unit and so that its characteristic does not divide m. Let A be an F-algebra and do consider H_m. Suppose further A being $\mathbb{Z}_m$-graded. Then A is an H_m-module algebra too if you consider the trivial skew-derivation and let us denote such a structure by A^{gr}. Consider now the H_m-module algebras UT_2^{gr} and UT_2, where the last is one so that the skew-derivation does not act trivially. In this case

$$UT_2^{gr} \in \mathcal{V}^{H_m}(UT_2).$$

At the light of the previous remark and the discussion above, we get the next result.

Proposition 20. *The H_m-module algebra UT_2 does not generate a variety of almost polynomial growth.*

6. A note on actions of pointed cocommutative Hopf algebras on finite dimensional algebras

In this section, we study the structure of H-module algebras of a particular subset of pointed cocommutative Hopf algebras H. Every algebra here is supposed to be associative.

We consider the following structure generated by the action of a Hopf algebra H.

Definition 2. Let A be an H-module algebra over a field F. Then the *smash product* algebra $H\#A$ is defined as follows: as a vector space $H\#A = H \otimes A$ and we write $h\#a$ instead of $h \otimes a$ while the multiplication is given by

$$(h\#a)(k\#b) = \sum h_2k\#a(h_1 \cdot b),$$

for all $a, b \in A$, $h, k \in H$.

It is easy to see $A \cong 1\#A$ and $H \cong H\#1$. We also recall for a given Hopf algebra (H, Δ, ϵ) we define the set of *group-like* elements as

$$G(H) := \{h \in H | \Delta(h) = h \otimes h\},$$

while we define the set of *primitive elements* as

$$P(H) := \{h \in H | \Delta(h) = h \otimes 1 + 1 \otimes h\}.$$

Notice that if $\mathfrak{g}$ is a Lie algebra and $U(\mathfrak{g})$ is its universal enveloping algebra, then $P(U(\mathfrak{g})) = \mathfrak{g}$.

Moreover a Hopf algebra is said to be *pointed* if every simple subcoalgebra has dimension 1 whereas it is said to be *connected* if the sum of its simple subcoalgebras has dimension 1. We have a nice description of pointed cocommutative Hopf algebras attributed to Cartier and Gabriel in [8] and to Konstant in [26].

Theorem 21. *Let H be a Hopf algebra with $G = G(H)$, then if H is pointed cocommutative we get $FG\#H_1 \cong H$ via $x\#h \mapsto xh$ for any $x \in G$, $h \in H$, where H_1 is a suitable sub Hopf algebra of H containing the unit element.* □

In the connected case we have the next result due to the independent works by Cartier [5] and Kostant which remained unpublished.

Theorem 22. *Let H be a cocommutative connected Hopf algebra over a field of characteristic 0. Then $H \cong U(\mathfrak{g})$ for $\mathfrak{g} = P(H)$.* □

Keeping in mind these last two classical results, we assume F is an algebraically closed field of characteristic zero and $H = FG\#H_1$, where $G = G(H)$ is a finite abelian group, H_1 is a FG-module algebra via $g \cdot h = ghg^{-1}$ for $g \in G$, $h \in H_1$, and $H_1 = U(\mathfrak{g})$, where $\mathfrak{g} = P(H_1)$ (the set of primitive elements).

We first note that $G \cdot \mathfrak{g} \subseteq \mathfrak{g}$. Thus $\mathfrak{g}$ is a G-graded algebra and, since G is abelian, this G-grading on $\mathfrak{g}$ induces a G-grading on $U(\mathfrak{g})$.

Now let A be a finite dimensional H-module algebra. Then A is a G-graded algebra because, as remarked before, FG can be identified as a subalgebra of H. Moreover, the G-grading on A induces naturally a G-grading on $\mathrm{End}_F(A)$ and then a G-grading on $\mathrm{Der}(A)$, the set of all derivations of A. Also, we get $\mathfrak{g}$ acts as a set of derivations on A. This means we have a Lie homomorphism

$$\iota : \mathfrak{g} \to \mathrm{Der}(A)$$

which is a graded homomorphism too. Hence, we have a graded homomorphism $U(\mathfrak{g}) \to \mathrm{End}_F(A)$.

Conversely, a G-grading on A and on $\mathfrak{g}$ and a graded Lie homomorphism $\mathfrak{g} \to \mathrm{Der}(A)$ defines a structure of H-module algebra on A. This is the content of the next result.

Theorem 23. *Let F be an algebraically closed field of characteristic zero. Let $H = FG\#U(\mathfrak{g})$, then a structure of H-module algebra on a finite-dimensional algebra A is uniquely determined by*

(i) *a G-grading on A,*
(ii) *a G-grading on $\mathfrak{g}$,*
(iii) *a graded Lie homomorphism $\mathfrak{g} \to \mathrm{Der}(A)$.*

□

Looking back at the previous result, we can also consider *graded differential polynomials*, that is, graded polynomials under the action of the graded Lie algebra $\mathfrak{g}$. In this case, using Theorem 23, the H_m-identities coincide with the differential graded polynomial identities. More precisely, we get the next result.

Proposition 24. *Let $\mathfrak{g}$ be a graded Lie algebra and let A be a finite-dimensional associative $\mathfrak{g}$-module algebra. Then*

$$Id^H(A) = Id^{gr,U(\mathfrak{g})}(A).$$

Because of the work [29] every G-grading on UT_n is elementary thus is generated by an n-tuple $(g_1, \ldots, g_n) \in G^n$. As mentioned before (see Proposition 10), every derivation of UT_n is inner. This leads us to the next conclusion.

