Steinberg slices in quasi-Poisson varieties

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Poisson transversals

M a complex manifold

$\pi \in \Gamma(\wedge^2 T_M)$ Poisson bivector

$\rightsquigarrow$ stratification by symplectic leaves $M = \sqcup L$

Definition

A submanifold $X \subset M$ is a Poisson transversal if, for each symplectic leaf (L, ω),

- $X \pitchfork L$;
- $\omega_{|X \cap L}$ is symplectic.

Then

- there is an induced Poisson structure $\pi_X \in \Gamma(\wedge^2 T_X)$;
- the symplectic leaves of (X, π_X) are $\{(X \cap L, \omega_{|X \cap L})\}$.

Poisson transversals

G semisimple algebraic group of adjoint type over $\mathbb{C}$

$\mathfrak{g} = \text{Lie } G$

$l = \text{rk}(\mathfrak{g})$

$\mathfrak{g}^* \cong \mathfrak{g} \quad \rightsquigarrow \quad \pi_{KKS}$ is a Poisson structure on $\mathfrak{g}$

The regular locus of $\mathfrak{g}$ is

$$\mathfrak{g}^{\mathrm{r}} = \{x \in \mathfrak{g} \mid \dim G^x = l\}.$$

- x regular semisimple $\rightsquigarrow G^x$ is a maximal torus
- x regular nilpotent $\rightsquigarrow G^x$ is a abelian group $\cong \mathbb{C}^l$

Poisson transversals

Let $\{e, h, f\} \subset \mathfrak{g}$ be a regular $\mathfrak{sl}_2$-triple.

Theorem [Kostant]

The principal slice

$$\mathcal{S} = f + \mathfrak{g}^e \subset \mathfrak{g}^r$$

meets each regular G-orbit on $\mathfrak{g}$ exactly once, transversally.

$\rightsquigarrow$ $\mathcal{S}$ is a Poisson transversal with $\pi_{\mathcal{S}} = 0$.

The universal centralizer

$$\begin{array}{ccc} G \times G \circlearrowright T^*_G \cong G \times \mathfrak{g} & & (a, x) \\ \downarrow \mu & & \downarrow \\ \mathfrak{g} \times \mathfrak{g} & & (a \cdot x, -x) \end{array}$$

μ is a Poisson map with image $\{(x, y) \in \mathfrak{g} \times \mathfrak{g} \mid -y \in G \cdot x\}$

$$\Rightarrow \quad \mu^{-1}(\mathcal{S} \times -\mathcal{S}) = \mu^{-1}(\mathcal{S}_\Delta) = \{(a, x) \in G \times \mathcal{S} \mid a \in G^x\} =: \mathcal{Z}.$$

The universal centralizer $\mathcal{Z}$ is a Poisson transversal (=symplectic submanifold) in T^*_G.

The universal centralizer

G has a canonical smooth compactification $\overline{G}$, called the wonderful compactification.

Plan

Compactify the centralizer fibers of $\mathcal{Z}$ in $\overline{G}$.

$$G \rightsquigarrow \overline{G}$$

$$T^*_G \rightsquigarrow T^*_{\overline{G},D}$$

Extend the symplectic structure on $\mathcal{Z}$ to a log-symplectic structure on its partial compactification.

The wonderful compactification

Let $\tilde{G}$ be the simply-connected cover of G,
V a regular irreducible $\tilde{G}$-representation.

Definition [DeConcini–Procesi]

$$\begin{array}{ccc} \tilde{G} & \longrightarrow & (\mathrm{End}\ V)\backslash\{0\} \\ \downarrow & & \downarrow \\ G & \overset{\varphi}{\hookrightarrow} & \mathbb{P}(\mathrm{End}\ V). \end{array}$$

The wonderful compactification of G is $\overline{G} := \overline{\varphi(G)}$.

- independent of V
- smooth projective $G \times G$-variety
- $D := \overline{G} \backslash G$ is a simple normal crossing divisor

The wonderful compactification

Example

Let $G = PGL_2 \quad \rightsquigarrow \quad \tilde{G} = SL_2, \quad V = \mathbb{C}^2$. Then

$$\varphi(G) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}(M_{2\times 2}) \mid ad - bc \neq 0 \right\},$$

and $\overline{G} = \mathbb{P}(M_{2\times 2}) \cong \mathbb{P}^3$.

$$D = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}(M_{2\times 2}) \mid ad - bc = 0 \right\} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Non-example

Let $G = PGL_n$ for $n \geqslant 3$. Then $V = \mathbb{C}^n$ is not a regular rep of $\tilde{G} = SL_n$, and

$$\overline{G} \ncong \mathbb{P}^{n^2-1}.$$

The wonderful compactification

$T^*_{\overline{G},D}$ logarithmic cotangent bundle of $\overline{G}$

- sections are logarithmic differential forms with poles along D
- canonical log-symplectic Poisson structure
  - top wedge power of Poisson bivector vanishes with minimal multiplicity on D
- open dense symplectic leaf is $T^*_G \subset T^*_{\overline{G},D}$

$T^*_{\overline{G},D}$ sits as a subbundle

$$\begin{array}{ccccc} T^*_G & \hookrightarrow & T^*_{\overline{G},D} & \hookrightarrow & \overline{G} \times \mathfrak{g} \times \mathfrak{g} \\ & \searrow & & \searrow & \downarrow \\ & & G & \hookrightarrow & \overline{G}. \end{array}$$

The partial compactification $\overline{\mathcal{Z}}$

$$
\begin{array}{ccc}
G \times G \circlearrowright \quad T^*_{\overline{G},D} & \hookleftarrow & T^*_G \\
\downarrow \overline{\mu} & \swarrow \mu & \\
\mathfrak{g} \times \mathfrak{g} & &
\end{array}
$$

$\overline{\mu}$ is a Poisson map with image $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$.

$$\Rightarrow \quad \overline{\mu}^{-1}(\mathcal{S} \times -\mathcal{S}) = \overline{\mu}^{-1}(\mathcal{S}_\Delta)$$

is a smooth submanifold of $T^*_{\overline{G},D}$ with an induced log-symplectic Poisson structure.

Theorem [B.]

$$\overline{\mu}^{-1}(\mathcal{S}_\Delta) \cong \{(a, x) \in \overline{G} \times \mathcal{S} \mid a \in \overline{G^x}\} =: \overline{\mathcal{Z}}$$

is a smooth, log-symplectic partial compactification of $\mathcal{Z}$.

A multiplicative analogue

$$\mathfrak{g} \rightsquigarrow \tilde{G}$$

$G \circlearrowright \tilde{G}$ by conjugation

Theorem [Steinberg]

There is an l-dimensional affine subspace

$$\Sigma \subset \tilde{G}^{\mathrm{r}}$$

which meets each regular conjugacy class in $\tilde{G}$ exactly once, transversally.

A multiplicative analogue

Definition

The (multiplicative) universal centralizer of $\tilde{G}$ is

$$\mathfrak{Z} := \left\{(a, h) \in G \times \Sigma \mid aha^{-1} = h\right\}$$

$$\mathfrak{g} \rightsquigarrow \tilde{G}$$

$$\mathcal{S} \rightsquigarrow \Sigma$$

$$\mathcal{Z} \rightsquigarrow \mathfrak{Z}$$

A multiplicative analogue

Definition

The (multiplicative) universal centralizer of $\tilde{G}$ is

$$\mathfrak{Z} := \left\{(a, h) \in G \times \Sigma \mid h \in \Sigma, aha^{-1} = h\right\}$$

$$\mathfrak{g} \rightsquigarrow \tilde{G}$$

$$\mathcal{S} \rightsquigarrow \Sigma$$

$$\mathcal{Z} \rightsquigarrow \mathfrak{Z}$$

$$T^*_G,\ T^*_{\overline{G},D} \rightsquigarrow ??$$

Quasi-Poisson structures

$$\tilde{G} \circlearrowright M \rightsquigarrow \begin{array}{rcl} \mathfrak{g} & \longrightarrow & \Gamma(T_M) \\ \xi & \longmapsto & \xi_M \end{array}$$

There is a canonical invariant Cartan 3-tensor $\chi \in \wedge^3 \mathfrak{g}$

$$\rightsquigarrow \chi_M \in \Gamma(\wedge^3 T_M)^{\tilde{G}}.$$

Definition [Alekseev–Kosmann-Schwartzbach–Meinrenken]

A quasi-Poisson structure on M is a bivector $\pi \in \Gamma(\wedge^2 T_M)^{\tilde{G}}$ such that

$$[\pi, \pi] = \chi_M.$$

The q-Poisson manifold (M, π) is Hamiltonian if it is equipped with a group-valued moment map

$$\Phi : M \longrightarrow \tilde{G}.$$

Quasi-Poisson structures

Definition [Alekseev–Kosmann-Schwartzbach–Meinrenken]

The Hamiltonian q-Poisson manifold (M,π) is nondegenerate if

$$\begin{aligned}\Psi : T_M^* \oplus \mathfrak{g} &\longrightarrow T_M \\ (\alpha,\xi) &\longrightarrow \pi^{\#}(\alpha) + \xi_M\end{aligned}$$

is surjective. In general, $\operatorname{im}\Psi \subset T_M$ is an integrable distribution

$\rightsquigarrow$ M is stratified by nondegenerate q-Poisson leaves.