Theorem 25. *Let F be an algebraically closed field of characteristic zero. Let H be a Hopf algebra as above. Then a structure of H-module algebra on UT_n is uniquely determined by*

(i) *an n-tuple $(g_1, \ldots, g_n) \in G^n$,*
(ii) *a G-grading on $\mathfrak{g}$,*
(iii) *a graded Lie homomorphism $\mathfrak{g} \to UT_n^{(-)}$.*

7. Acknowledgments

The research of the first author is partially supported by FAPESP grant 2018/02108-7 and by CNPq grant 308800/2018-4. The second author was financed by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

References

[1] E. Aljadeff, A. Giambruno, *Multialternating graded polynomials and growth of polynomial identities*, Proc. Amer. Math. Soc. **141**(9) (2013), 3055-3065.
[2] E. Aljadeff, A. Giambruno, D. La Mattina, *Graded polynomial identities and exponential growth*, J. Reine Angew. Math. **650** (2011), 83-100.
[3] Y. Bahturin, F. Yasumura, *Distinguishing simple algebras by means of polynomial identities*, São Paulo J. Math. Sci. **13**(1) (2019), 39–72.
[4] F. Benanti, A. Giambruno, M. Pipitone, *Polynomial identities on superalgebras and exponential growth*, J. Algebra **269** (2003), 422-438.
[5] P. Cartier, Groupes algébriques et groupes formels, in Coll. sur la théorie des groupes algébriques, Bruxelles, CBRM, 1962.
[6] L. Centrone, *Action of Pontryagin dual of semilattices on graded algebras*, Comm. Algebra **42**(8) (2014), 3491-3506.
[7] A. Davidov, *Full Centre of an H-Module Algebra*, Comm. Algebra **40**(1) (2012), 273-290.
[8] J. Dieudonné, Introduction to the Theory of Formal Groups, Marcel Dekker, New York, 1973.
[9] O. Di Vincenzo, P. Koshlukov, A. Valenti, *Gradings on the algebra of upper triangular matrices and their graded identities*, J. Algebra **275** (2004), 550–566.
[10] V. Drensky, Free algebras and PI-algebras. Graduate course in algebra, Springer-Verlag Singapore, Singapore, 2000.
[11] A. Elduque, M. Kochetov, *Gradings on simple Lie algebras*, Mathematical Surveys and Monographs, vol. 189, American Mathematical Society, 2013.
[12] P. Etingof, D. Nikshych, V. Ostrik, *On fusion categories*, Ann. Math. **162**(2) (2005), 581-642.
[13] A. Giambruno, D. La Mattina, *Graded polynomial identities and codimensions: computing the exponential growth*, Adv. Math. **225** (2010), 859-881.
[14] A. Giambruno, M. Zaicev, Polynomial identities and asymptotic methods, Math. Surveys and Monographs 122, AMS, Providence, RI, 2005.
[15] A. Giambruno, M. V. Zaicev, *Exponential codimension growth of P.I. algebras: an exact estimate*, Adv. Math. **142** (1999), 221-243.
[16] A. S. Gordienko, *Algebras simple with respect to a Sweedler's algebra action*, J. Alg. Appl. **14**(1) (2015), 1–15.
[17] A. S. Gordienko, *Algebras simple with respect to a Taft algebra action*, J. Pure and Appl. Alg. **219**(8) (2015), 3279–3291.
[18] A. Gordienko, *Lie algebras simple with respect to a Taft algebra action*, J. Algebra **517** (2019), 249-275.
[19] N. Jacobson, Basic algebra. II. Second edition. W. H. Freeman and Company, New York, 1989.

[20] Y. Karasik, *Kemer's theory for H-module algebras with application to the PI exponent*, J. Algebra **457** (2016), 194-227.
[21] L. Kong, I. Runkel, *Morita classes of algebras in modular tensor categories*, Adv. Math. **219**(2008), 1548-1576.
[22] P. Koshlukov, A. Valenti, *Graded identities for the algebra of $n \times n$ upper triangular matrices over an infinite field*, Internat. J. Algebra Comput. **13**(5) (2003), 517–526.
[23] S. Montgomery, H.-J. Schneider, *Skew derivations of finite-dimensional algebras and actions of the double of the Taft Hopf algebra*, Tsukuba J. Math. **25**(2) (2001), 337-358
[24] S.-H. Ng, *Non-semisimple Hopf algebras of dimension p^2*, Journal of Algebra **255** (2002), 182–197.
[25] B. E. Sagan, The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions, Graduate Texts in Mathematics 203, Springer Verlag, 2000.
[26] M. E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[27] A. Valenti, *The graded identities of upper triangular matrices of size two*, J. Pure Appl. Algebra **172** (2002), 325–335.
[28] A. Valenti, *Group graded algebras and almost polynomial growth*, J. Algebra **334**(1) (2011), 247-254.
[29] A. Valenti, M. Zaicev, *Group gradings on upper triangular matrices*, Arc. Math. **89** (2007), 33-40.

Campus Universitario "Ernesto Quagliarello"-Università degli Studi di Bari Aldo Moro, via E. Orabona 4 - 70125 Bari - Italy

IMECC, Universidade Estadual de Campinas, Rua sérgio Buarque de Holanda 651-13083-859 Campinas (SP)-Brazil
Email address: lucio.centrone@uniba.it, centrone@unicamp.br

Department of Mathematics, Instituto de Matemática e Estatística, Universidade de São Paulo, Brazil
Email address: fyyasumura@ime.usp.br