Example

$\tilde{G} \circlearrowright \tilde{G}$ by conjugation

$\tilde{G}$ has a q-Poisson bivector $\pi_{AKM} \in \Gamma(\wedge^2 T_{\tilde{G}})^{\tilde{G}}$

moment map $\Phi = \mathrm{Id}$

nondegenerate leaves $\leftrightarrow$ conjugacy classes

Example

The double $\mathbb{D}_G := G \times \tilde{G}$ has a nondegenerate q-Poisson structure relative to the action

$$(g_1, g_2) \cdot (a, h) = (g_1 a g_2^{-1}, g_2 h g_2^{-1}).$$

The group-valued moment map is

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} \circlearrowright & \mathbb{D}_G & (a, h) \\ & \downarrow \mu & \downarrow \\ & \tilde{G} \times \tilde{G} & (aha^{-1}, h^{-1}) \end{array}$$

Steinberg slices

Proposition [B.]

Let (M, π) be a q-Poisson $\tilde{G}$-manifold,

$M = \sqcup L$ its stratification by nondegenerate leaves,

and with moment map

$$\Phi : M \longrightarrow \tilde{G}.$$

Then $M_\Sigma := \Phi^{-1}(\Sigma)$ is a smooth submanifold of M,

with a natural induced Poisson structure $\pi_\Sigma \in \Gamma(\wedge^2 T_{M_\Sigma})$,

whose symplectic leaves are $\{M_\Sigma \cap L\}$.

Steinberg slices

Example

$$
\begin{array}{cccc}
\tilde{G} \times \tilde{G} \circlearrowright & \mathbb{D}_G & & (a, h) \\
& \Big\downarrow \mu & & \Big\downarrow \\
& \tilde{G} \times \tilde{G} & & (aha^{-1}, h^{-1})
\end{array}
$$

The image of μ is $\{(g, h) \in \tilde{G} \times \tilde{G} \mid g \in G \cdot h^{-1}\}$

$$
\begin{aligned}
\Rightarrow \quad \mu^{-1}(\Sigma \times \iota(\Sigma)) &= \mu^{-1}(\Sigma_\Delta) \\
&= \{(a, h) \in G \times \Sigma \mid aha^{-1} = h\} = \mathfrak{Z}.
\end{aligned}
$$

$\rightsquigarrow$ $\mathfrak{Z}$ is a symplectic manifold.

The partial compactification $\overline{\mathfrak{Z}}$

Recall the inclusions

$$\begin{array}{ccccc}
T^*_G & \hookrightarrow & T^*_{\overline{G},D} & \hookrightarrow & \overline{G} \times \mathfrak{g} \times \mathfrak{g} \\
& \searrow & & \searrow & \downarrow \\
& & G & \hookrightarrow & \overline{G}.
\end{array}$$

Proposition [B.]

$\mathbb{D}_G$ extends to a smooth, q-Poisson variety

$$\begin{array}{ccccc}
\mathbb{D}_G & \hookrightarrow & \mathbb{D}_{\overline{G}} & \hookrightarrow & \overline{G} \times \tilde{G} \times \tilde{G} \\
& \searrow & & \searrow & \downarrow \\
& & G & \hookrightarrow & \overline{G}.
\end{array}$$

The partial compactification $\overline{\mathfrak{Z}}$

$$
\begin{array}{ccccc}
\tilde{G} \times \tilde{G} \circlearrowright & \mathbb{D}_{\overline{G}} & \hookleftarrow & \mathbb{D}_G \\
 & {\scriptstyle \overline{\mu}} \downarrow & \swarrow {\scriptstyle \mu} & \\
 & \tilde{G} \times \tilde{G} & &
\end{array}
$$

$\overline{\mu}$ is a q-Poisson moment map with image $\tilde{G} \times_{\tilde{G}//G} \tilde{G}$.

$$\Rightarrow \quad \overline{\mu}^{-1}(\Sigma \times \iota(\Sigma)) = \overline{\mu}^{-1}(\Sigma_\Delta)$$

is a smooth submanifold of $\mathbb{D}_{\overline{G}}$ with an induced Poisson structure.

Theorem [B.]

$$\overline{\mu}^{-1}(\Sigma_\Delta) \cong \{(a, g) \in \overline{G} \times \Sigma \mid a \in \overline{G^g}\} =: \overline{\mathfrak{Z}}$$

is a smooth, log-symplectic partial compactification of $\mathfrak{Z}$.